Progress in persistence for shape analysis

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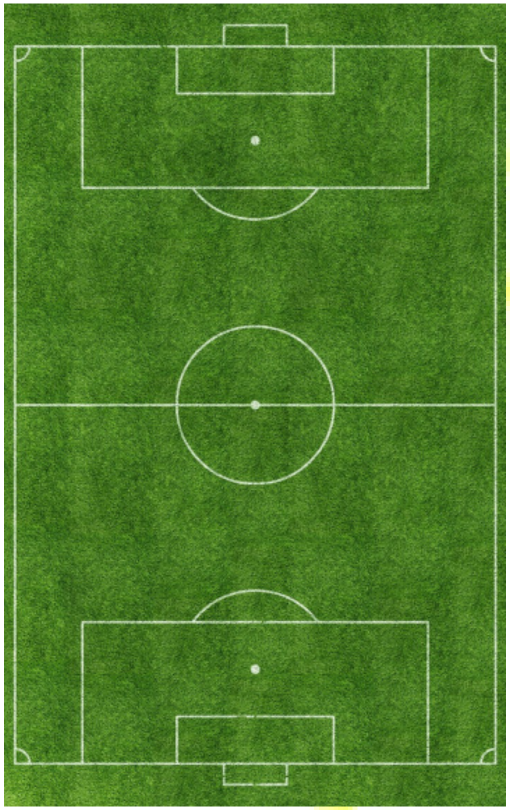
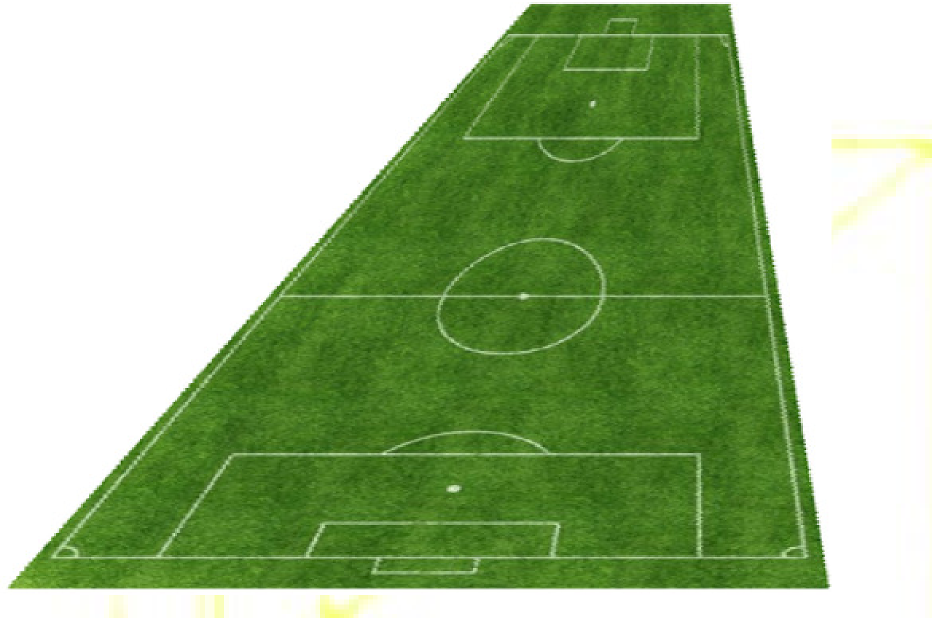
- **Persistent topology**
- Image processing
- Shape analysis
- Theoretical progress
- Not only images
- Conclusion

Persistent topology



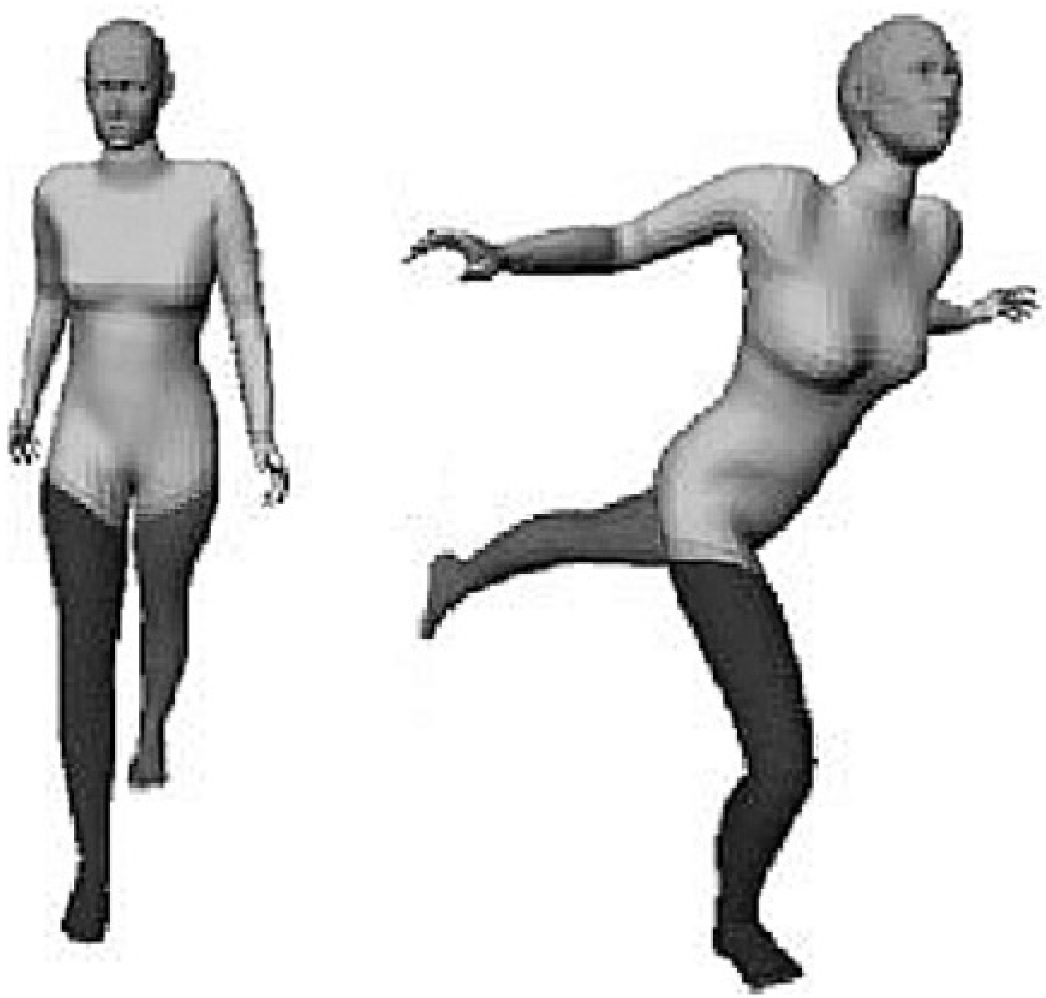


$$\begin{cases} x' = ax + by + et \\ y' = cx + dy + ft \\ t' = gx + hy + kt \end{cases}$$

$$adk + bfg + ech - edg - afh - bck \neq 0$$

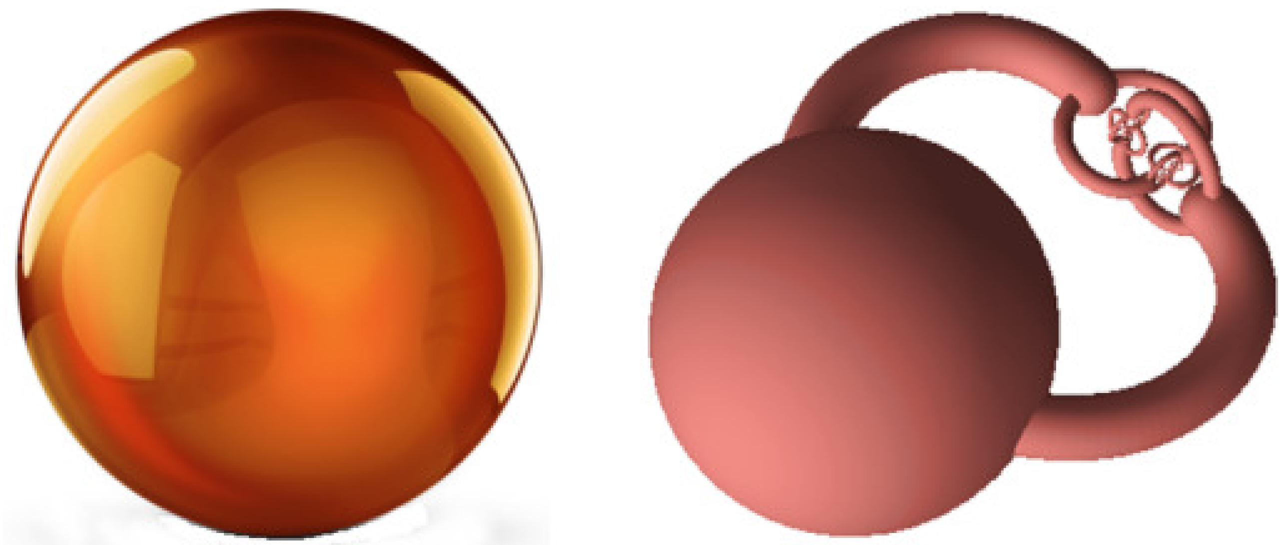
In image recognition, linear similarity is performing well

