Homological Farrell invariants for embedded graphs

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Abstract

A very general recursive graph invariant, introduced by E.J. Farrell, is revisited and extended to graphs embedded into surfaces, by use of the surface homology.

1 Introduction

Two very interesting areas in graph theory are the combinatorial study of recursive invariants and the topological investigation of embedded graphs. The former was initiated by H. Whitney in a wonderful, short paper [8] where he introduced various recursive invariants (e.g. the famous chromatic polynomial); it got a powerful push from the invention of the Tutte polynomial [7] and other related invariants (see, e.g., [1]); a fruitful variation was found by E.J. Farrell in [2] (see also [3]). The latter theory is intertwined with graph theory since its birth, and the Four Colour Theorem is its most representative aspect. Exactly in the study of graph colorations did the two streams flow together with the already quoted chromatic polynomial; they interacted again within the theory of recursive knot invariants (see [5]).

E.J. Farrell tied the concept of recursive invariants to the one of covers of a graph, by subgraphs taken into special graph classes. Farrell exploited his idea in a very rich line of research papers but, as far as we know, never faced an extension to graphs embedded into a surface. This is what we do in the present article (Section 3). We mean that we deal with a suitable extension of Farrell's idea, to define not only new embedding related classes of graphs, but also a new invariant built on the homology of the surface.

Reasonably enough, we first need to go through Farrell's work in a slightly new setting (Section 2).

2 Farrell's recursion theorem revisited

By the term *graph* we shall mean a pseudograph, i.e. a triple $G = (V(G), E(G), f)$, where $V(G)$ is the set of *vertices*, $E(G)$ (disjoint from $V(G)$) is the set of *edges*, and the function f associates to each edge the set of (either one or two) *end–points*. This means that *multiple edges* (having the same end–point set) and *loops* (having just one end–point) are allowed. We shall always deal with finite graphs, i.e. with finite $V(G)$ and $E(G)$. Often, G will not simply be a graph, but a graph endowed with more structure (e.g. colourings, embeddings).

Particular cases of graphs are the *geometric* graphs, i.e. 1–dimensional CW–complexes [6, 7.6]. For every finite graph G, there are geometric graphs isomorphic to it, called the *geometric realizations* of G.

For all undefined notions of graph theory, we refer to [4].

2.1 Restricted graphs, F-covers

A *restricted* graph will be a pair $\mathcal{G} = (G, U)$, where G is a graph and $U \subset E(G)$ will be called the set of *restricted edges*. U can also be empty: In that case we shall identify G with the *unrestricted* graph $(G, \emptyset)$.

In what follows, F will denote either 1) a set of isomorphism classes of finite, connected graphs, or 2) a set of geometric graphs. In the first case, when we say that a graph *belongs* to F, we shall actually mean that it belongs to an isomorphism class belonging to F. Let moreover a ring R be given. We shall consider a fixed map, called *weight*, $\varphi : F \to R$.

Let a restricted graph $\mathcal{G} = (G, U)$ be given. An F*–cover* H of $\mathcal{G}$ will be a spanning subgraph of G whose connected components all belong to F and such that $U \subset E(H)$. Define $\varphi(H)$ as the element of R

$$\varphi(H) = \prod_{\substack{C \text{ is a connected} \\ \text{component of } H}} \varphi(C).$$

Then the *F–invariant* of $\mathcal{G}$ will be the element of R defined as

$$F(\mathcal{G};\varphi) = \sum_{\substack{H \text{ is an} \\ F\text{-cover of } \mathcal{G}}} \varphi(H).$$

Remark - The concept of restricted graph is the central, absolutely innovative idea of Farrell in [2]. Also F-covers and weights were defined and used in that paper; however, only rings of polynomials were considered there, so limiting the power of the construction. One might wonder what interest there could be in invariants of restricted graphs. Actually, restriction is mostly a technical contrivance by which invariants of "unrestricted" graphs (i.e. with $U = \emptyset$) are defined and dealt with.

2.2 Recursion Theorem

We now recall the main theorem of the theory, with its straightforward proof, as it is in the original paper [2], up to notation and to a more thorough proving argument.

Let $\mathcal{G} = (G, U)$ be a restricted graph and $e \in E(G)$. Then set $G \setminus e = (V(G), E(G) \setminus \{e\})$ and $\mathcal{G} \setminus \{e\} = (G \setminus e, U \setminus \{e\})$.

