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Projective Pose Estimation of Linear and Quadratic Primitives in Monocular Computer Vision

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In this paper the relevance of perspective geometry for 3D scene analysis from a single view is asserted. Analytic procedures for perspective inversion of special primitive configurations are presented. Four configurations are treated: (1) four coplanar segments; (2) three orthogonal segments; (3) a circle arc; (4) a quadric of revolution. A complete and thorough illustration of the developed methodologies is given. The importance of the selected primitives is illustrated in different application contexts. Experimental results on real images are provided for configurations (3) and (4). © 1993 Academic Press, Inc.

1. INTRODUCTION

Monocular computer vision is one of the most challenging approaches for 3D scene analysis. The basic idea of monocular vision is to understand in which situations and under which conditions a single 2D image can provide enough information for a 3D interpretation of the scene. A set of paradigms, known as "shape from X," has been developed within this reference, such as shape from shading, shape from texture, and shape from contours.

As is extensively documented in the computer vision literature, this approach is especially suitable for model-based object recognition and spatial localization, where a strong a-priori knowledge about objects is available. In this context, single image based methodologies are often competitive with "effective" 3D methodologies, like stereovision or active vision, both for computational cost and for performances.

Within industrial robotics, applications like object recognition and manipulation are good examples of situations in which a lot of information (often in the form of CAD models) is available both on the objects and on the environment structure. Anyway, at present other new applications are emerging that seem as well suited for the monocular approach, like the auto-positioning and the landmark-based navigation of autonomous mobile robots.

Among the methodologies of monocular computer vision, perspective inversion as a tool to infer 3D information from 2D data plays a very important role, especially for its consolidated mathematical foundations and for its large applicability. Independently of the kind of applications, the basic problem of perspective inversion is to recover the 3D orientation of some scene elements, referred to as primitives, starting from their 2D projection in the image plane, exploiting model knowledge to obtain the necessary constraints. The choice of the involved primitives is a crucial point: they must be as much general as possible, in order to be useful in many different applications, robust to noise, and efficiently detectable with conventional low-level vision modules.

This paper presents some mathematical procedures which allow us to compute the perspective inversion of particular configurations of primitives, obtaining completely analytic solutions. We shall consider four configurations:

(a) four coplanar segments projecting to four image segments;

(b) three orthogonal segments projecting to three image segments;

(c) a circle arc projecting to an elliptic arc in the image;

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(d) a quadric of revolution projecting to a region bounded by a conic in the image.

Two configurations, (a) and (c), lie on a plane in space, the other two are strictly three-dimensional. Two, (a) and (b), are linear, the other two are quadratic.

About the primitives, some considerations can be made. Primitives must be "relevant" to the application, in the sense that a significant, although incomplete description of the analyzed scene must be obtained through their use. Besides, they must be "detectable" in an efficient and robust way from images.

In the man-made mechanical object recognition domain, primitives like circles and segments are often present; furthermore the possibility to have only partial views of model primitives (for instance, for occlusion) is taken into account both at low-level (during the feature detection) and at high-level (inside the back-projection methodologies). Something more about image primitives detection will be said in the application section. Not only straight segments and circles, but also special geometric 3D primitives like quadrics of revolution can be thought of as important components of object models. Moreover, there are also some interesting situations, especially in the field of self-localization of mobile robots, in which structures that can be almost completely described in terms of such primitives are very important for the task exploitation.

Section 2 gives some general information about our approach and solutions also, in comparison with other works in the same context. Sections 3, 4, 5, and 6 are devoted to the complete exposition of our solutions for each considered configuration. Section 7 describes briefly a range of applications in which the presented methodologies are or can be used. Experimental results on real data are presented in Section 8, together with some details on the adopted low-level processing and with some considerations about the accuracy of the methods. Conclusions and planned developments are exposed in Section 9. In the Appendix some basic definitions and properties of projective geometry are reported.

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2. THE PERSPECTIVE INVERSION APPROACH

In general terms, perspective inversion, or backprojection, consists in determining the possible 3D configurations of scene elements that can project a given 2D image. While the direct perspective transformation is immediate, the inverse problem is more complex and cannot be generally solved, due to its nonlinearity and intrinsic ambiguity that result in a large (often infinite) number of possible solutions.

According to the paradigm "shape from X," this work can be classified in the class "shape from contour," in the sense that the used primitives correspond to occluding or pictorial boundaries of the scene objects. The image primitives we have selected are straight segments and elliptic arcs. In the following (except for the last configuration), it will tacitly be assumed that straight image segments are projections of straight scene segments, and elliptic arcs of circle arcs. This roughly corresponds to the "viewpoint general position assumption," that has been largely adopted in the application of perspective geometry to image analysis.

Due to the intrinsic ambiguity of the problem, also analytic procedures like those described in the paper can yield more than one mathematically feasible solution. The number and the type of these spurious solutions will be discussed in some detail for each case. An important point is also the relevance of the correspondence among scene and image primitives, that is essentially a preliminary phase for the application of perspective backprojection. Although most of the algorithms presented here, as many others reported in literature, need such information, a study of the robustness of them in the presence of erroneous correspondences will be tried.

The first configuration, four coplanar segments, gives a unique solution, if the corresponce hypothesis is correct. When this hypothesis is wrong, it may happen that no solution is possible. As the adjacency between segments is not requested, the case of four points in a plane with relative known positions is included in the solution. Other works in literature, like [1–3], deal with coplanar segments or special configurations of points. In [4], it is proved that for three lines the solution comes from a fourth-degree equation. Anyway, no examples are known to the authors of approaches similar to the present one in terms of adopted mathematical tools.

For the second configuration, three orthogonal segments, the orientation problem can be solved up to a finite ambiguity: two sets of directions will be obtained (see also [5]). Also in this case it may happen that the orthogonal interpretation cannot be found if the hypothesis on segments' orthogonality is not correct. The three segments need not meet in a point (unlike for similar results in [6]). In [4, 7] the problem of three arbitrary oriented lines has been reduced to the solution of an algebraic equation of degree 8. In the particular case of three orthogonal segments it reduces to a degree 4 equation, while the method of the present paper gives a degree 2 equation. Moreover, in this method the orthogonality constraint can be relaxed by allowing one of the three angles involved not to be right. In this case the degree of the equation that must be solved raises to 4.

In the third case, the backprojection of an elliptic arc, the computation allows determining two possible orientations of the lying plane without specific model knowledge. The only assumption is that the elliptic arc in the image comes from a circle arc. When the radius of the circle is known, it is possible to compute the absolute position. This procedure can be applied also to a set of points (five or more points) inscriptable in a circle, but in this case the hypothesis of punctual correspondence between image and scene primitives is required.

A previous analytic solution to the same problem has been already exposed by two of us in [8], but here the same results are obtained more directly using more powerful and general mathematical tools.

In literature, the problem of perspective inversion of conics (together with polygons and parametric curves) has been faced for the first time by Haralick and Chu in [9]. In their paper, the authors decompose the problem in a first optimization phase which determines the three rotation parameters and in a successive algebraic computation of the position of the geometric figure with respect to the camera. Recently Dhome *et al.* in [10, 11] have given another solution of the same problem, completely different from the one presented here.

The last configuration, the quadrics of revolution, consists of several cases, for three of which (spheres, circular cones, and cilinders) explicit formulae will be given. The basic idea is recognizing the position in space of a quadric surface of revolution from the projection of its contour in the image plane. A different approach to the problem of positioning objects of revolution starting from their occluding boundary can be found in [10].

As a general comment, it can be said that the various proposed methodologies have as common denominator the use of some mathematical tools belonging to perspective geometry not very conventional in the computer vision community. In the Appendix the basic terminology of projective geometry is recalled and some concepts that are useful for our proofs are reviewed.

As for notation, $\cdot$ will mean matrix product, $\langle\ ,\ \rangle$ scalar (or inner) product of vectors, $\wedge$ vector (or exterior) product, A^T the transpose matrix of A, V^T the column vector transpose of the row vector V, I_n the identity matrix of order n; if P and Q are points on an oriented straight line, $d(P,Q)$ is their signed distance, i.e., the length with sign of the segment PQ.

3. FOUR COPLANAR LINES

Let four segments be given in the image and assume that they are the projection of four coplanar segments in the scene (Fig. 1). A natural enough way to solve the pose estimation problem for the scene segments could be to use the coordinates of the end-points of the image segments, together with a priori knowledge on mutual relations of the end-points in the scene. But this method turns out to be somewhat unreliable.