Persistent topology



still sometimes you need homeomorphism

Persistent topology

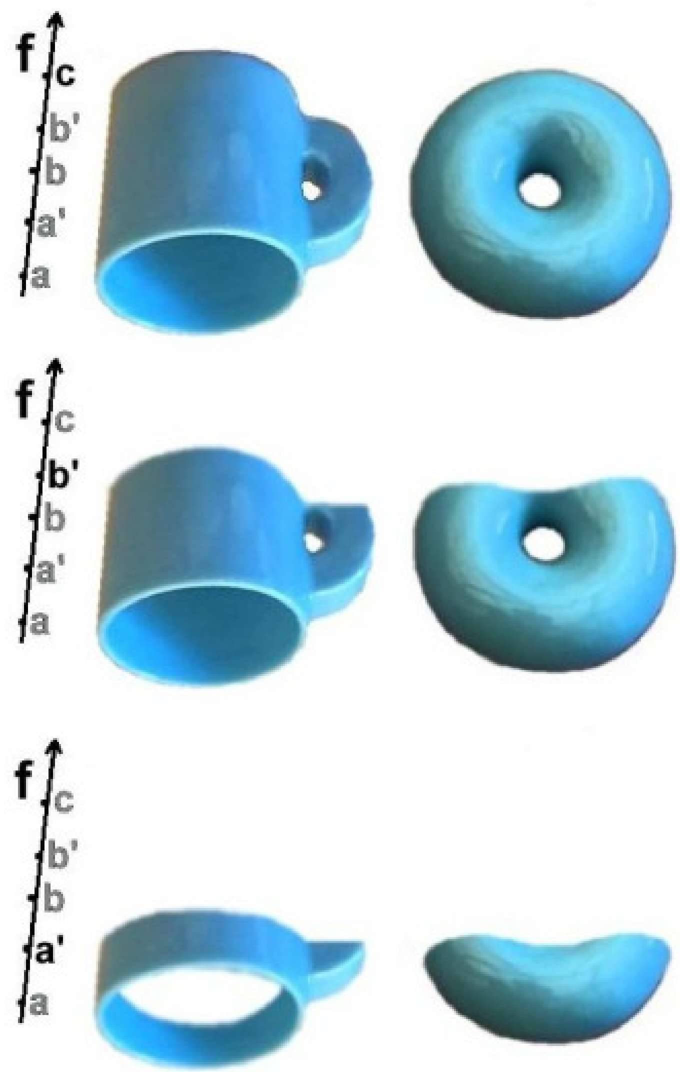


but here there is a homeomorphism too!

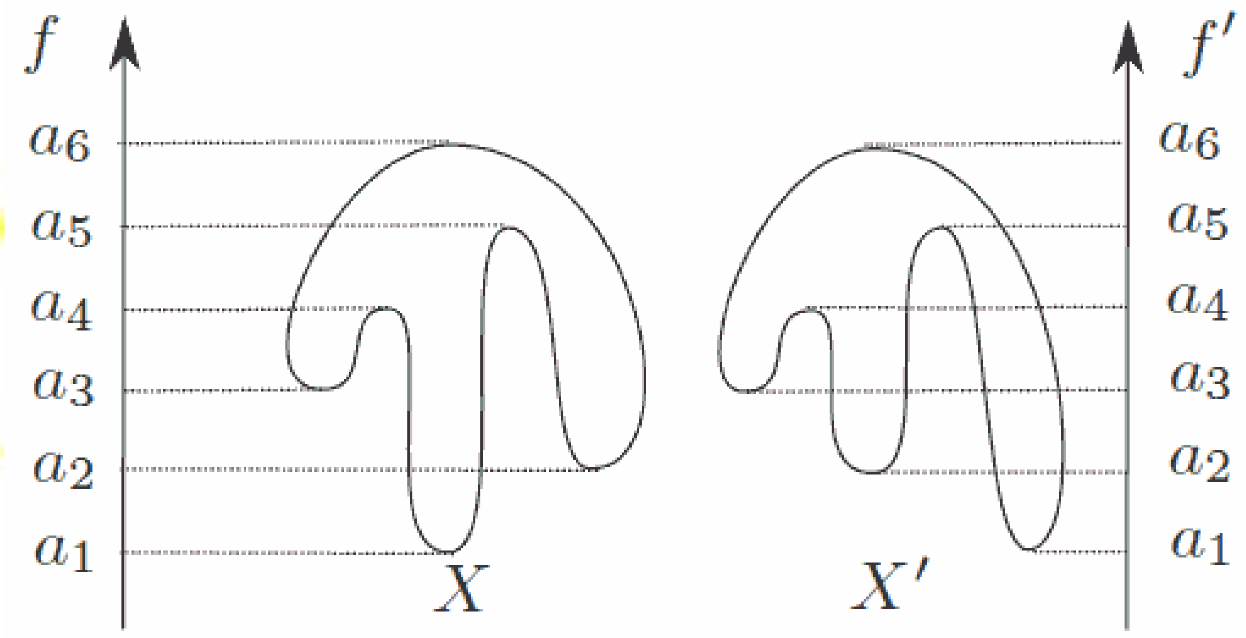
Persistent topology

The key idea of persistence is to consider not just a space *X*, but a filtration of it, e.g. through sublevel spaces of a *filtering map f:X->*$\mathbf{R}^n$.
(Mostly *n*=1)

With *f* = - ordinate we can then distinguish mug and doughnut, while recognizing different mugs and different doughnuts as similar.



Persistent topology



How much do these two curves differ w.r.t. ordinate as a filtering function?

Our proposal: the minimum cost, in terms of *f* and *f′*, of transforming one in the other.

Persistent topology

Definition

The natural pseudo-distance between the size pairs $(M, \boldsymbol{\varphi})$ and $(N, \boldsymbol{\psi})$ is

$$d\big((M,\varphi),(N,\psi)\big) = \begin{cases} \inf_{h \in H(M,N)} \max_{P \in M} \|\varphi(P) - \psi(h(P))\|_\infty, \\ +\infty \qquad \text{if } H(M,N) = \emptyset, \end{cases}$$

$H(M, N)$ being the set of all the homeomorphisms between M and N.

P. Frosini, M. Mulazzani, *Size homotopy groups for computation of natural size distances*, Bull. of the Belgian Math. Soc. - Simon Stevin, 6 (1999), 455-464.

Persistent topology

But the natural pseudodistance is difficult (or even impossible) to compute.

Therefore we need a computable lower bound for it.

Luckily, we have it: the *matching distance* (or *bottleneck distance*) between *Persistent Betti Number* functions of the size pairs.

Persistent topology

For each $i \in \mathbb{Z}$, the i-th *Persistent Betti Number (PBN) function* of $(X, \vec{f})$ is $\rho_{(X,\vec{f},i)} : \Delta^+ \to \mathbb{N}$ defined as
$\rho_{(X,\vec{f},i)}(\vec{u}, \vec{v}) = \dim(Im f_{\vec{u}}^{\vec{v}}),\ \vec{u} \prec \vec{v}$ with

$$f_{\vec{u}}^{\vec{v}} : H_i(X\langle \vec{f} \preceq \vec{u} \rangle) \to H_i(X\langle \vec{f} \preceq \vec{v} \rangle),$$

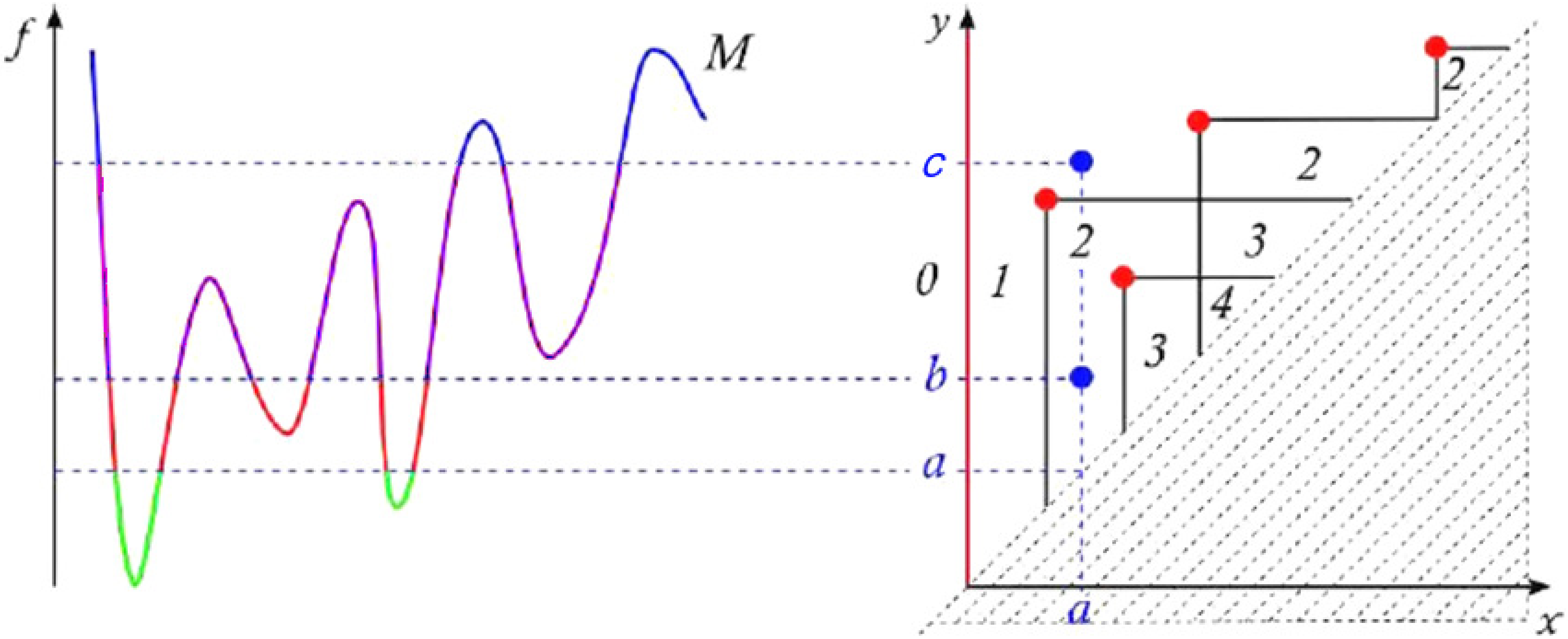
where $f_{\vec{u}}^{\vec{v}}$ is the homomorphism induced by the inclusion map of lower level sets $X\langle \vec{f} \preceq \vec{u} \rangle \subseteq X\langle \vec{f} \preceq \vec{v} \rangle$