With this notation, we can now state the Recursion Theorem:

Proposition 2.1 [2, Thm. 1] *Let $e \in E(G) \setminus U$ and $\mathcal{G}' = (G, U \cup \{e\})$. Then, for any F, φ,*

$$F(\mathcal{G}, \varphi) = F(\mathcal{G} \setminus e, \varphi) + F(\mathcal{G}', \varphi).$$

Proof - Any F–cover H of $\mathcal{G}$ such that $e \in E(H)$ is an F–cover of $\mathcal{G}'$ but not of $\mathcal{G} \setminus e$; any F–cover H of $\mathcal{G}$ such that $e \notin E(H)$ is an F–cover of $\mathcal{G} \setminus e$ but not of $\mathcal{G}'$. So, the sum defining $F(\mathcal{G}, \varphi)$ splits into the sum of the two terms of the right–hand side of the thesis. □

Figure 1 shows the use of the Recursion Theorem. Dashed lines represent restricted edges.

2.3 Multiplicity, F–polynomials

We define the *multiplicity* of the vertices of a graph G as any map which associates to each $x \in V(G)$ a pair of integers (h_x, k_x). Unless otherwise stated, every vertex of every graph will be endowed with multiplicity $(1, 0)$.

Figure 1: *The recursion process for a restricted graph.*

Let now R be the ring of polynomials with real coefficients, in the indeterminates $w_{i,j}$ with i, j nonnegative integers.

A first example of F–invariant is a polynomial, where for each $C \in F$ we set $\varphi(C) = w_{i,j}$ with

$$i = \sum_{x \in V(C)} h_x \qquad j = \sum_{x \in V(C)} k_x + \#E(C).$$

A ring homomorphism (sending each $w_{i,j}$ to the one–index indeterminate w_i) takes this invariant to the *general F–polynomial*. A further ring homomorphism taking each w_i (or $w_{i,j}$) to the single indeterminate w gives rise to the *simple F–polynomial*. All three polynomials were already in [2]. We shall not report here the great number of invariants obtained from these polynomials by choosing suitable sets F or by suitable evaluations: See [3].

2.4 Deletion and contraction

Perhaps the most interesting application of Farrell's technique is the recursion in terms of deleted and contracted edges. Deletion of an edge has already been defined: It is the passage from G to $G \setminus e$. Contraction needs more attention to the specific case. The framework of the passage from a graph G to a graph G/e obtained by contraction of an edge e is always the following. In $G = (V(G), E(G), f)$ let e be an edge with $f(e) = \{a, b\}$, where $a \neq b$. Then let c be an object (to be considered as a vertex) not belonging to $V(G)$. In the definition of G/e, we shall always consider $V(G/e) = (V(G) \setminus \{a, b\}) \cup \{c\}$.

The definition of $E(G/e)$ is more complicated and is context dependent. 1) It coincides with $E(G)$ on the subset of the edges whose end–points are different from a and b. 2) There is a bijection between the set of edges connecting a vertex in $V(G) \setminus \{a, b\}$ with one in $\{a, b\}$ and the set of edges connecting a vertex in $V(G/e) \setminus \{c\}$ and c. 3) Edges of G, different from e, which connect a with b, become loops with the only end–point c in G/e. 4) Finally, it depends on the particular instance, whether e simply desappears in G/e, or becomes a loop with the only end–point c as well; if not explicitly stated otherwise, we shall assume that e desappears. 5) Also additional structures, if present, have to be defined for the single instances when passing from G to G/e.

In particular, the multiplicity of vertex c will be:

$$(h_c, k_c) = (h_a, k_a) + (h_b, k_b) + (0, 1).$$

2.5 Stable sets and weights, Contraction Theorem

A set F of graphs will be said to be *contraction stable* (or simply *stable*) if $\forall C \in F$,

- $\forall e \in E(C)$ (e not a loop), also $C/e \in F$, and
- $\forall C'$ such that $\exists e' \in E(C')$ (e not a loop) for which $C = C'/e'$, also $C' \in F$.

Important examples of stable graph sets are the sets of

- all connected graphs;
- all graphs with exactly k independent cycles (k a nonnegative integer);
- all graphs with at most k independent cycles (k a positive integer).

Given a stable set F, a (*contraction*) *stable* weight φ on F is one such that

$$\forall C \in F,\ \forall e \in E(C),\ \varphi(C/e) = \varphi(C).$$

Of course, the weights defined in Subsection 2.3 are so conceived as to be stable.