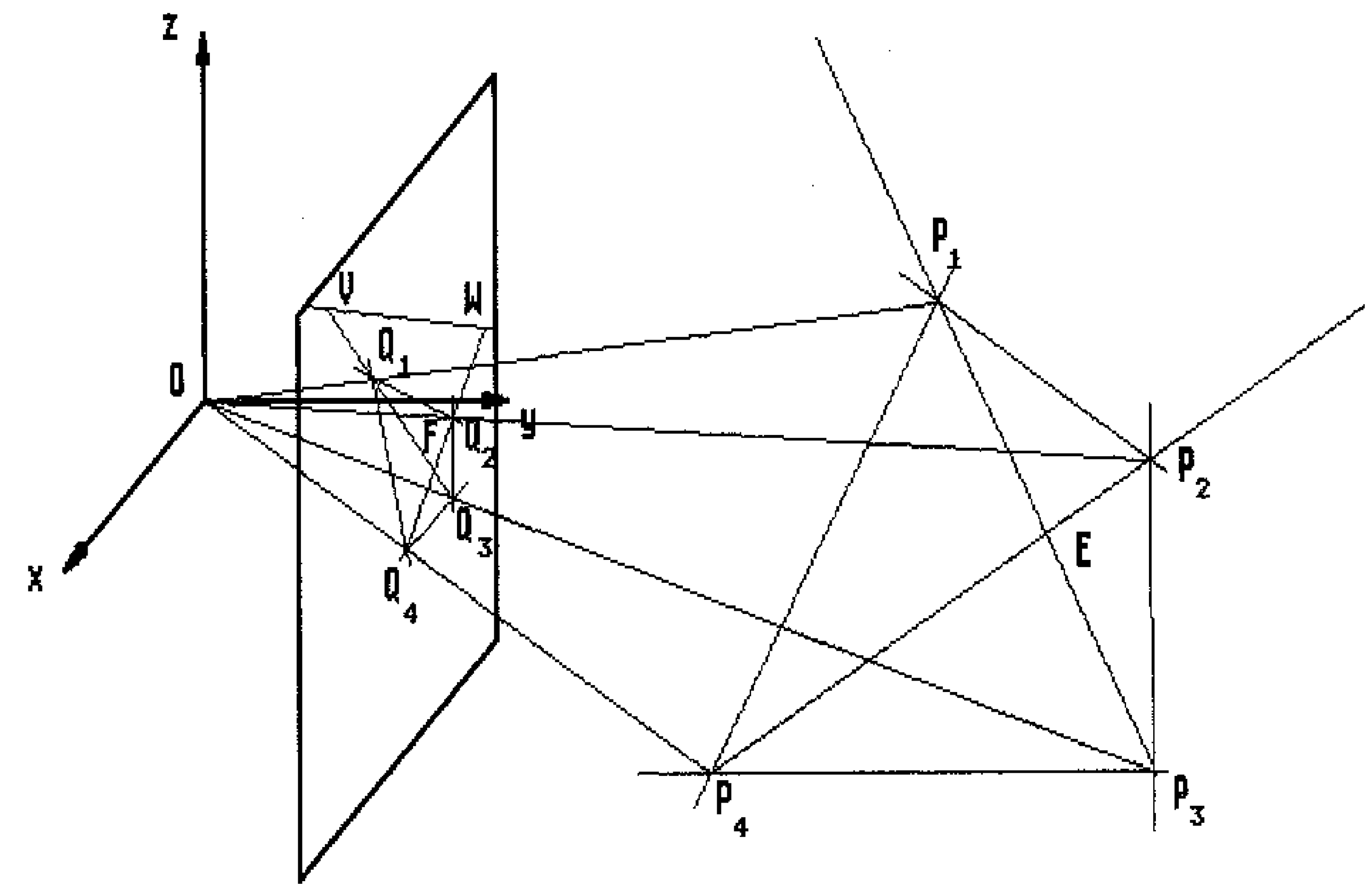

FIGURE 1

A more robust algorithm can be based on the extraction of the lines to which the segments belong. The actual computation will then be carried on with points as data: not the eight end-points of the segments, but four intersection points of line pairs.

In the half-space of positive y, let four coplanar points P_i $(i = 1, \ldots, 4)$ be given, such that they are the vertices of their quadrilateral convex hull, whose boundary segments are P_1P_2, P_2P_3, P_3P_4, P_4P_1; assume that they are not coplanar with the origin O of the reference frame. Let also E be the intersection point of the lines P_1P_3 and P_2P_4. Let further Q_1 be the projection of each P_i on the image plane $y = f$ from O, and F be the projection of E. Necessarily, the points Q_i are vertices of a quadrilateral and F is the intersection of the lines Q_1Q_3 and Q_2Q_4.

Assume the coordinates in the plane $y = f$ of each Q_i to be (X_i, Z_i); these are the image data. Assume also $d_i = d(P_i, E)$ to be known for each i, from model knowledge, with respect to an arbitrary orientation of the lines P_1P_3 and P_2P_4. Finally, set

$$b_1 = \begin{vmatrix} (X_3 - X_2) & (X_4 - X_2) \\ (Z_3 - Z_2) & (Z_4 - Z_2) \end{vmatrix}, \quad b_2 = \begin{vmatrix} (X_4 - X_1) & (X_3 - X_1) \\ (Z_4 - Z_1) & (Z_3 - Z_1) \end{vmatrix}$$

$$b_3 = \begin{vmatrix} (X_1 - X_2) & (X_4 - X_2) \\ (Z_1 - Z_2) & (Z_4 - Z_2) \end{vmatrix}, \quad b_4 = \begin{vmatrix} (X_2 - X_1) & (X_3 - X_1) \\ (Z_2 - Z_1) & (Z_3 - Z_1) \end{vmatrix}.$$

PROPOSITION 3.1. *Let points P_i and Q_i, and (X_i, Z_i), d_i, b_i, E, F be given as above ($i \in \{1, 2, 3, 4\}$). Then the plane containing the points P_i is parallel to the one of equation*

$$\begin{vmatrix} x & y & z \\ (X_1d_1b_1 + X_3d_3b_3) & f(d_1b_1 + d_3b_3) & (Z_1d_1b_1 + Z_3d_3b_3) \\ (X_2d_2b_2 + X_4d_4b_4) & f(d_2b_2 + d_4b_4) & (Z_2d_2b_2 + Z_4d_4b_4) \end{vmatrix} = 0.$$

Proof. By intersecting the lines Q_1Q_3 and Q_2Q_4 one can compute the coordinates of F as

$$(X_F, Z_F)^{\mathrm{T}} = \frac{1}{\begin{vmatrix} (Z_3 - Z_1)(X_1 - X_3) \\ (Z_4 - Z_2)(X_2 - X_4) \end{vmatrix}} \begin{pmatrix} (Z_3 - Z_1)(X_2 - X_4)X_1 - (X_1 - X_3)(Z_4 - Z_2)X_2 + (X_1 - X_3)(X_2 - X_4)(Z_1 - Z_2) \\ (Z_3 - Z_1)(Z_4 - Z_2)(X_2 - X_1) + (Z_3 - Z_1)(X_2 - X_4)Z_2 - (X_1 - X_3)(Z_4 - Z_2)Z_1 \end{pmatrix}.$$

Now let V_∞, W_∞ be the ideal points of the lines P_1P_3 and P_2P_4, respectively (i.e., their "points at infinity"; see the Appendix), and let $V \equiv (X_V, Y_V)$, $W \equiv (X_W, Y_W)$ be the vanishing points of those lines, i.e., the projections of V_∞ and of W_∞, respectively, on the image plane from O. Then from the above computation and from the invariance of the cross ratio (see Theorem A.1 and Remark A.1 of the Appendix) we obtain

$$R_{VFQ_1Q_3} = R_{V_\infty EP_1P_3};$$

by developing the computations in the context of Remark A.2 of the Appendix and gathering the quantities b_i we obtain

$$\frac{(X_1 - X_V)\, b_1}{-(X_3 - X_V)\, b_3} = \frac{d_3}{d_1};$$

the same equality of cross ratios yields also an analogous relation between the Z coordinates of the same points; analogous relations come from the equality $R_{WFQ_2Q_4} = R_{W_\infty EP_2P_4}$. From all this we obtain

$$(X_W, Z_W) = (X_2d_2b_2 + X_4d_4b_4, Z_2d_2b_2 + Z_4d_4b_4)/(d_2b_2 + d_4b_4).$$

$$(X_W, Z_W) = (X_2d_2b_2 + X_4d_4b_4, Z_2d_2b_2 + Z_4d_4b_4)/(d_2b_2 + d_4b_4).$$

By the definition of V and W, the line VW is the vanishing line of the plane containing the points P_i, i.e., the projection of its ideal line (or "line at infinity") from O. So the plane Π passing through O, V, and W has the same ideal line; otherwise stated, it is parallel to the required plane. The equation of Π is the one given in the statement. ■

Remark 3.1. The indeterminacy in position of the plane can be solved by using the actual length of the segments in the scene, or the mutual distances of their end-points, known from the model.

Now an example of computation follows; although it is rather artificial, it hopefully can convey the idea of the precision of the method.

Let the focal length be $f = 1$, and let the four corner points at the intersections of the image lines be $Q_1 \equiv (-\frac{2}{9}, 1, 0)$, $Q_2 \equiv (\frac{2}{3}, 1, 0)$, $Q_3 \equiv (\frac{1}{3}, 1, \frac{1}{3})$, $Q_4 \equiv (-\frac{1}{8}, 1, \frac{1}{8})$. Further let the distances $d_1 = 3\sqrt{29}/7$, $d_2 = 6\sqrt{30}/7$, $d_3 = 4\sqrt{29}/7$, $d_4 = \sqrt{30}/7$ be known from the model. From the coordinates we obtain $b_1 = \frac{2}{9}$, $b_2 = -\frac{1}{27}$, $b_3 = -\frac{1}{9}$, $b_4 = \frac{8}{27}$. From all this we obtain the equation

$$\frac{2\sqrt{870}}{5103}(x + 2y + z) = 0$$

of the plane through the origin O, parallel to the one on which the scene points P_i lie. The latter plane will then have equation of the form $x + 2y + z = k$.

Now, intersecting the plane $x + 2y + z = k$ with the line OQ_1 and the line OQ_3 we obtain for each such plane, two "candidate" points of coordinates $(-k/8, 9k/16, 0)$ and $(k/8, 3k/8, k/8)$, respectively. The square of their Euclidean distance is $29k^2/256$; it must equal $(d_1 + d_3)^2 = 29$, so we obtain the equation $k^2 = 256$; of the solutions ± 16, the only physically acceptable is the one for which the points P_i are "visible", i.e., have positive y. Therefore the resulting plane is $x + 2y + z = 16$; the same $k = 16$ gives at once $P_1 \equiv (-2, 9, 0)$ and $P_3 \equiv (2, 6, 2)$. The remaining two points $P_2 \equiv (4, 6, 0)$ and $P_4 \equiv (-1, 8, 1)$ come from the intersection of the plane with the lines OQ_2 and OQ_4.