H. Edelsbrunner, D. Letscher and A. Zomorodian, *Topological Persistence and Simplification*, Proc. 41st Ann. IEEE Sympos. Found Comput. Sci. (2000), 454–463.

G. Carlsson and A. Zomorodian, *The Theory of Multidimensional Persistence*, Symposium on Computational Geometry, June 6–8, 2007, Gyeongiu, South Korea (2007) 184–193.

Persistent topology

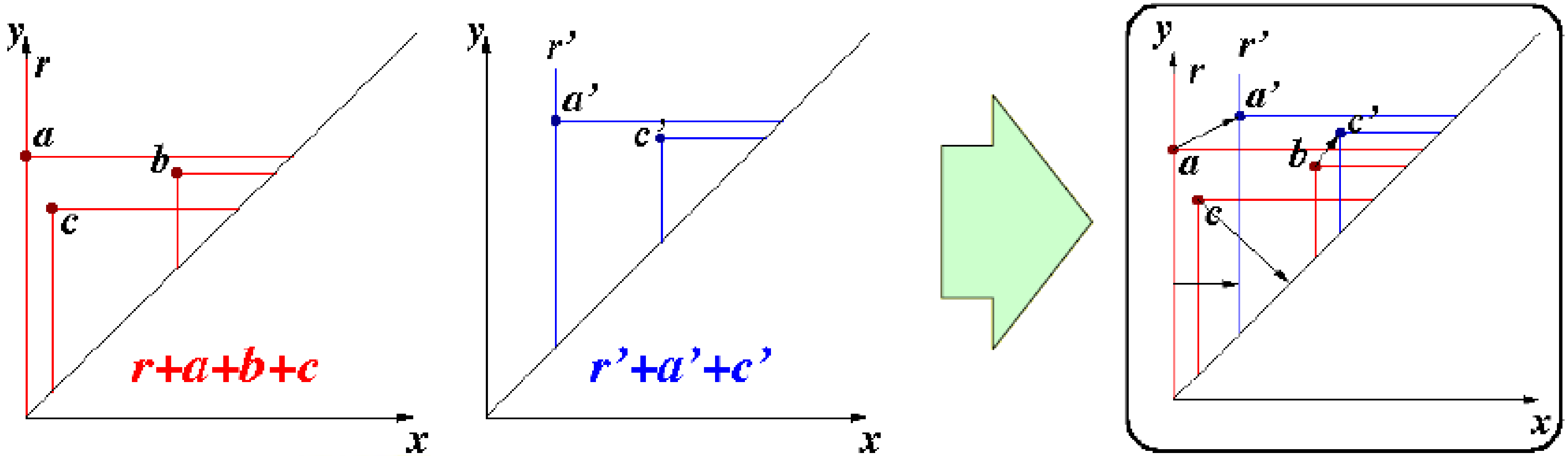
An easy example of 0-degree Persistent Betti Number function (also called *size function*) with $f : M \to \mathbb{R}$:



P. Frosini *Measuring shapes by size functions*, Proc. of SPIE, Intelligent Robots and Computer Vision X: Algorithms and Techniques, Boston, MA 1607 (1991).

Persistent topology

All information carried by a PBN function can be condensed in the formal series of its *cornerpoints*



The *matching distance*

P. Frosini, C. Landi, *Size functions and formal series*, Appl. Algebra Engrg. Comm. Comput. 12 (2001), 327-349.

Persistent topology

It turns out that:

$$d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{N},\psi)}) \leq d((\mathcal{M},\varphi),(\mathcal{N},\psi))$$

i.e. the matching distance between size functions yields a lower bound to the natural pseudodistance.

S. Biasotti, A. Cerri, P. Frosini, D. Giorgi, C. Landi *Multidimensional size functions for shape comparison* Journal of Mathematical Imaging and Vision 32 (2008), 161–179.

A. Cerri, B. Di Fabio, M. Ferri, P. Frosini, C. Landi *Betti numbers in multidimensional persistent homology are stable functions* Math. Meth. Appl. Sci. DOI: 10.1002/mma.2704 (2012).

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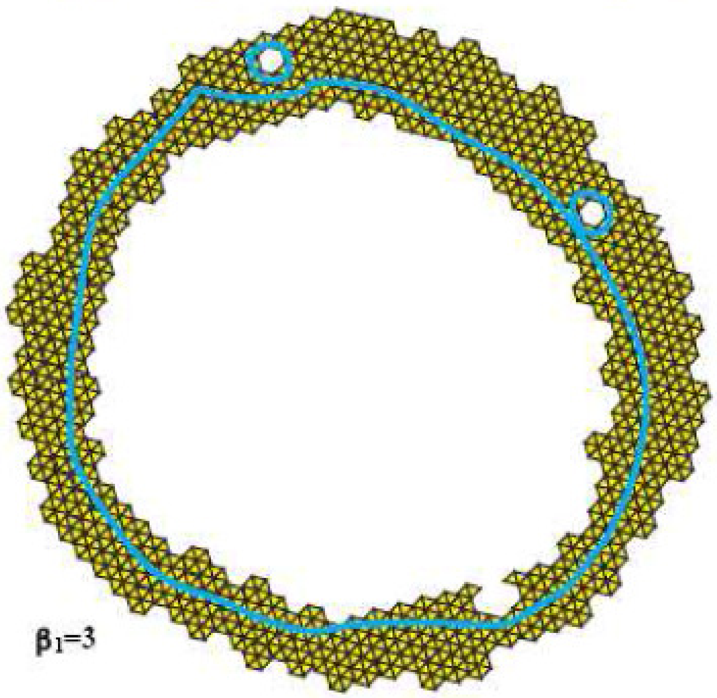
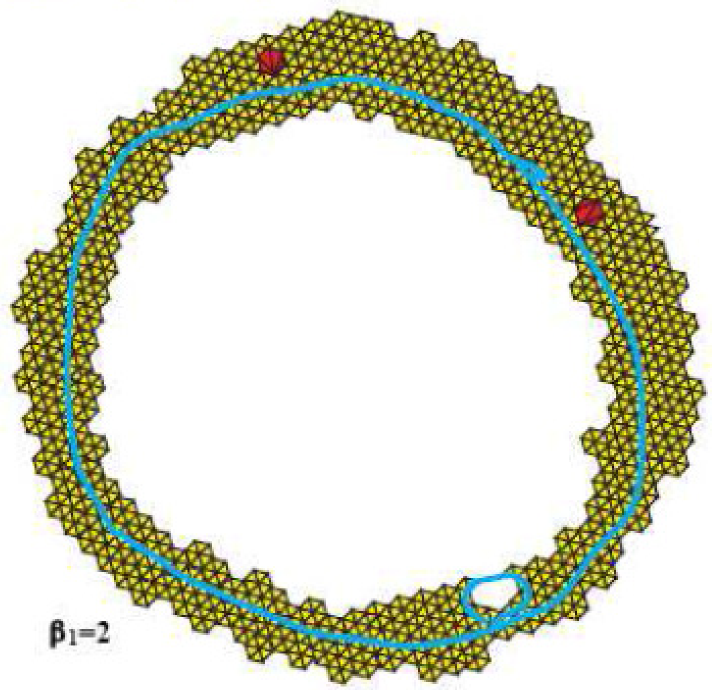
Image processing

Persistence was (re)invented in Stanford for guessing the topology of a sampled space:





G. Carlsson, A. Collins, L. Guibas and A. Zomorodian. *Persistence barcodes for shapes*. In Proc. 2nd Sympos. Geometry Process., 2004, 127–138.

Image processing

In this setting, robustness to noise has been the object of extensive research of various teams.

Cohen-Steiner, David, Herbert Edelsbrunner, and John Harer. *Stability of persistence diagrams*. Discrete & Computational Geometry 37.1 (2007), 103-120.