If $\mathcal{G} = (G, U)$ is a restricted graph, we define $\mathcal{G}/e$ naturally as $(G/e, U \setminus \{e\})$. Contraction and restriction are related together by the following *Contraction Lemma*:

PROPOSITION 2.2 *Let F be a stable set. Let $\mathcal{G} = (G, U)$ be a restricted graph, and let $e \in U$. Then for any stable weight φ*

$$F(\mathcal{G}; \varphi) = F(\mathcal{G}/e; \varphi).$$

Proof - In any F–cover of $\mathcal{G}$, there is exactly one component C such that $e \in E(C)$. The subgraph of G obtained by substituting C with C/e is an F–cover of $\mathcal{G}/e$, and has the same weight. Conversely, each F–cover of $\mathcal{G}/e$ comes by contraction from an F–cover of $\mathcal{G}$ of equal weight. More precisely, contraction induces a bijection from the set of F–covers of $\mathcal{G}$ to the set of F–covers of $\mathcal{G}/e$ which preserves weight. □

This leads us to the following *Contraction Theorem*, which permits us to obtain powerful invariants of unrestricted graphs, which respect the classical recursion treated in [1].

Proposition 2.3 *Let F be a contaction stable set. Let $\mathcal{G} = (G, U)$ be a restricted graph, and let $e \notin U$. Then for any stable weight φ*

$$F(\mathcal{G}; \varphi) = F(\mathcal{G} \setminus e; \varphi) + F(\mathcal{G}/e; \varphi).$$

Proof - It is an immediate corollary of Propositions 2.1 and 2.2. □

Remark - The ideas of stable sets and weights were implicit in the work of Farrell, who proved the Contraction Lemma and Theorem in important, special cases. Here we have just given them a more accurate and general setting.

3 Embedded graphs

The main goal of this paper is to define recursive invariants for graphs embedded into surfaces (i.e. 2–dimensional manifolds). All graphs in this Section will be geometric; the manifolds will be piecewise–linear. Given a (geometric) graph $G = (V(G), E(G))$ and its underlying topological space (called the *space* of G) $|G|$, an *embedding* of G into a surface S is a map $h : |G| \to S$ which is a homeomorphism onto its image. In particular, we shall assume that $|G| \subset S$ and that h is the inclusion. Moreover, it is always possible to consider a CW–complex decomposition of S such that G is a subcomplex of it.

Given an embedded graph G and one of its edges e, the embedded graph $G \setminus e$ is easily defined, by using the restriction of the embedding h to $|G \setminus e|$. Contraction is not so immediate.

3.1 Contraction for embedded graphs

Let G be a graph embedded in a surface S, e be one of its edges, which is not a loop, and a, b be its end–points (see Figure 2). Let N be a regular neighbourhood of e in S, such that e is the only edge of G completely contained in N, and the only edges of G with nonempty intersection with N are the ones with either a or b as an end–point. Such a neighbourhood can always be found. We now consider a fixed homeomorphism of pairs:

$$\Phi : (N, e) \rightarrow (D^2, I)$$

where $I = \{0\} \times (-1/2, +1/2) \subset D^2 = \{(x, y) \in \mathbf{R}^2 \,|\, x^2 + y^2 \leq 1\}$.

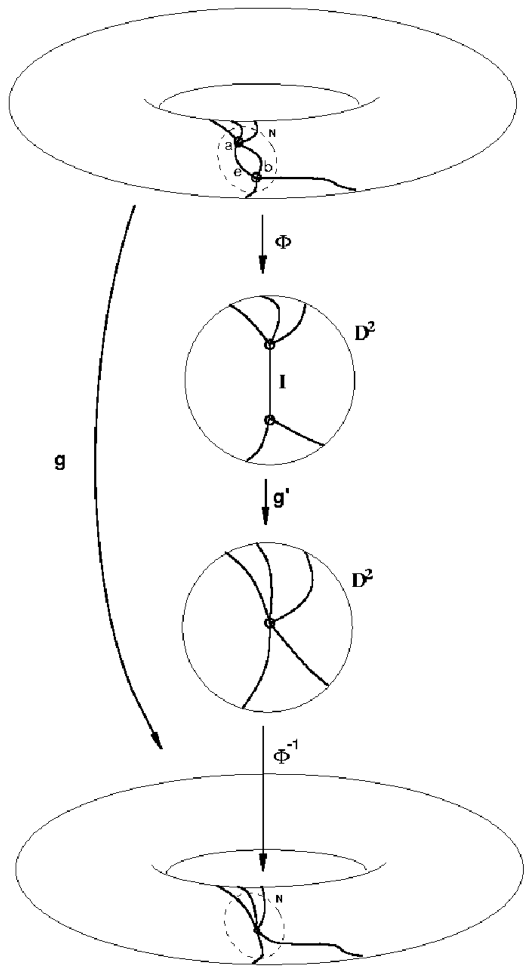

Figure 2: *Contraction for an embedded graph.*

There exists at least one map $g' : D^2 \rightarrow D^2$ such that $g'(I) = \{(0, 0)\}$, $g'|_{\partial D^2} = \mathbf{1}|_{\partial D^2}$, and such that $g'|_{D^2 \setminus I}$ is a homeomorphism on $D^2 \setminus \{(0, 0)\}$.