4. THREE ORTHOGONAL LINES

When three orthogonal straight lines are present in a scene, they project to three straight lines on the image (assuming general position of the viewpoint). Generally these projected lines are not orthogonal (see Fig. 2). The problem is to compute the spatial orientations of the three scene lines, starting from their projections. In the real case, of course, only segments will be in the scene and in the image, but it is easy to extract from them the equations of the lines they lie on.

The presented method does not require the three lines to meet in a common point, allowing consideration of model primitives from different faces of the object. In fact, the problem can be solved in the ideal plane, so only directions are considered. In addition, an extension is proposed to allow one of the lines to be not orthogonal to the other two (Remark 4.3).

PROPOSITION 4.1. *Let three straight lines s_i ($i = 1, 2, 3$) be given on the plane $y = f$, and let their equations be respectively $a_ix + c_iz = -b_if$. Let also $W_i = (a_i, b_i, c_i)$*

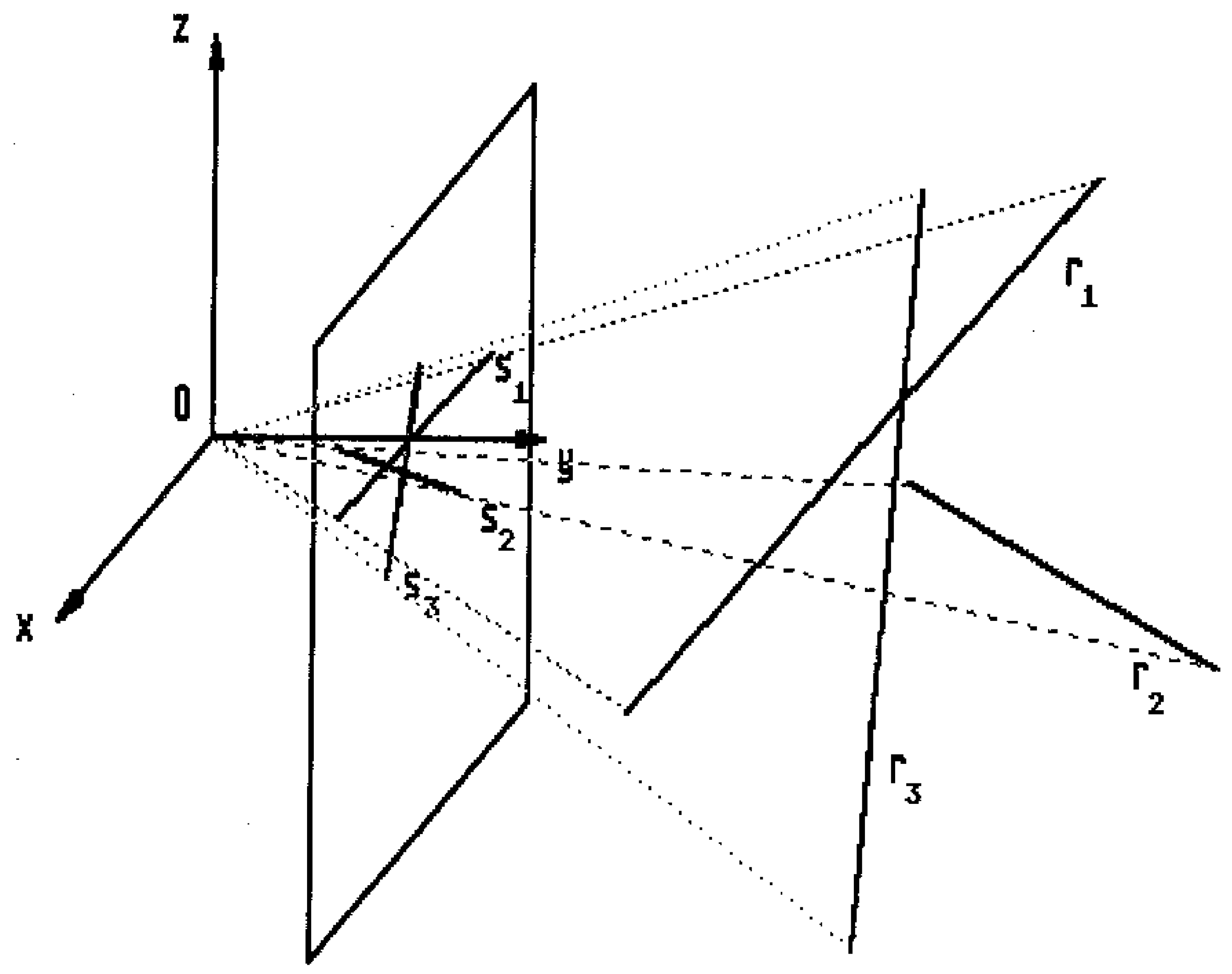

FIGURE 2

($i = 1, 2, 3$) *and* A, B *be any two (but nonproportional and nonnull) of the three triples* $(-b_3, a_3, 0)$, $(-c_3, 0, a_3)$, $(0, -c_3, b_3)$.

Then a triple of orthogonal straight lines r_i *exists in space, such that* s_i *is the projection of* r_i *for each i, iff the quadratic equation*

$$(A \cdot M \cdot A^{\mathrm{T}})\lambda^2 + 2(A \cdot M \cdot B^{\mathrm{T}})\lambda\mu + (B \cdot M \cdot B^{\mathrm{T}})\mu^2 = 0,$$

where

$$M = (W_1 \cdot W_2^{\mathrm{T}})I_3 - (W_1^{\mathrm{T}} \cdot W_2 + W_2^{\mathrm{T}} \cdot W_1)/2$$

admits real solutions. In that case, for such a solution (λ', μ') *the ideal point of* r_3 *has homogeneous coordinates* $(0, X_3, Y_3, Z_3)$, *where* $(X_3, Y_3, Z_3) = \lambda' A + \mu' B$.

Proof. Assuming that three orthogonal lines r_i exist, which project to the image lines s_i, they must each lie on the plane Π_i passing through the origin and intersecting the image plane $y = f$ in s_i. These planes respectively have equations $a_i x + b_i y + c_i z = 0$, and the ideal points V_i of the lines r_i must belong to the ideal lines of such planes, since the lines r_i themselves lie on the planes Π_i.

In the following argument all points and lines will lie on the ideal plane Π_0, so we shall always skip the equation $t = 0$ and the first homogeneous coordinate (equal to zero). So the ideal points of the lines r_i will be represented by $V_i = (X_i, Y_i, Z_i)$, and the ideal lines of the planes will be denoted by the homogeneous equations of the planes themselves.

V_1 and V_2 must be orthogonal to V_3, so they must satisfy the equation

$$X_3 x + Y_3 y + Z_3 z = 0;$$

they also represent ideal points lying on the first two ideal lines, respectively. Therefore V_1 is determined, up to proportionality, as a nonnull solution of the linear homogeneous system

$$a_1 x + b_1 y + c_1 z = 0$$
$$X_3 x + Y_3 y + Z_3 z = 0$$

and V_2 as a nonnull solution of

$$a_2 x + b_2 y + c_2 z = 0$$
$$X_3 x + Y_3 y + Z_3 z = 0;$$

so, up to proportionality, $V_i = W_i \wedge V_3 \quad (i = 1, 2)$.

Finally, there is the orthogonality condition for V_1 and V_2, i.e., $\langle V_1, V_2 \rangle = 0$:

$$\langle (W_1 \wedge V_3), (W_2 \wedge V_3) \rangle = 0$$

whence, by standard formulae,

$$\langle W_1, W_2 \rangle \langle V_3, V_3 \rangle - \langle W_1, V_3 \rangle \langle W_2, V_3 \rangle = 0;$$

this is a homogeneous quadratic equation in X_3, Y_3, Z_3, with coefficients given by the image data a_i, b_i, c_i ($i = 1, 2$). Explicit computation of the quadratic form on the left-hand side shows that the corresponding matrix is

$$M = (W_1 \cdot W_2^{\mathrm{T}})I_3 - (W_1^{\mathrm{T}} \cdot W_2 + W_2^{\mathrm{T}} \cdot W_1)/2;$$

i.e., the equality expressed above in terms of scalar products can be rephrased as the fact that V_3 must satisfy the equation

$$(x\ y\ z) \cdot M \cdot (x\ y\ z)^{\mathrm{T}} = 0.$$

This equation can be seen as representative of a quadric in space or (together with the equation $t = 0$ of Π_0) as representative of the conic at infinity $\mathscr{C}_0$ of the same quadric. So V_3 (or, better said, the corresponding ideal point) must belong to the conic at infinity $\mathscr{C}_0$. On the other hand, V_3 must belong to the ideal line t_3 of Π_3, which has equation ($t = 0$ always understood)

$$a_3 x + b_3 y + c_3 z = 0.$$

So the required ideal point must belong to the intersection of $\mathscr{C}_0$ and t_3; now, by standard methods the intersection points can be found as $\lambda' A + \mu' B$ with A and B distinct points of the line and (λ', μ') a nonnull solution of the equation

$$(A \cdot M \cdot A^{\mathrm{T}})\lambda^2 + 2(A \cdot M \cdot B^{\mathrm{T}})\lambda\mu + (B \cdot M \cdot B^{\mathrm{T}})\mu^2 = 0;$$

at least two of the triples $(-b_3, a_3, 0)$, $(-c_3, 0, a_3)$, $(0, -c_3, b_3)$ are nonnull and not proportional solutions of

$a_3x + b_3y + c_3z = 0$, so they represent such distinct points A, B.

Conversely, if a solution of the equation exists, then there is an intersection of the line t_3 and the conic $\mathscr{C}_0$, so that there exists a triple of orthogonal directions of lines in space which project to lines parallel to s_1, s_2, s_3. In this case, by parallel displacement it is possible to find three lines which project exactly to s_1, s_2, s_3 (actually for each s_i there are infinitely many possible r_i projecting to it). ■

Remark 4.1. A suitable change of basis in $\mathbb{R}^3$ (e.g., one for which W_1 assumes second and third components equal to zero) shows that M has two equal eigenvalues, and a third eigenvalue which either vanishes or has a sign opposite to the one of the other eigenvalue. In the former case the conic $\mathscr{C}_0$ reduces to a point; in the latter it has infinitely many points.