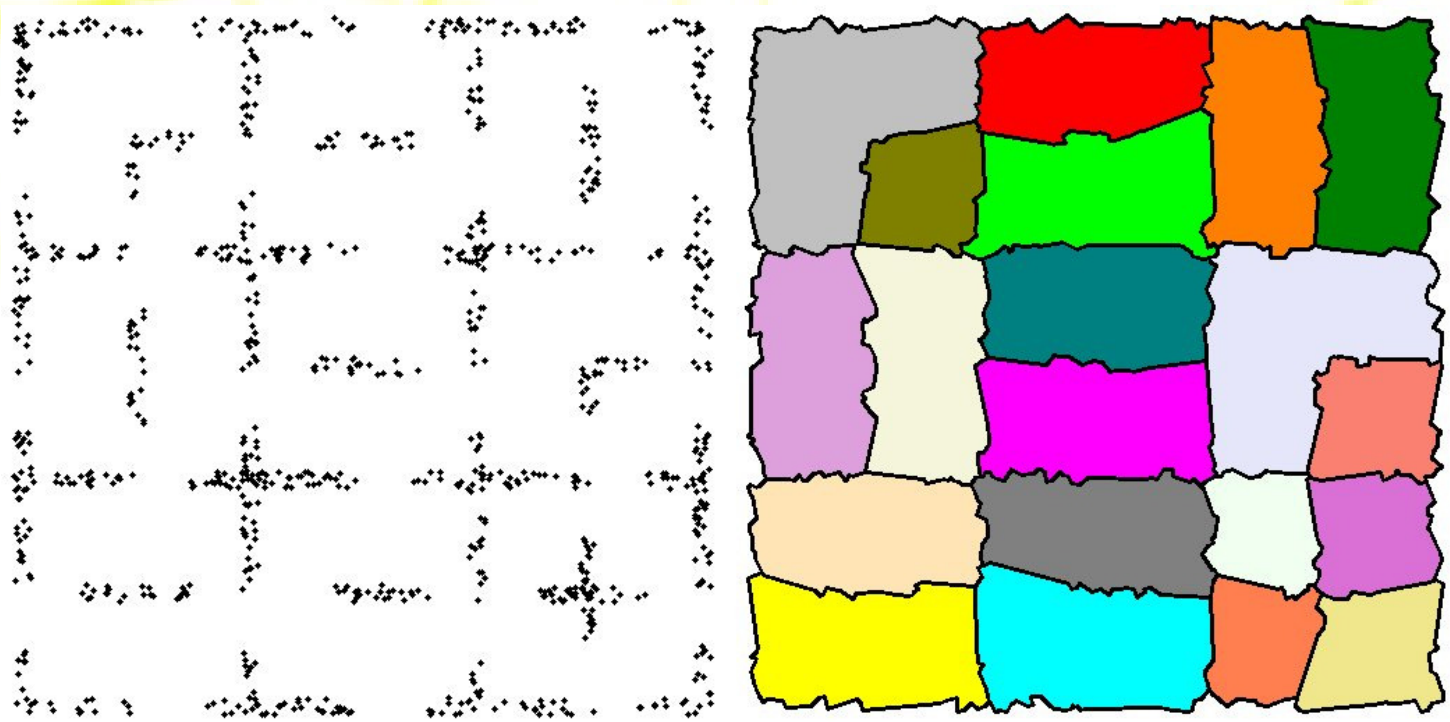
Chazal, F., De Silva, V., Glisse, M., and Oudot, S. (2012). *The structure and stability of persistence modules*. To appear as a Monograph in SpringerBriefs, 2016.

Bendich, P., Edelsbrunner, H., Morozov, D., and Patel, A. *Homology and robustness of level and interlevel sets*. Homology, Homotopy and Applications 15.1 (2013), 51-72.

Frosini, P., and Landi, C. *Persistent betti numbers for a noise tolerant shape-based approach to image retrieval*. Pattern Recognition Letters 34.8 (2013), 863-872.

Image processing

A hierarchy of gaps between cornerpoints gives rise to amazing possibilities for segmentation out of very noisy data.



Kurlin, V. *A fast persistence-based segmentation of noisy 2D clouds with provable guarantees*. Pattern Recognition Letters (in press).

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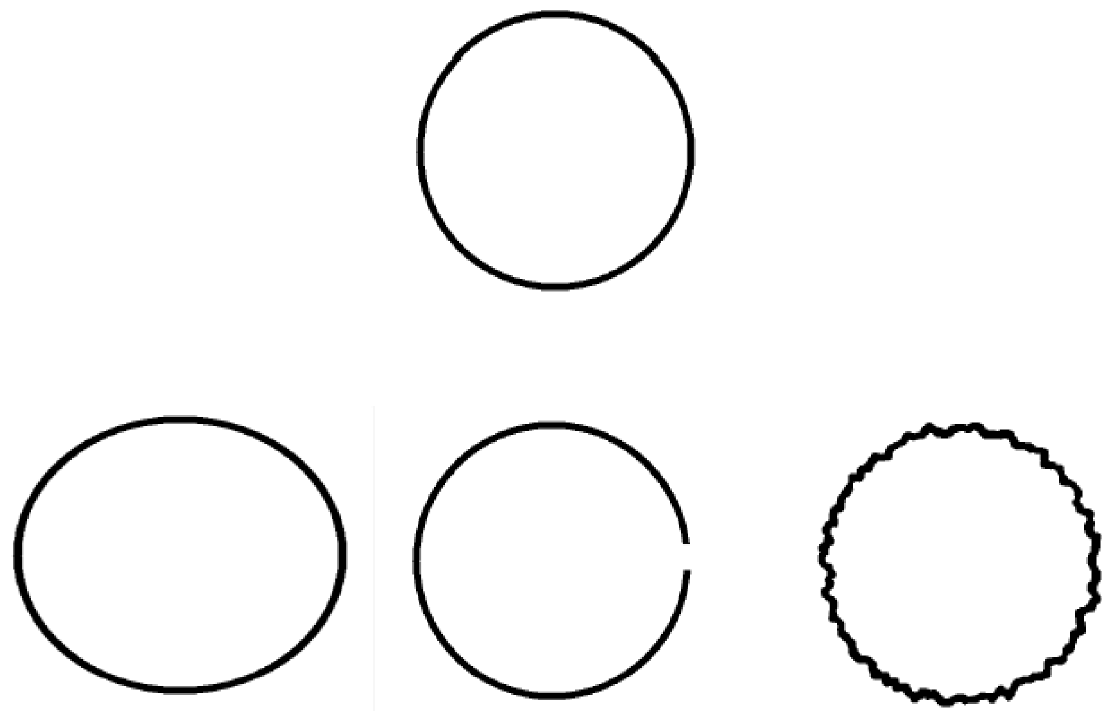
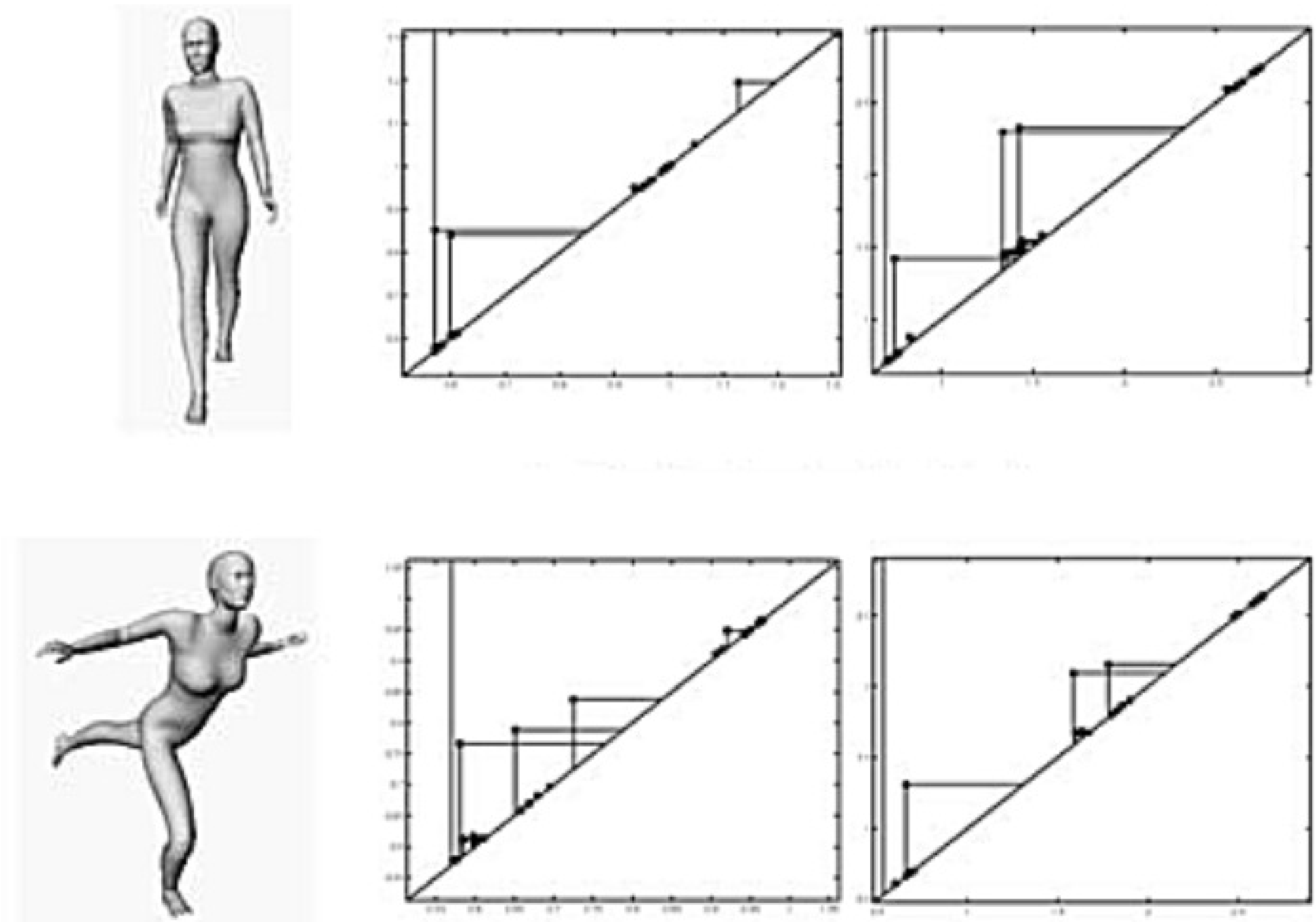
Shape analysis