Now, let $g : S \to S$ be the map which extends $\Phi^{-1} \circ g' \circ \Phi$ by the identity outside N.

We define G/e: $V(G/e) = (V(G) \setminus \{a, b\}) \cup \{g(e)\}$, $E(G/e) = \{g(e') \mid e' \in E(G) \setminus \{e\}\}$; i.e., in this definition of contraction e desappears. The assignment of the end–points is defined as in Subsection 2.4.

3.2 Topologically defined stable sets

In what follows, homology will always be simplicial, and its operation, however commutative, will be dealt with in multiplicative notation.

In a geometric graph, be it embedded or not, every cycle (in the sense of the theory of undirected graphs) can give rise to a homological cycle $[z]$ (of the graph itself, or of the space into which it is embedded) and to its inverse $[z^{-1}]$, according to the orientation given to its edges: In fact there are two such coherent orientations.

Let now a surface S be given. We define a family of graph sets: For all integers m, n, r, s with $0 \leq m \leq n$ and $0 \leq r \leq s$, let F_{mr}^{ns} be the set of connected graphs embedded in S, containing i cycles, trivial in $H_1(S)$, and j cycles, nontrivial in $H_1(S)$, $m \leq i \leq n$, $r \leq j \leq s$.

Proposition 3.1 *For all admitted m, n, r, s, the set F_{mr}^{ns} is contraction stable.*

Proof - The contraction operation, as defined in the previous subsection, does not transform a cycle into a noncycle (nor conversely), and does not alter its homology class. □

It is possible to extend this definition, by considering a subset W of $H_1(S)$, closed under inversion, and imposing to the cycles to belong (respectively not to belong) to W. The above defined family is re–obtained by setting W equal to the trivial subgroup.

Figure 3 shows a graph embedded in a torus, and the computation of its F_{00}^{01}–polynomial (yielding $w_{10}^2 + 3w_{21} + 2w_{22}$) both as a sum over all F_{00}^{01}–covers and by recursion. Note that this polynomial distinguishes the embedding of the picture from one in which both cycles are null–homologous: In the latter case the F_{00}^{01}–polynomial would be $w_{10}^2 + 3w_{21}$.