Therefore, however the first two image lines are given, a third image line always exists such that an interpretation of the triple as the projection of three orthogonal space lines is possible. Actually, this interpretation is unique (up to parallelism) in the case of a zero eigenvalue or in the case of tangency between the conic and the line at infinity. In all other cases, two possible triples of orthogonal directions can be found. On the other hand, there always exist also "third lines" s_3 which make this interpretation impossible; this is the case of an ideal line t_3 external to the conic at infinity $\mathscr{C}_0$.

Note that the homogeneous equation in λ and μ, when solvable, always admits infinitely many solutions. This does not give a further ambiguity, because these solutions form either one or two classes of proportionality; replacing a solution (λ', μ') with a proportional one does not change the resulting point, since we are working with homogeneous coordinates.

Remark 4.2. When solutions exist, two "degrees of freedom" can be eliminated by using the mutual distances of the lines in the model, but the finite ambiguity due to the possible two solution classes and the remaining simple infinity of interpretations can only be solved by using the length of the actual segments.

Remark 4.2. The orthogonality condition can be relaxed by allowing the directions V_1 and V_2 to form a given angle α. Then the condition $\langle V_1, V_2\rangle = 0$ is replaced by the equality

$$\langle V_1, V_2\rangle^2 = (\cos\alpha)^2\langle V_1, V_1\rangle\langle V_2, V_2\rangle.$$

Again, by substituting V_i with $W_i \wedge V_3$ ($i = 1, 2$) one obtains the equation, in X_3, Y_3, Z_3, of a curve at infinity; this is no longer a conic, but a quartic. This brings with it a greater difficulty of solution for the system yielding the intersection with the third ideal line, and a greater ambiguity. Still, this is the same degree of the equation obtained in [4] for the simpler case of three right angles.

Remark 4.4. Again in comparison with the results of [4], it should be noted that Proposition 4.1 provides the direction of the lines with no requirements of further computations. Of course, [4] maintains the advantage of a greater generality.

Again a mathematically minded example: Let $f = 1$, and let the three image lines s_1, s_2, s_3 have equations $x - z = 0$, $2x + z - 1 = 0$, $3x + 5z - 1 = 0$, respectively. Consequently, one has $W_1 = (1, 0, 1)$, $W_2 = (2, -1, 1)$, $W_3 = (3, -1, 5)$. It is now necessary to decide whether a triple of orthogonal lines in space exists with the given projections, and if this is the case, one wants to determine the direction of the line r_3 projecting to s_3. Obviously, also r_1 and r_2 can be analogously determined by permuting indices.

The three triples of Proposition 4.1 are (1, 3, 0), (−5, 0, 3), (0, −5, −1); as A and B choose, e.g., the second and third triple, respectively. The outcoming matrix is then

$$M = \begin{pmatrix} -1 & \frac{1}{2} & -1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ - & -\frac{1}{2} & 3 \end{pmatrix}.$$

The equation to discuss is

$$-22\lambda^2 + 33\lambda\mu + 22\mu^2 = 0;$$

this actually is solvable, and its solutions are $\lambda = -\mu/2$ and $\lambda = 2\mu$. The first solution yields (setting $\mu = 2$) the triple $(-5, 0, 3) - 2(0, -5, -1) = (-5, 10, 5)$; the second solution gives (setting $\mu = 1$) the triple $(-10, -5, 5)$. These are two possible directional vectors of the line r_3. Another way of expressing it is to say that r_3 has an ideal point of coordinates either (0, −5, 10, 5) or (0, −10, −5, 5). Since directional vectors (and homogeneous coordinates) are defined up to proportionality, we can use (0, −1, 2, 1) and (0, −2, −1, 1) as well.

The same procedure yields (0, 0, 1, 1) and (0, −1, −1, 1) for r_2, and (0, 1, 0, 1), (0, −1, 1, −1) for r_1. By the required orthogonality of the lines, we finally have either (0, 1, 0, 1), (0, −1, −1, 1), (0, −1, 2, 1) or (0, −1, 1, −1), (0, 0, 1, 1), (0, −1, 2, 1) as triples of ideal points.

5. CIRCLE IN SPACE

The problem of inverting the perspective projection for an ellipse Γ, which is known as coming from a circle in the scene, is reduced to finding out those planes whose intersection with the cone over the ellipse and with the

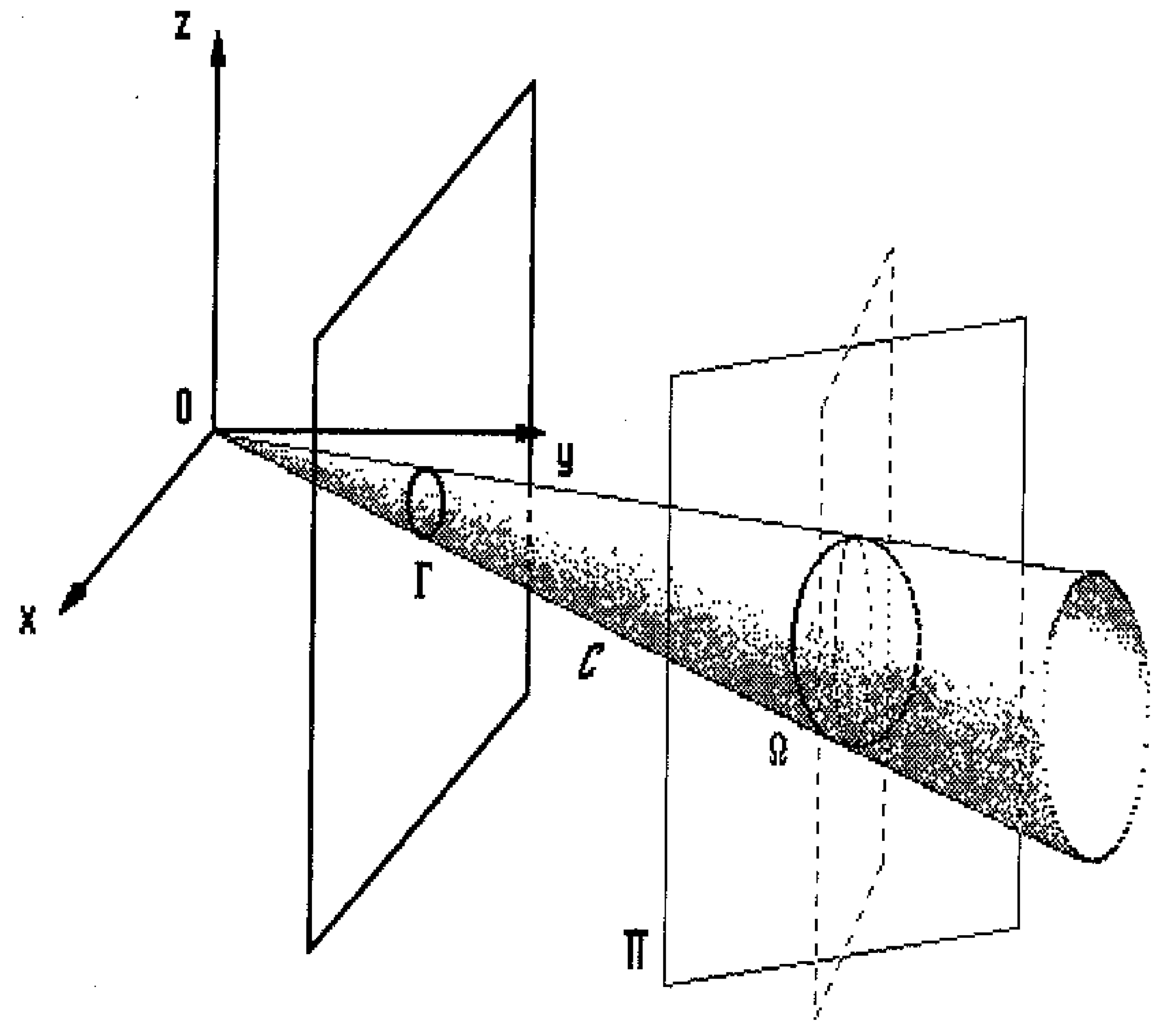

FIGURE 3

vertex in the origin are circles (see Fig. 3). The radius value will allow choosing, among an infinity of parallel planes, the actual one. Except for particular settings, two pencils of parallel planes are possible for a given ellipse.

In the following t, x, y, z will be used as homogeneous coordinates, so that the usual Cartesian coordinates x, y, z come from setting $t = 1$, and the ideal plane Π_0 (i.e., the plane at infinity) has equation $t = 0$.