Which object has "the same shape" as the upper circle?

In our opinion, this depends on the observer (his/her viewpoint, interest, tasks…). This can be formalized by a *filtering function* defined on the object of interest.

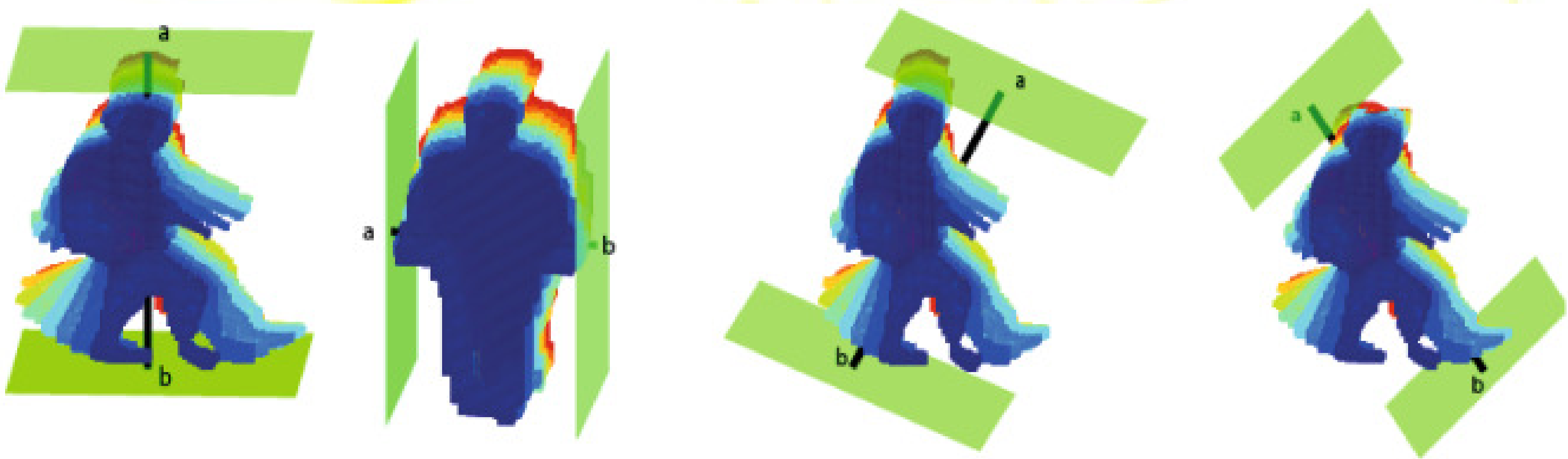
Shape analysis



Different filtering functions for different tasks and viewpoints

Shape analysis

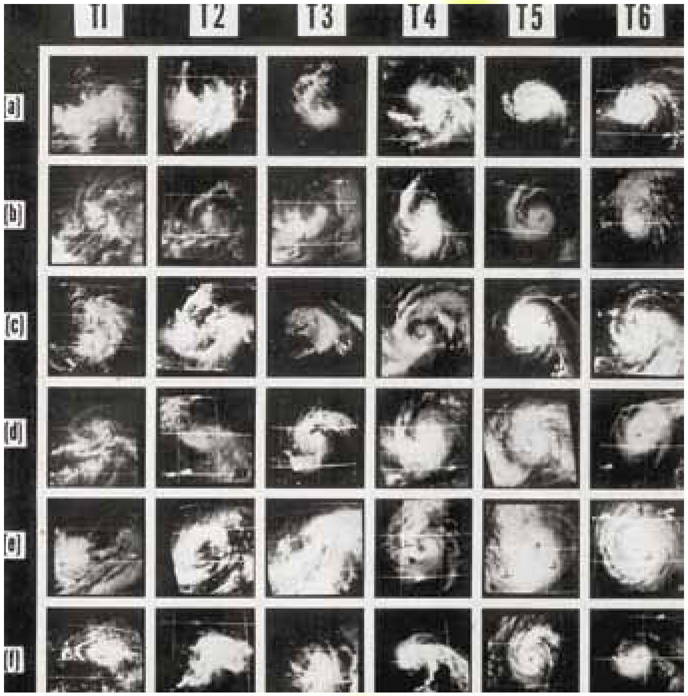
Considering a stack of silhouettes as a 3D object, and using four different filtering functions, makes 0- and 1-degree persistent homology a tool for identifying people through their gait.

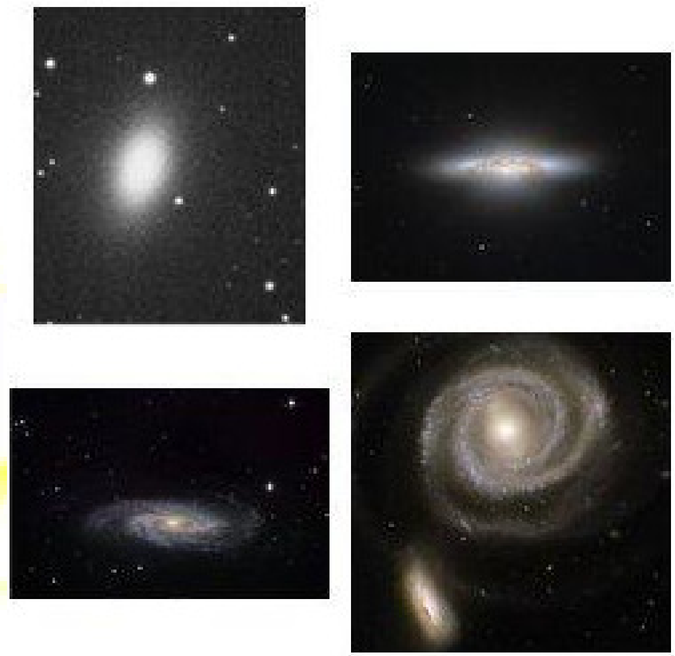


Lamar-León, J., García-Reyes, E.B., and Gonzalez-Diaz, R. *Human gait identification using persistent homology*. Progress in Pattern Recognition, Image Analysis, Computer Vision, and Applications. Springer Berlin Heidelberg (2012), 244-251.

Shape analysis

Two different types of spirals: hurricanes and galaxies; both are analyzed through persistence.