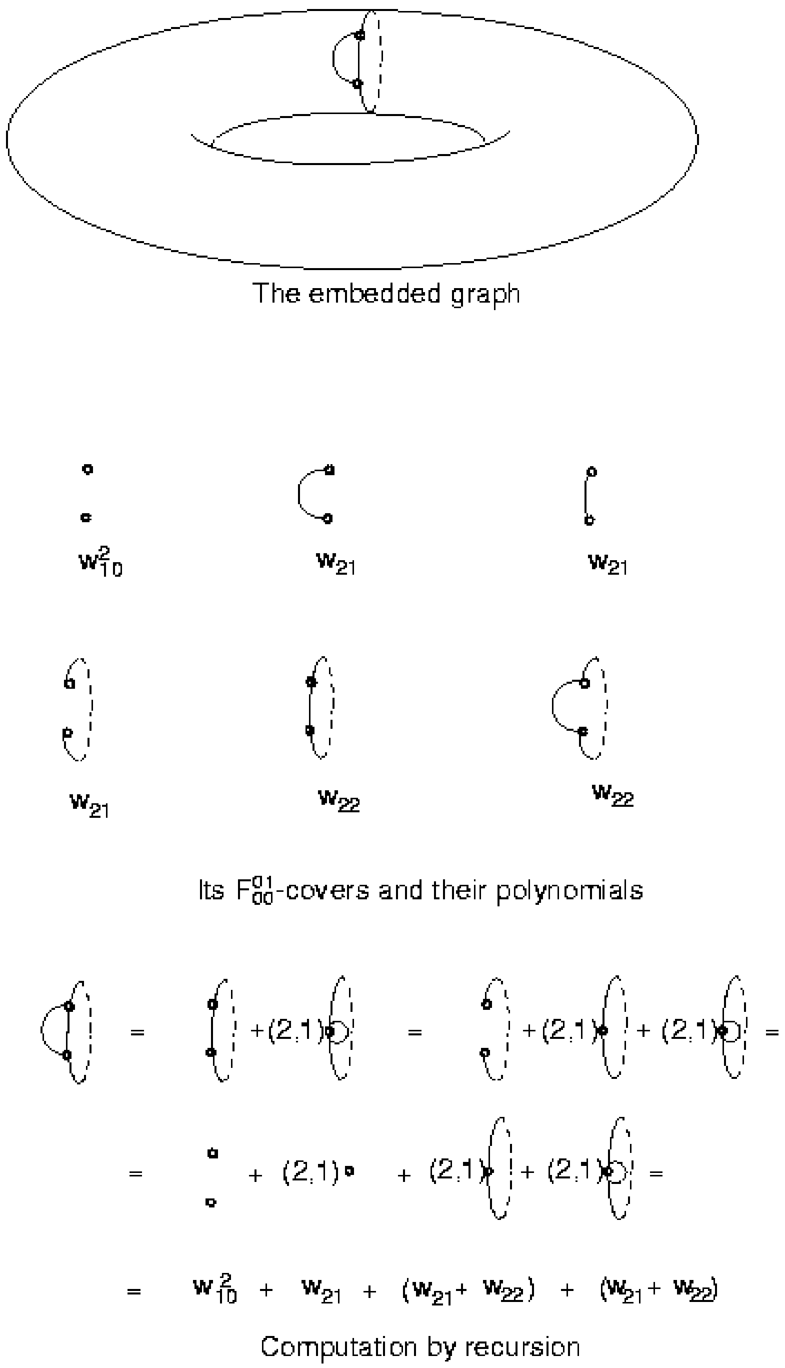

Figure 3: *Covers and recursion for an embedded graph.*

3.3 Homological invariants

In the previous subsection we have used the homology of S only for selecting stable sets of embedded graphs, with a criterion strictly dependent on the embedding itself, and not simply on the combinatorial structure of the graphs. Here we want to go further, and make use of the freedom we have in defining weights.

Let $R_1(S)$ be the group–ring of $H_1(S;\mathbf{Z}_2)$. (Passing to the group–ring is the reason why we have chosen to use the multiplicative notation for the operation in $H_1(S)$). Then, given any set F of embedded graphs containing at most one cycle, we can define a weight $\varphi : F \to R_1(S)$ as follows. Given a graph $K \in F$, its cycle z and its embedding $h : |K| \to S$ (possibly the inclusion map), set

$$\varphi(K) = h_*(\{z\}) \in H_1(S;\mathbf{Z}_2).$$

PROPOSITION 3.2 *For each contraction stable set F, the above defined weight φ is contraction stable.* $\square$

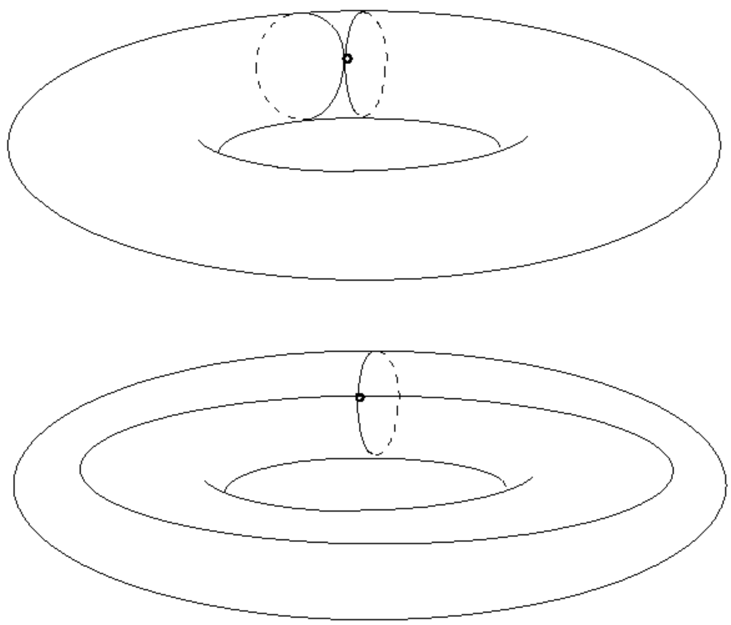

Figure 4: *Two different embeddings of the same graph into a torus.*

This weight distinguishes the two embeddings (Figure 4) of the same graph into a torus, since the homology classes of the loops differ. This would not have been the case with the F–polynomial defined on any of the F_{mr}^{ns} sets of Section 3.2. Of course, a suitable choice of a subset W (see Section 3.2) would have done; but this would mean knowing the embeddings in advance, whereas the weight in $R_1(S)$ reveals the differences at once.

Acknowledgements

Work performed under support of INdAM–GNSAGA of Italy and of the University of Bologna, funds for selected research topics.

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