Let Γ be the conic section of the focal plane $y = f$ represented by the equations (in Cartesian coordinates)

$$ax^2 + cxz + gz^2 + bx + ez + d = 0$$
$$y = f.$$

Then the cone $\mathscr{C}$ over Γ with the origin as vertex has equation (again in Cartesian, but actually also in homogeneous coordinates) $\Phi = 0$, where

$$\Phi = \alpha x^2 + 2\gamma xz + \eta z^2 + 2\beta xy + 2\varepsilon yz + \delta y^2$$

and where $\alpha = a$, $\gamma = c/2$, $\eta = g$, $\beta = b/2f$, $\varepsilon = e/2f$, $\delta = d/f^2$. This is easily verified: Φ is homogeneous quadratic, so $\Phi = 0$ represents a cone with the origin as vertex, and the intersection with the focal plane yields Γ. Now consider the matrix

$$M = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \delta & \varepsilon \\ \gamma & \varepsilon & \eta \end{pmatrix}.$$

PROPOSITION 5.1. *The matrix M admits real eigenvalues $k_1 \leq k_2 \leq k_3$. The quadratic equation $\Phi - k_2\Theta = 0$ represents the union of two (possibly coincident) planes Π_1, Π_2 passing through the origin. All and only the planes parallel to Π_1 or to Π_2 intersect the cone $\mathscr{C}$ in circles.*

Proof. Every circle can be considered as the intersection of its plane Π with a suitable sphere Σ; therefore the intersection $\Pi \cap \Sigma \cap \Pi_0$ of the circle with the ideal plane Π_0 coincides with the intersection $\Pi \cap \mathscr{H}_0$ of the plane Π (or of its ideal line) with the absolute circle (Theorem A.2 of the Appendix). Thus, if a plane Π meets the cone $\mathscr{C}$ in a circle, then $\Pi \cap \mathscr{C} \cap \Pi_0 = \Pi \cap \mathscr{H}_0$; this means that the ideal line of Π passes through two points common to the absolute circle $\mathscr{H}_0$ and to $\mathscr{C}_0 = \mathscr{C} \cap \Pi_0$, the *conic at infinity* of $\mathscr{C}$. If Π meets $\mathscr{C}$ in a circle, so does every parallel plane; therefore one just has to look for the ideal lines of the interested planes. The argument will be entirely developed in the ideal plane Π_0, so the equation $t = 0$ will always be skipped. So now $\Phi = 0$ represents, in Π_0, the conic at infinity $\mathscr{C}_0$ of $\mathscr{C}$.

It is recommended to follow the next argument on Fig. 4a (respectively on Fig. 4b for the exceptional case of bitangency). Although a drawing is impossible, since $\mathscr{H}_0$ and most other elements are imaginary, the picture can be a useful scheme.

Consider the pencil $\mathscr{P}$ of conics (in Π_0) generated by $\mathscr{C}_0$ and $\mathscr{H}_0$, i.e., the set of all conics represented by equations which are linear combinations of the equations of $\mathscr{C}_0$ and

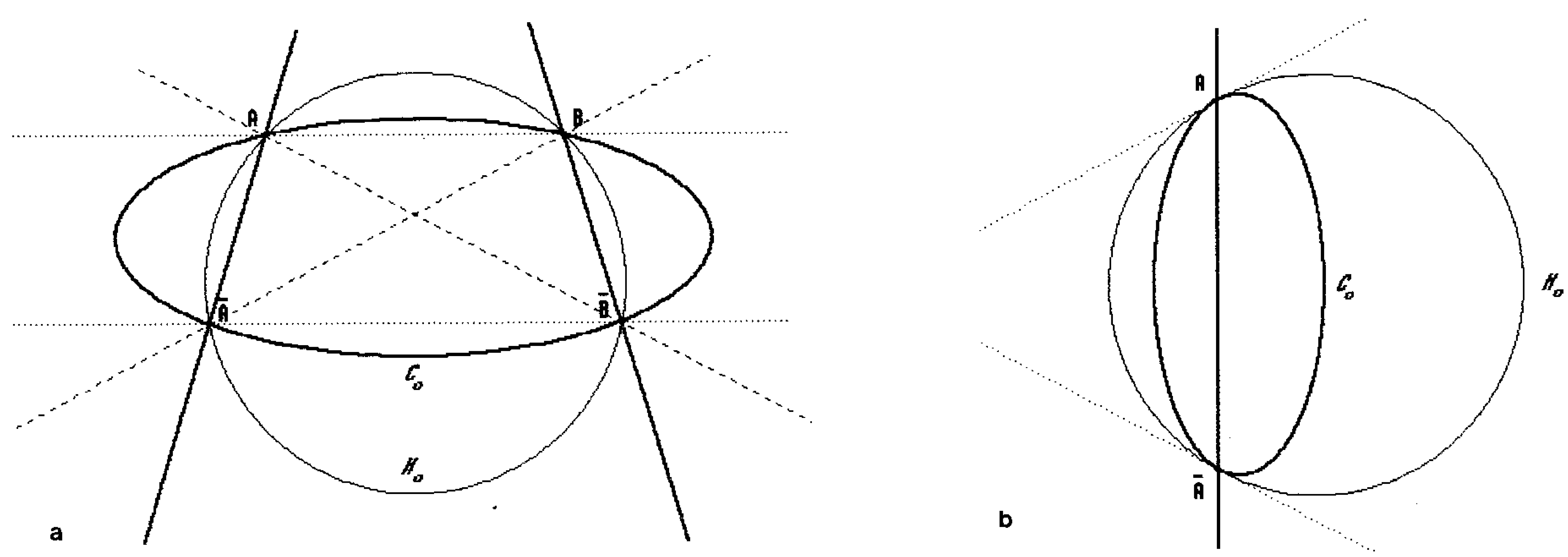

FIGURE 4

$\mathcal{H}_0$. All conics of $\mathcal{P}$ pass through the common points of $\mathcal{C}_0$ and $\mathcal{H}_0$. These points are necessarily nonreal, so they come in conjugate pairs $A, \bar{A}, B, \bar{B}$ (with the possible coincidence $A = B$ and $\bar{A} = \bar{B}$, in which case $\mathcal{C}_0$ and $\mathcal{H}_0$ are bitangent). Then $\mathcal{P}$ contains three degenerate conics: $AB \cup \bar{A}\bar{B}, A\bar{A} \cup B\bar{B}, A\bar{B} \cup \bar{A}B$ (in the bitangency case one has two: the union of the two common tangents and the line $A\bar{A}$ "counted twice"). Since a real line containing a nonreal point must also contain its conjugate, only the second degenerate conic splits into two real lines (bitangency case: the line counted twice is real, the two tangents are not.) The real lines of this degenerate conic are the requested ideal lines (in the bitangency case one has only one line, so only one pencil of planes).

Each conic of $\mathcal{P}$ (except $\mathcal{H}_0$) is represented by the equation $\Phi - k\Theta = 0$ for a suitable $k \in \mathbb{R}$. The discriminant of $\mathcal{C}_0$ is M, so the one of the generic conic of $\mathcal{P}$ is $(M - kI_3)$. The degenerate conics correspond to null determinants of the discriminants; therefore, the previous geometric reasonment implies the existence either of three distinct real roots of $|M - kI_3|$, or of two real roots, of which one has multiplicity two; these roots actually are the eigenvalues of M.

One is now left with the problem of determining which of the three (or two) roots corresponds to the conic of interest. The case of two roots (bitangency) is immediate, since the line counted twice corresponds to the double root.

There remains the case of three distinct roots: one of them yields the requested conic, formed by two distinct real lines. Now, the discriminant of such a conic has three eigenvalues: one null, one positive, and one negative. The characteristic polynomial of $(M - kI_3)$ is

$$\begin{aligned}|M - kI_3 - \lambda I_3| = {} & -\lambda^3 + \lambda^2(-3k + \alpha + \delta + \eta) + \lambda[-3k^2 \\ & + 2k(\alpha + \delta + \eta) - \alpha\delta - \alpha\eta - \delta\eta + \beta^2 + \gamma^2 + \varepsilon^2] \\ & -k + k^2(\alpha + \delta + \eta) - k(\alpha\delta + \alpha\eta + \delta\eta - \beta^2 - \gamma^2 - \varepsilon^2) \\ & + (\alpha\delta\eta + 2\beta\gamma\varepsilon - \alpha\varepsilon^2 - \delta\gamma^2 - \eta\beta^2).\end{aligned}$$

Call b_0 the term of degree zero (in λ) and b_1 the coefficient of λ. Then b_0 obviously is $|M - kI_3|$, and $-b_1$ is the sum of the pairwise products of the eigenvalues of $(M - kI_3)$. Therefore, one is led to select those values of k which make b_0 null and b_1 positive. On the other hand, b_1 is the derivative of b_0 with respect to k; b_0, as a function of k, is a cubic polynomial with three distinct real roots and with negative leading coefficient, so the root at which the derivative is positive is the intermediate one k_2.

Once the quadratic form $(\Phi - k_2\Theta)$ has been decomposed into a product of two linear forms (or recognized as a square), the arising linear equations represent ideal lines in Π_0, but in 3-space they can be seen as equations of planes passing through the origin and meeting Π_0 in those ideal lines. ■

Remark 5.1. The pencils of parallel planes which intersect $\mathcal{C}$ in circles are obtained by adding constants to the linear forms of the decomposition of $(\Phi - k_2\Theta)$ and equating to zero.

From a computational point of view, Proposition 4.1 provides a procedure which consists in computing the eigenvalues of M by solving a third-degree polynomial equation. The second step is to decompose the quadratic form into a product of two linear forms; this can be accomplished by solving the following system:

$$\begin{aligned} v_1 w_1 &= \alpha - k_2 \\ v_2 w_2 &= \delta - k_2 \\ v_3 w_3 &= \eta - k_2 \\ v_1 w_2 + v_2 w_1 &= 2\beta \\ v_2 w_3 + v_3 w_2 &= 2\varepsilon \\ v_1 w_3 + v_3 w_1 &= 2\gamma, \end{aligned}$$

where (v_1, v_2, v_3) and (w_1, w_2, w_3) are free vectors normal to the planes. Solving the system is rather simple, since the variables appearing in the system are separable.

Let Ω be a circle, $\mathcal{C}$ be the circumscribed cone from O to it, and M be the matrix associated with the conic at infinity of $\mathcal{C}$ with respect to the chosen reference frame S of origin O. Then let (u_1, u_2, u_3) be a free vector of norm one, orthogonal to the plane on which Ω lies, oriented so that, if applied in O, it points towards the plane. Further, let E be an orthogonal matrix having (u_1, u_2, u_3) as the second row, so it is a matrix change of an orthogonal reference frame from S to a frame S' having the y' axis orthogonal to the plane of Ω.