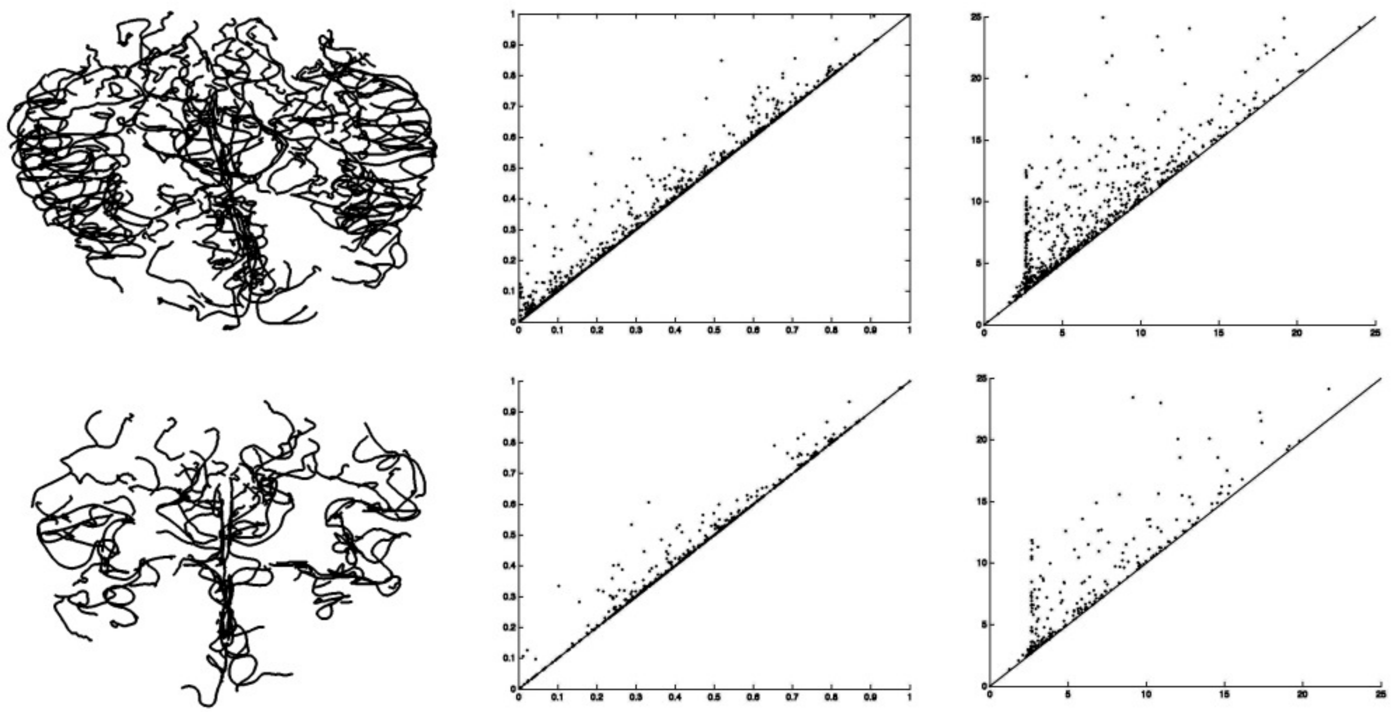


Banerjee, S. *Size Functions in the Study of the Evolution of Cyclones*. International Journal of Meteorology 36.358 (2011), 39-46.

Banerjee, S. *Size Functions In Galaxy Morphology Classification*. International Journal of Computer Applications 100.3 (2014), 1-4.

Shape analysis

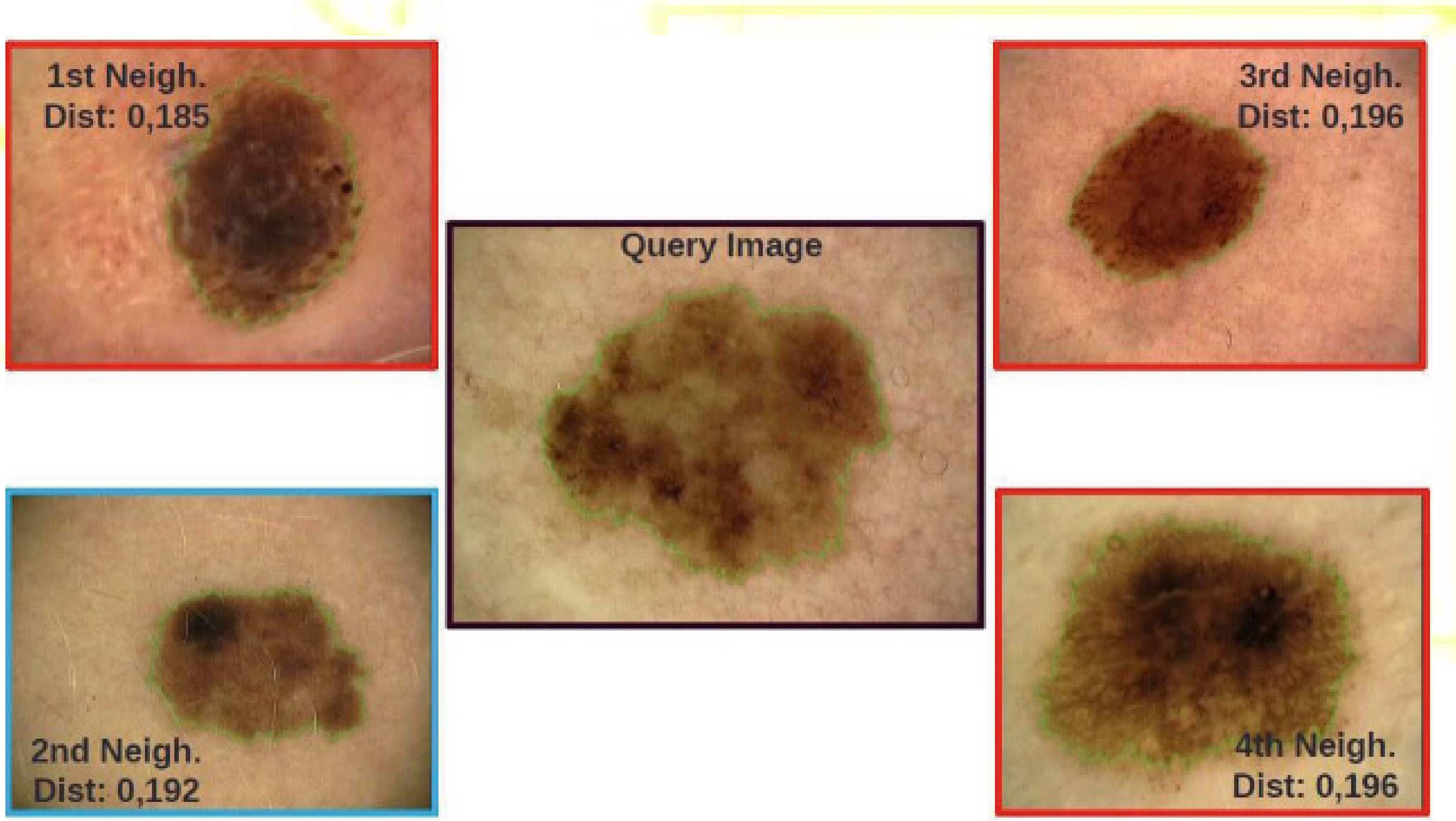
Besides density and thickness, bending of brain arteries is a sign of age. Its assessment is best done through 0- and 1-degree persistent homology.



Bendich, P., Marron, J. S., Miller, E., Pieloch, A. and Skwerer, S. *Persistent homology analysis of brain artery trees*. The Annals of Applied Statistics 10 (2014), 198-218.

Shape analysis

Several colour-based filtering functions are the key to image retrieval for dermatologists.



Ferri, M., Tomba, I., Visotti, A. and Stanganelli, I. *A feasibility study for a persistent homology based k-Nearest Neighbor search algorithm in melanoma detection*. arXiv preprint arXiv:1605.09781 (2016).

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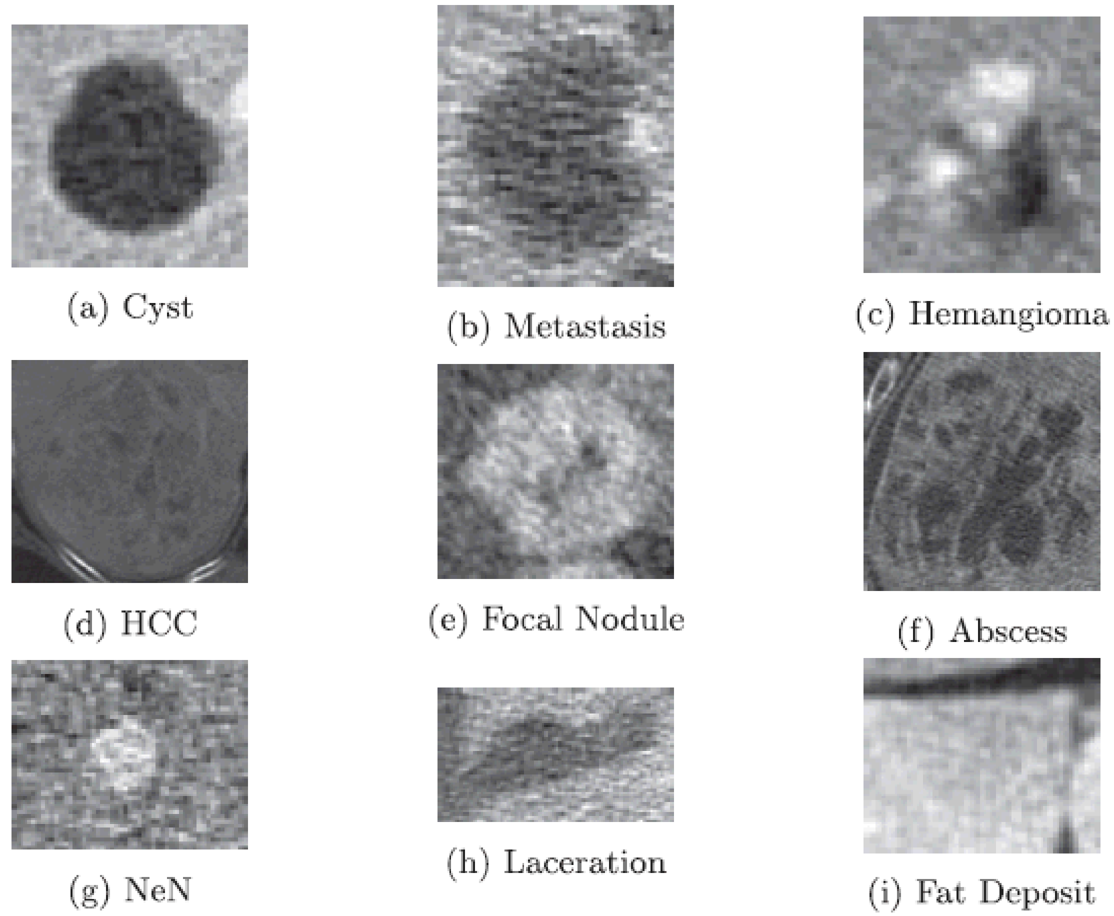
Theoretical progress

Experimental evidence (on hepatic lesions) confirms that filtering functions with an nD range carry more information than their single components.



(a) Cyst (b) Metastasis (c) Hemangioma
(d) HCC (e) Focal Nodule (f) Abscess
(g) NeN (h) Laceration (i) Fat Deposit

Adcock, A., Rubin, D. and Carlsson, G. *Classification of hepatic lesions using the matching metric*. Computer vision and image understanding 121 (2014), 36-42.

Theoretical progress

With nD filtering functions, cornerpoints are replaced by (2n-2)-D objects.

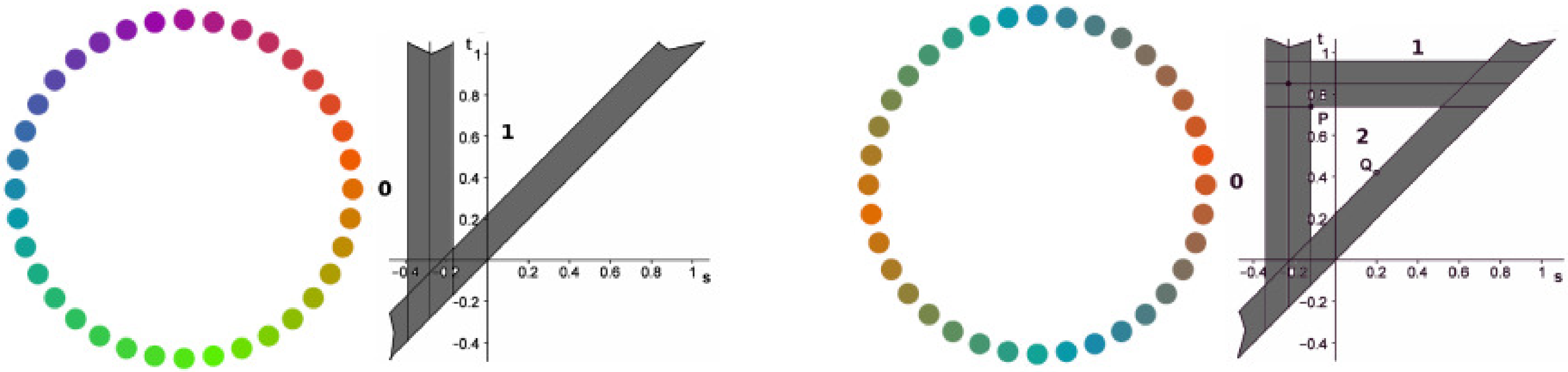
The study of discontinuities becomes essential. In particular, a monodromy phenomenon comes into play.

Cerri, A., & Frosini, P. *Necessary conditions for discontinuities of multidimensional persistent Betti numbers.* Mathematical Methods in the Applied Sciences 38.4 (2015), 617-629.

Cerri, A., Ethier, M. and Frosini, P. *A Study of Monodromy in the Computation of Multidimensional Persistence*. DGCI. 2013, 192-202.

Theoretical progress

Assessing the discontinuities of PBNs through a sampling is possible also in the nD case, so providing lower bounds for the natural pseudodistance.



$$u = (-0.28, 0.12),\ v = (0.32, 0.72),\quad u' = (-0.06, 0.34),\ v' = (0.1, 0.5)$$

$$\beta_{(Y,g,0)}(u,v) = 2 > 1 = \beta_{(X,f,0)}(u',v')\ \delta\left((X,f),(Y,g)\right) \geq 0.22$$

Cavazza, N., Ferri, M. and Landi, C. *Estimating multidimensional persistent homology through a finite sampling*. Int. J. of Computational Geometry & Applications 25.03 (2015): 187-205.

Theoretical progress

- The *interleaving distance* between persistence modules has been extended to nD indexing.

- In the definition of natural pseudodistance, all possible homeomorphisms are taken into account.
  However, there are settings in which it is convenient to limit the analysis to a subgroup of homeomorphisms.

- This is an aspect of a new framework for shape analysis.

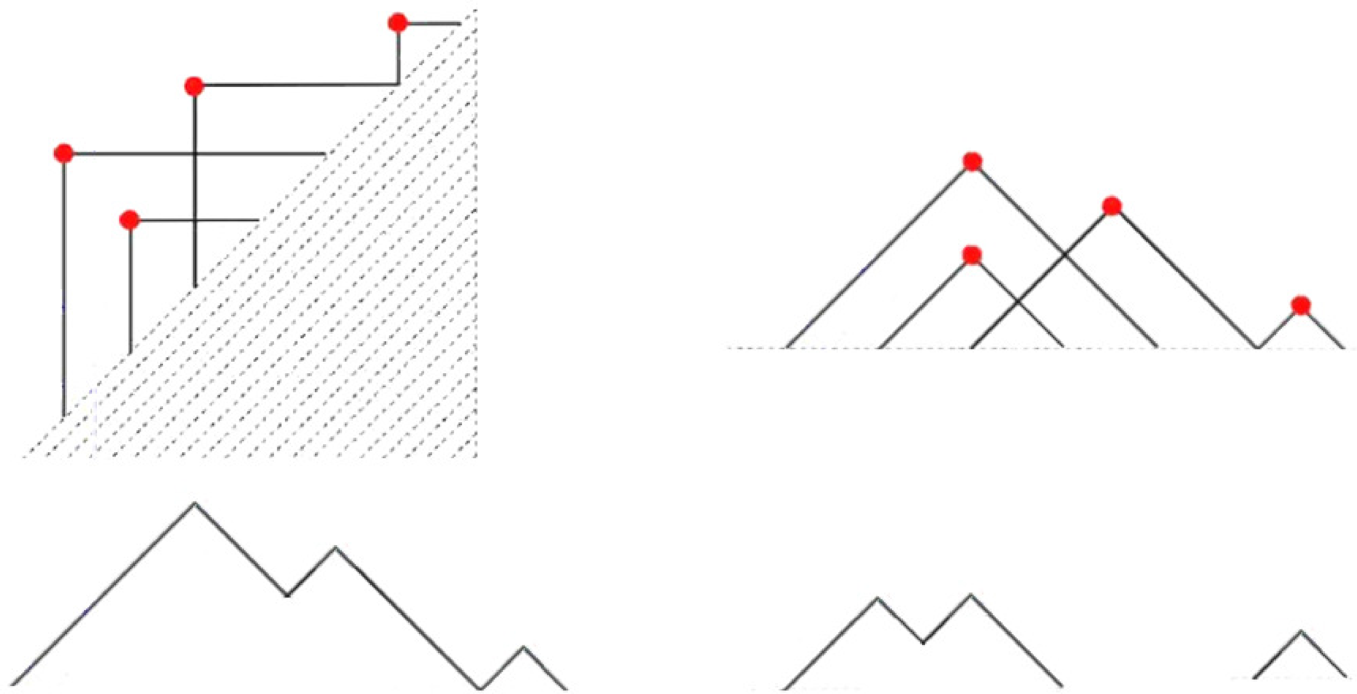
Lesnick, M. *The theory of the interleaving distance on multidimensional persistence modules.* Foundations of Computational Mathematics 15.3 (2015): 613-650.

Frosini, P. and Jablonski, G. *Combining persistent homology and invariance groups for shape comparison*. Discrete & Computational Geometry 55 (2016), 373-409.

Frosini, P. *Position paper: Towards an observer-oriented theory of shape comparison*. Proc. of the 8th Eurographics Workshop on 3D Object Retrieval, Lisbon, Portugal, May 7-8, 2016, A. Ferreira, A. Giachetti, and D. Giorgi Eds. (2016), 5-8.

Theoretical progress

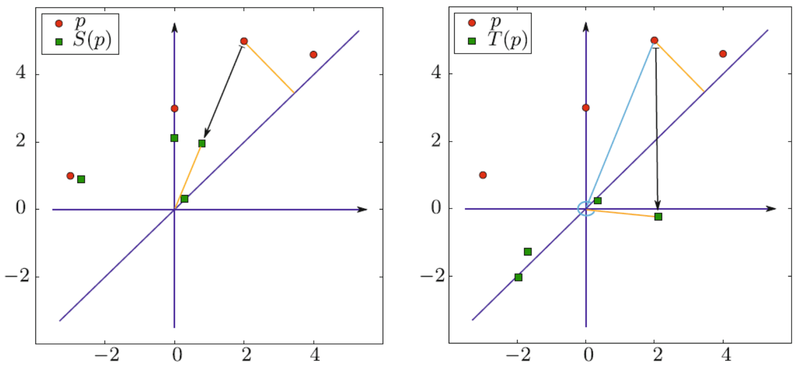
PBNs functions can be encoded into linear maps, *landscapes*, ready for averaging and for topological statistics.



Bubenik, P. and Dlotko P. *A persistence landscapes toolbox for topological statistics*. Journal of Symbolic Computation, Elsevier (2016). <hal-01258875>

Theoretical progress

Another coding is: consider cornerpoints as complex numbers (after a warping taking the diagonal to 0) and form the polynomial which has them as roots.