PROPOSITION 5.2. *Let M, Ω, (u_1, u_2, u_3), $\mathcal{C}$, E be as above, and let R be the radius of Ω. Setting $M' = E \cdot M \cdot E^T = (m'_{ij})$, then for the coordinates (X_c, Y_c, Z_c) of the centre of Ω, with respect to S, it holds:*

$$(X_c\ Y_c\ Z_c) = \frac{sign(m'_{11})\, R}{\sqrt{m'^2_{12} + m'^2_{23} - m'_{11} m'_{22}}}(-m'_{12}\ m'_{11}\ -m'_{23}) \cdot E.$$

Proof. In the new reference frame S', let (X'_c, Y'_c, Z'_c) be the coordinates of the centre of Ω; so, the plane where Ω lies has the equation $y' = Y'_c (>0)$; the equation of the cone $\mathcal{C}$ with respect to S' is then (with $\lambda \neq 0$)

$$\lambda\left(x'^2 - 2\frac{X'_c}{Y'_c}x'y' + z'^2 - 2\frac{Z'_c}{Y'_c}y'z' + \frac{(X'^2_c + Z'^2_c - R^2)}{Y'^2_c}y'^2\right) = 0$$

and the associated matrix $M' = \mathrm{E} \cdot \mathrm{M} \cdot \mathrm{E}^{\mathrm{T}}$ is

$$M' = \lambda \begin{pmatrix} 1 & -X'_c/Y'_c & 0 \\ -X'_c/Y'_c & (X'^2_c + Z'^2_c - R^2)/Y'^2_c & -Z'_c/Y'_c \\ 0 & -Z'_c/Y'_c & 1 \end{pmatrix};$$

therefore,

$$Y'_c = \frac{R|m'_{11}|}{\sqrt{m'^2_{12} + m'^2_{23} - m'_{11}m'_{22}}},$$
$$X'_c = -Y'_c m'_{12}/m'_{11}, \quad Z'_c = -Y'_c m'_{23}/m'_{11}$$

and the final result comes from multiplication with the matrix E of the change of frame. ■

Remark 5.2. A matrix E as the one required is easily found by building an orthonormal basis of the orthogonal complement of (u_1, u_2, u_3).

Remark 5.3. Unlike the earlier paper [8] by two of us, the article [10] uses a starting point which is very similar to the one presented here. However, it should be noted that our Proposition 5.1 provides the pencils of planes out of just one of the eigenvalues of M, thereby diminishing the effect of approximation in the solution of the characteristic equation. Moreover, here inversion of trigonometric functions is never required.

As an example, with focal length $f = 1$, assume that the image ellipse has equation

$$17x^2 + z^2 - 22x + 7 = 0$$

and that the circle projecting to it has radius two. The matrix is

$$M = \begin{pmatrix} 17 & -11 & 0 \\ -11 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and has eigenvalues 1, $\pm\sqrt{146}$; the intermediate eigenvalue is then one. One can now substitute it into k in the equation

$$(17x^2 + z^2 - 22x + 7) - k(x^2 + y^2 + z^2) = 0$$

and obtain

$$16x^2 - 22xy + 6y^2 = 0.$$

This can be decomposed into two plane equations by solving the system

$$\begin{aligned} v_1w_1 &= 16 \\ v_2w_2 &= 6 \\ v_3w_3 &= 0 \\ v_1w_2 + v_2w_1 &= -22 \\ v_2w_3 + v_3w_2 &= 0 \\ v_1w_3 + v_3w_1 &= 0; \end{aligned}$$

this yields $(-8, 3, 0)$ and $(-2, 2, 0)$ as vectors normal to planes which contain circles projecting to the image ellipse.

Assume now that, possibly by the use of other primitives, one can determine that $(-2, 2, 0)$ is the correct vector. Then one obtains

$$E = \begin{pmatrix} +1/\sqrt{2} & +1/\sqrt{2} & 0 \\ -1/\sqrt{2} & +1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{whence} \quad M' = \begin{pmatrix} 1 & -5 & 0 \\ -5 & 23 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, by Proposition 5.2, one can compute the coordinates of the circle centre as $(4, 6, 0)$.

6. QUADRICS OF REVOLUTION

The next problem to be faced is the one of recognizing the position in space of a quadric surface of revolution Q from the projection Γ of its contour on the image plane. The same notation as above will be used for the conic Γ, the cone $\mathscr{C}$ projecting Γ from the origin O, and the related coefficients.

Two orthogonal reference frames will be used (see, e.g., Fig. 6): the standard one S with the vantage point O as origin and $y = f$ as the equation of the image plane, and a new frame S' with the origin O' in the center of Q (or anywhere on the rotation axis, if Q is a cylinder) and with the z' axis coinciding with the rotation axis. Moreover, assume that the vantage point O lies in the $x' = 0$ plane and that its second coordinate with respect to S' is negative.

The purpose is to detect the position of O' and the orientation of the z' axis with respect to S. In order to accomplish that, one just has to compute eigenvalues and eigenvectors of a form representing $\mathscr{C}$, from the canonical form of Q; these are invariant (up to multiplicative factors) under frame change: the ratios between eigenvalues will yield information on the position of O′, while a normalized set of eigenvectors will lead to the reciprocal orientation of the coordinate axes of the two reference frames.

From now on the quadric Q will have, with respect to S', equation

$$x'^2 + y'^2 + rz'^2 + s = 0$$

and $(0, \tau, \nu)$, with $\tau < 0$, will be the coordinates of the vantage point O.

6.1. *The General Case*

If D is the matrix associated with a quadric, then the circumscribed cone from a fixed point of homogeneous coordinates $\mathbf{P} = (\bar{t}', \bar{x}', \bar{y}', \bar{z}')$ (i.e., the locus of the tangent lines to the quadric through the point) has equation

$$(P \cdot D \cdot X'^{T})^2 - (P \cdot D \cdot P^{T})(X' \cdot D \cdot X'^{T}) = 0$$

with $X' = (t', x', y', z')$ as the row of unknowns (see, e.g., [17, p. 587]).

In our case, we have $P = (1, 0, \tau, \nu)$ and

$$D = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

from which the cone $\mathscr{C}$ circumscribed from O has equation

$$(-r\nu^2 - s - \tau^2)x'^2 + (-r\nu^2 - s)y'^2 + (-rs - r\tau^2)z'^2 \\ +2(r\tau\nu)y'z' + 2s\tau y' + 2rs\nu z' + (-rs\nu^2 - s\tau^2) = 0$$

and its associated matrix is

$$C = \begin{pmatrix} -(rs\nu^2 + s\tau^2) & 0 & s\tau & rs\nu \\ 0 & -(r\nu^2 + s + \tau^2) & 0 & 0 \\ s\tau & 0 & -(r\nu^2 + s) & r\tau\nu \\ rs\nu & 0 & r\tau\nu & -(rs + r\tau^2) \end{pmatrix}.$$

The directions normal to the planes of orthogonal symmetry of $\mathscr{C}$ are given by the eigenvectors of the minor

$$N = \begin{pmatrix} -(r\nu^2 + s + \tau^2) & 0 & 0 \\ 0 & -(r\nu^2 + s) & r\tau\nu \\ 0 & r\tau\nu & -(rs + r\tau^2) \end{pmatrix}$$

whose eigenvalues are

$$\lambda_1 = -(r\nu^2 + s + \tau^2)$$

$$\lambda_2 = \frac{1}{2}(-(s(r+1) + r(\tau^2 + \nu^2)) \\ + \sqrt{(s(r-1) + r(\tau^2 - \nu^2))^2 + 4r^2\tau^2\nu^2}\,)$$

$$\lambda_3 = \frac{1}{2}(-(s(r+1) + r(\tau^2 + \nu^2)) \\ - \sqrt{(s(r-1) + r(\tau^2 - \nu^2))^2 + 4r^2\tau^2\nu^2}\,).$$

Setting $\omega = s(r - 1) + r(\tau^2 - \nu^2)$ and $\psi = \sqrt{\omega^2 + 4r^2\tau^2\nu^2}$, one has mutually orthogonal eigenvectors

$$\mathbf{v}_1 \equiv (1, 0, 0)$$

$$\mathbf{v}_2 \equiv (0, -r\tau\nu, (\omega - \psi)/2)$$

$$\mathbf{v}_3 \equiv (0, -r\tau\nu, (\omega, + \psi)/2),$$

where components are taken with respect to S'. Note that, for particular values of the parameters, the formula for either $\mathbf{v}_2$ or $\mathbf{v}_3$ (but not for both at the same time) may collapse to the null triple; in those cases a nonnull eigenvector can be recovered from the orthogonal complement of the other two.

The same vectors $\mathbf{v}_i$ are expressed, with respect to S, by the triples which are eigenvectors of the matrix

$$M = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \delta & \varepsilon \\ \gamma & \varepsilon & \eta \end{pmatrix},$$

where M represents the conic at infinity of $\mathscr{C}$ with respect to S, and is obtained from the data as in Section 5.

LEMMA 6.1.1. *Let B (resp. B') be the orthogonal matrix whose ith column is the triple of components of $\mathbf{v}_i$ with respect to S (resp. S') divided by its norm. Then the third row of $E = B' \cdot B^{-1} = B' \cdot B^{T}$ is the triple of components of a unit vector of the rotation axis with respect to S.*

Proof. B (resp. B') is the matrix of the change from the ordered basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ to the basis belonging to S (resp. S'). So, since the origin is the same in both reference frames, E is the orthogonal matrix of the change from S to S'. Therefore its third row is the triple of components of a unit vector of the z' axis with respect to S. ■

Note that the entries of E are functions of the data and of r, s, τ, and ν.

LEMMA 6.1.2. *There is an ordering of the eigenvalues* $(\lambda_1, \lambda_2, \lambda_3)$ *of N and an ordering of the eigenvalues* (μ_1, μ_2, μ_3) *of M such that the two triples are proportional. In particular, the eigenvalues of the two matrices have the same multiplicities.*

Proof. Up to multiplication by a nonzero factor, M and N are similar matrices. ■

Since r and s are known factors, the computation of τ and ν is the last step to the solution of the problem. Now, τ and ν can theoretically be obtained by equating two ratios of pairs of eigenvalues λ_i with the corresponding ratios of eigenvalues (μ_i say) of M. In the general case this involves some computational problems; however, there are three classes of quadrics for which essential simplifications occur.

Cones and cylinders of revolution and spheres are commonly used primitives, for which the values of r and s allow the simplification of the square root in ψ. The geometrical counterpart of this algebraic fact is the particularly simple form of the circumscribed cone: a circular cone for the case of spheres (Fig. 5), a pair of planes for cones and cylinders (Fig. 6). Examples of computation will be given, with focal length $f = 1$.

6.2. *Spheres*

PROPOSITION 6.2.1. *If Q is a sphere, then M has a multiple eigenvalue* μ_1 *and a simple one* μ_2. *If* $\mathbf{v}_2$ *is an eigenvector relative to* μ_2 *with positive second component (with respect to S), then*

$$O' - O = \sqrt{s(\mu_1/\mu_2 - 1)}\,\mathbf{v}_2/\|\mathbf{v}_2\|.$$

Proof. Spheres correspond to $r = 1$ and $s < 0$. The matrix N then has two eigenvalues, one of which has multiplicity 2:

$$\lambda_1 = \lambda_3 = -(s + \tau^2 + \nu^2)$$
$$\lambda_2 = -s.$$

The equality $\mu_1/\mu_2 = \lambda_1/\lambda_2$ coming from Lemma 5.1.2 yields

$$\tau^2 + \nu^2 = \|O' - O\|^2 = s(\mu_1/\mu_2 - 1).$$

The eigenspace of $\lambda_1 = \lambda_3$ is the linear closure of $\{(1, 0, 0), (0, \nu, -\tau)\}$. The statement then comes from the fact that the eigenvectors relative to λ_2 are generated by $(0, \tau, \nu)$ (i.e., they are proportional to $O' - O$). ■

EXAMPLE. Let $s = -1$ and the image conic Γ have equation

$$103x^2 + 108z^2 - 12xz - 60x - 40z + 12 = 0;$$

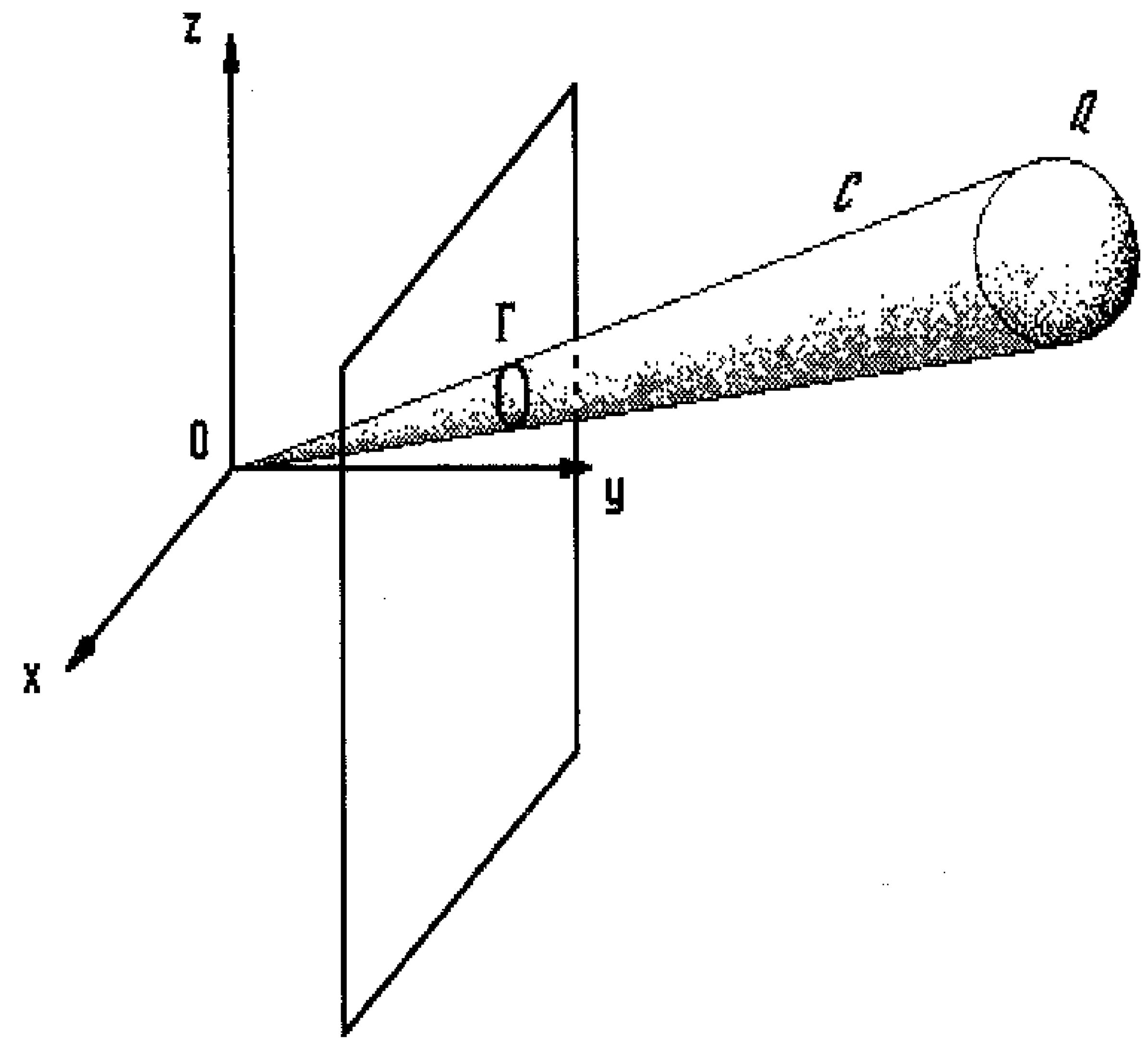

FIGURE 5

the corresponding matrix is

$$M = \begin{pmatrix} 103 & -30 & -6 \\ -30 & 12 & -20 \\ -6 & -20 & 108 \end{pmatrix}$$

and its characteristic equation is

$$\mu^3 - 223\mu^2 + 12320\mu + 12544 = 0.$$

The double eigenvalue is 112 and the simple one is -1; an eigenvector relative to the latter value is (3, 10, 2). This implies, by Proposition 6.2.1, that the coordinates of the sphere centre are

$$\sqrt{113}\,\frac{(3, 10, 2)}{\sqrt{113}} = (3, 10, 2).$$

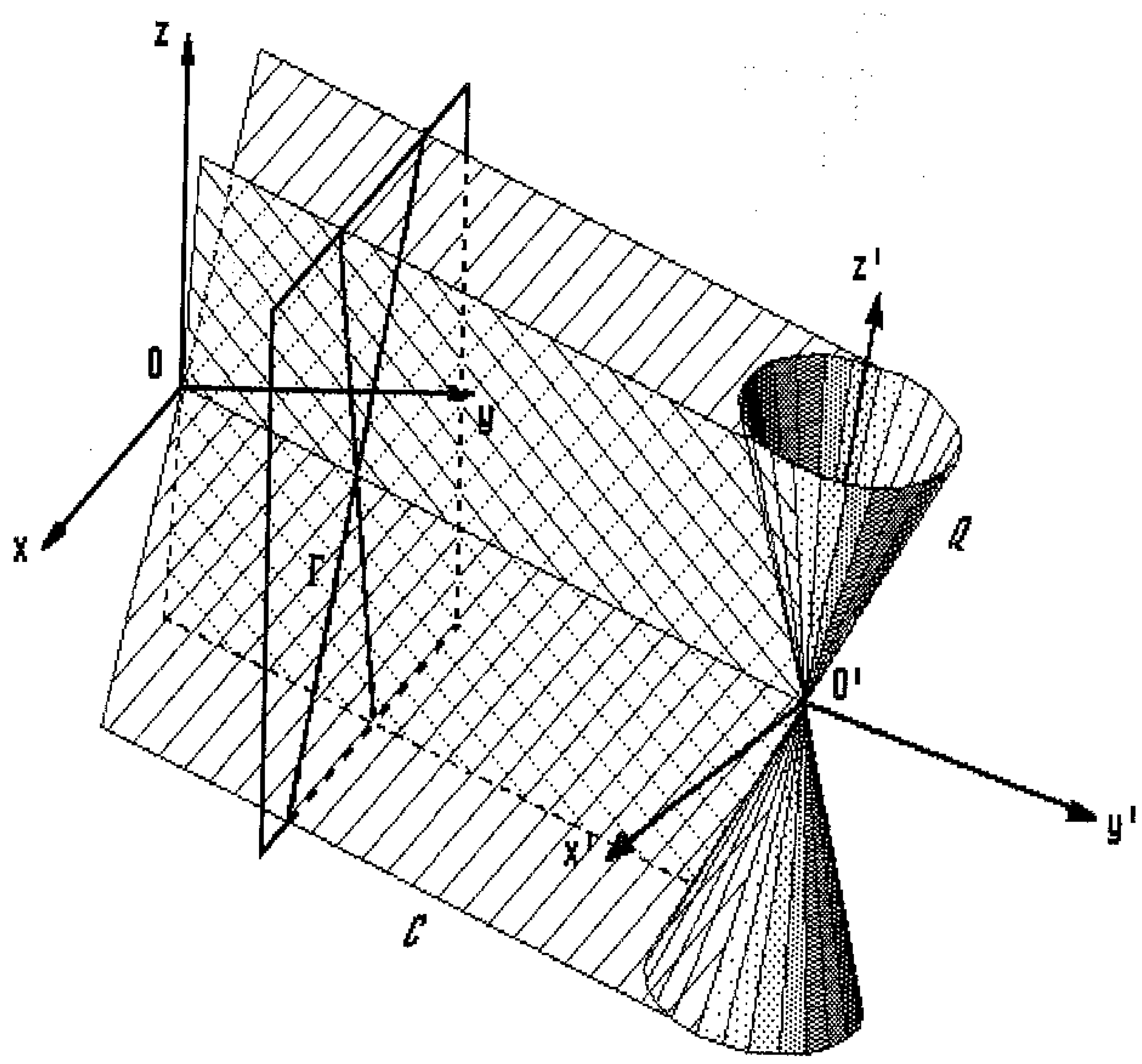

FIGURE 6

6.3. *Cones and Cylinders*

Contours of cones and cylinders project to pairs of straight lines in the image plane, so we shall extract information in detail not from the equation of a conic but from the most natural data, i.e., the equations of the lines. Let