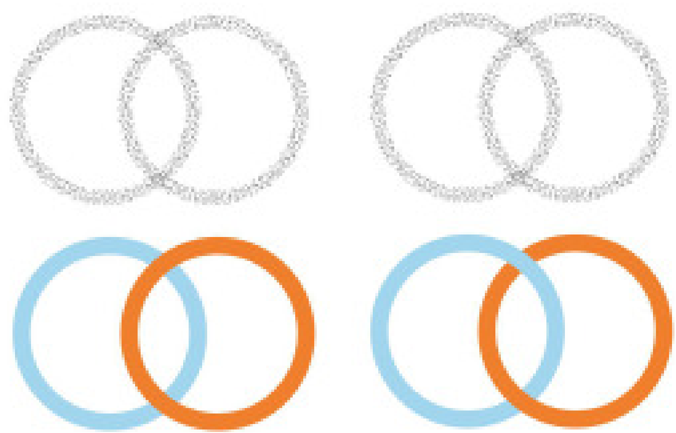
A dramatic progress in the computation of the bottleneck distance has made this coding less interesting for speed; still, it can be useful for skimming in data retrieval.

Di Fabio, B., and Ferri, M. *Comparing persistence diagrams through complex vectors*. Image Analysis and Processing—ICIAP 2015. Springer International Publishing (2015), 294-305.

Kerber, M., Morozov, D. and Nigmetov, A. *Geometry Helps to Compare Persistence Diagrams*. In: ALENEX 2016, SIAM (2016), 103-112.

Theoretical progress

An extension of the theory, *A∞ - persistence*, makes it possible (among other) to connect different dimensions, e.g. by cup and Massey products.



Point clouds sampling (thick) trivial or Hopf links can then be distinguished.

Belchí , F. and Murillo, A. *A∞ - persistence*. Applicable Algebra in Engineering, Communication and Computing 26 (2015), 121-139.

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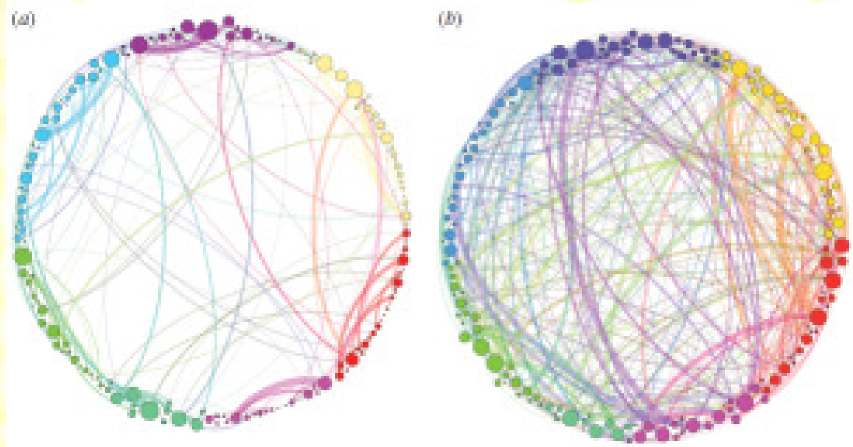
Not only images

- Cycle persistence is at the base of two applications in the biological domain:
- Through Vietoris-Rips complexes for isolating genetic pathways to Coronary Artery Disease
- Through clique complexes for analyzing the effect of psilocybin on brain networks.



Platt, D. E., Basu, S., Zalloua, P. A. and Parida, L. *Characterizing redescriptions using persistent homology to isolate genetic pathways contributing to pathogenesis*. BMC Systems Biology 10 (2016), 107-119.

Petri, G., Expert, P., Turkheimer, F., Carhart-Harris, R., Nutt, D., Hellyer, P. J. and Vaccarino, F. *Homological scaffolds of brain functional networks*. Journal of The Royal Society Interface 11.101 (2014): 20140873.

Not only images

More applications:

- Detection of linguistic subfamilies (via syntactic proximity and Vietoris-Rips complexes) through 0-degree persistence; speculation on the meaning of 1-degree persistence
- Recognition of tonal, modal and atonal music is performed through alignment of time series and time evolution of persistence diagrams (work in progress).

Port, A., Gheorghita, I., Guth, D., Clark, J. M., Liang, C., Dasu, S. and Marcolli, M. *Persistent topology of syntax*. arXiv preprint arXiv:1507.05134 (2015).

Bergomi, M.G. *Dynamical and topological tools for (modern) music analysis*. Theses, Université Pierre et Marie Curie - Paris VI, December 2015. https://tel.archives-ouvertes.fr/tel-01293602

Not only images

Once more, persistence in Vietoris-Rips complexes, or approximations of them, appear in two applications to physics:

- String vacua

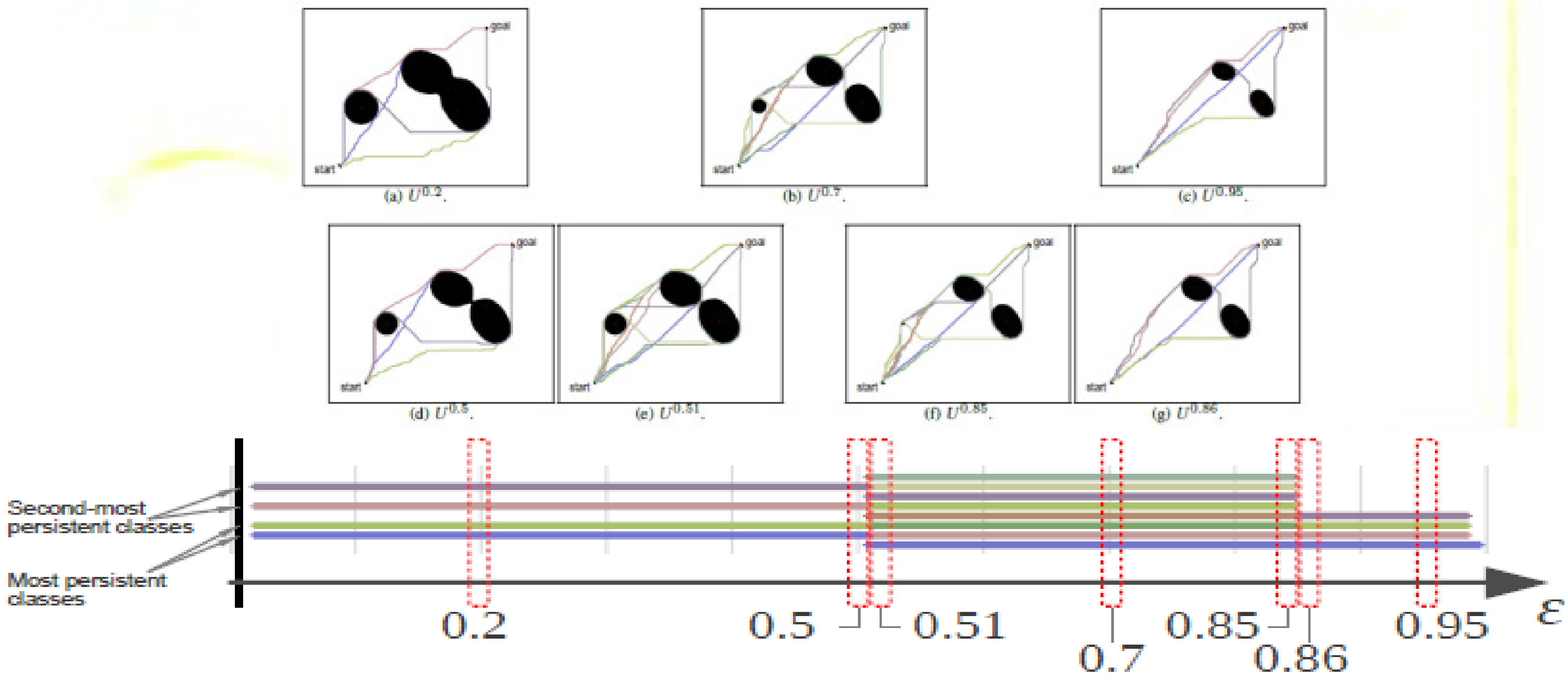
- Phase transitions

Cirafici, M. *Persistent homology and string vacua.* J. Of High Energy Physics 3(2016), 045.

Donato, I., Gori, M., Pettini, M., Petri, G., De Nigris, S., Franzosi, R. and Vaccarino, F. *Persistent homology analysis of phase transitions*. Physical Review E 93 052138 (2016).

Not only images

Robot navigation needs a map of obstacles; different thresholds of occupancy probability yield different maps. Preferred routes are the ones homologically persistent when varying the threshold.



Bhattacharya, S., Ghrist, R., & Kumar, V. *Persistent homology for path planning in uncertain environments*. IEEE Transactions on Robotics 31.3 (2015), 578-590.

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Conclusion

Whatever your idea of "shape", persistence may be a useful tool.

So far, attention has been paid mostly to image and mesh analysis; but the theory is ready for more ambitious tasks, for analysing, understanding, classifying shapes of objects in any dimension and of any nature.

THANKS FOR YOUR ATTENTION!

http://vis.dm.unibo.it

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