$$u_1x + v_1z + w_1 = 0, \quad u_2x + v_2z + w_2 = 0$$

be the equations of the lines composing Γ. Then the equation of Γ is

$$u_1u_2x^2 + v_1v_2z^2 + (u_1v_2 + u_2v_1)xz + (u_1w_2 + u_2w_1)x + (v_1w_2 + v_2w_1)z + w_1w_2 = 0;$$

call M the corresponding matrix. The vectors (in S components)

$$\mathbf{n}_1 \equiv (u_1, w_1/f, v_1), \quad \mathbf{n}_2 \equiv (u_2, w_2/f, v_2)$$

are normal to the planes through O and the two lines, respectively. One can assume that they are also of unitary norm, by a suitable choice of coefficients in their proportionality class. Set $c = \langle \mathbf{n}_1, \mathbf{n}_2 \rangle$. A lengthy, yet straightforward computation yields the following Lemma.

LEMMA 6.3.1. *For M, $\mathbf{n}_1$, $\mathbf{n}_2$, c as above, the eigenvalues of M are*

$$\mu_1 = 0, \quad \mu_2 = (c + 1)/2, \quad \mu_3 = (c - 1)/2$$

and eigenvectors corresponding to μ_2 and μ_3 respectively are

$$\mathbf{n}_1 + \mathbf{n}_2, \quad \mathbf{n}_2 - \mathbf{n}_1.$$

A third vector forming a triple of mutually orthogonal vectors is then $\mathbf{n}_1 \wedge \mathbf{n}_2$.

Remark 6.3.1. Note that the indices of the eigenvalues μ_i of the preceding lemma do not refer to the analogous indices of the eigenvalues λ_i of the matrix N.

One can consider the image plane as split into parts by the pair of lines; there are two cases, which both can occur for cones as well as for cylinders: (1) the lines are parallel, then we call "internal part" the strip bounded by the lines; (2) the lines are incident, then the "internal part" is that union of two opposite angles, delimited by the lines, which contains the projection of the quadric.

Analogously, the two planes forming $\mathscr{C}$ split the space into two parts (each the union of two opposite dihedra), of which only one, the "internal part," contains the quadric itself. It is always possible to initially choose the sign of the coefficients, so that the vector pointing to the internal part is exactly $\mathbf{n}_1 + \mathbf{n}_2$; assume that choice.

Cones. For cones, one has $r < 0$ and $s = 0$. Note that $-r$ is the tangent of the angle ϑ formed by the rotation axis and any generatrix of the cone Q. Therefore, a necessary and sufficient condition for the vantage point O to be exterior to Q (so that Q itself can be "seen") is that $\tau^2 > -r\nu^2$. Moreover, the convex angle between $\mathbf{n}_1$ and $-\mathbf{n}_2$ is greater than or equal to 2ϑ; therefore,

$$c = \cos \widehat{\mathbf{n}_1\mathbf{n}_2} = \cos (\pi - \widehat{\mathbf{n}_1(-\mathbf{n}_2)}) = -\cos \widehat{\mathbf{n}_1(-\mathbf{n}_2)} \geq -\cos 2\vartheta = (r + 1)/(r - 1).$$

PROPOSITION 6.3.1. *For a cone Q of vertex O', with r, $\mathbf{n}_1$, $\mathbf{n}_2$ as above, $O' - O$ is proportional to $\mathbf{n}_1 \wedge \mathbf{n}_2$. A free vector of the rotation axis of Q is*

$$\rho \sqrt{(c(r-1) - r - 1)/(c-1)}\, \mathbf{n}_1 \wedge \mathbf{n}_2 + \sqrt{-r}\, (\mathbf{n}_1 + \mathbf{n}_2),$$

where ρ is either $+1$ or -1.

Proof. The vertex O' lies on the intersection line of the two planes forming $\mathscr{C}$; on the other hand, $\mathbf{n}_1 \wedge \mathbf{n}_2$ is a free vector of this line, so $O' - O$ is proportional to it.

In order to determine the direction of the rotation axis, one can find eigenvectors of N. The eigenvalues of N are

$$\lambda_1 = -(r\nu^2 + \tau^2), \quad \lambda_2 = -r(\nu^2 + \tau^2), \quad \lambda_3 = 0$$

with eigenvectors, respectively (in S' components),

$$\mathbf{v}_1 \equiv (1, 0, 0), \quad \mathbf{v}_2 \equiv (0, \nu, -\tau), \quad \mathbf{v}_3 \equiv (0, \tau, \nu).$$

By comparing these with the orthogonal triple of Lemma 6.3.1, one obtains that λ_1 matches μ_3 and λ_2 matches μ_2. In fact $\mathbf{n}_1 \wedge \mathbf{n}_2$ is directed as $O' - O = -\mathbf{v}_3$, so its S' components are proportional to $(0, \tau, \nu)$. Thus, by Lemma 6.1.2,

$$\frac{\mu_2}{\mu_3} = \frac{-r(\nu^2 + \tau^2)}{-(r\nu^2 + \tau^2)}.$$

(Note that the ratio is negative, as necessary for $\mathscr{C}$ to be built of two real planes, because of the inequality $\tau^2 > -r\nu^2$ above). Consequently, from the values of μ_2 and μ_3 computed as in Lemma 6.3.1,

$$\left(\frac{\nu}{\tau}\right)^2 = \frac{(r-1)c - (r+1)}{2r}.$$

The ratio at the right-hand side is actually positive because of the previously seen inequality $c \geq (r + 1)/(r - 1)$.

Direct computation then shows that the matrix B of Lemma 6.1.1 has columns formed respectively by the triples of components of $(\mathbf{n}_2 - \mathbf{n}_1)/\sqrt{2(1 - c)}$, $(\mathbf{n}_1 + \mathbf{n}_2)/\sqrt{2(1 + c)}$, $\mathbf{n}_1 \wedge \mathbf{n}_2/\sqrt{1 - c^2}$, and

$$B' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho\sqrt{\dfrac{(r-1)c-(r+1)}{(r-1)(c+1)}} & -\sqrt{\dfrac{2r}{(r-1)(c+1)}} \\ 0 & \sqrt{\dfrac{2r}{(r-1)(c+1)}} & \rho\sqrt{\dfrac{(r-1)c-(r+1)}{(r-1)(c+1)}} \end{pmatrix},$$

where $\rho = \mathrm{sign}(\nu)$. From Lemma 6.1.1 one then obtains a unit vector of the rotation axis, in S components, by taking the third row of $B' \cdot \mathrm{B}^{\mathrm{T}}$; this turns out to be

$$\frac{1}{(c+1)\sqrt{1-r}}\left(\rho\sqrt{\frac{c(r-1)-r-1}{c-1}}\,\mathbf{n}_1 \wedge \mathbf{n}_2 + \sqrt{-r}\,(\mathbf{n}_1 + \mathbf{n}_2)\right)$$

and the result comes from neglecting the normalization factor. ■

Remark 6.3.1. It is not possible to determine the distance $O\,O'$ from these data only; this can be seen also by considering that the "circumscribed cone" is the same pair of planes no matter how the vantage point is displaced along the line $O\,O'$. Moreover, there is a finite ambiguity in that it is not possible to know the sign of ν from data only. This again is consistent with the physical situation. However, concrete models consist of truncated cones, so the indeterminacies can be solved by using segment measures and by integrating with the methods of Section 5.

EXAMPLE. Given a cone with $r = -2$; image lines, forming the contour of the cone projection, of equations

$$577x - (914 - 500\sqrt{3})z - (125 + 20\sqrt{3}) = 0,$$
$$577x - (914 + 500\sqrt{3})z - (125 - 20\sqrt{3}) = 0,$$

respectively, then

$$\mathbf{n}_1 = \frac{(577, -125 - 20\sqrt{3}, -914 + 500\sqrt{3})}{505\sqrt{6} - 450\sqrt{2}},$$
$$\mathbf{n}_2 = \frac{(577, -125 + 20\sqrt{3}, -914 - 500\sqrt{3})}{505\sqrt{6} + 450\sqrt{2}},$$

whence

$$\mathbf{n}_1 + \mathbf{n}_2 = \frac{\sqrt{6}}{195}(101, -25, -82), \quad \mathbf{n}_1 \wedge \mathbf{n}_2 = \frac{4\sqrt{3}}{195}(7, 25, 1)$$

so that, by Proposition 6.3.1, the vertex O' lies on the line passing through O, with directional vector (7, 25, 1). The formula of the same proposition yields

$$\frac{18\sqrt{3}}{65}(4, 0, -3), \quad \frac{2\sqrt{3}}{195}(94, -50, -83)$$

as the two possible directional vectors of the rotation axis.

Cylinders. Cylinders have $r = 0$ and $s < 0$. Because of their particular symmetry, O' can be chosen anywhere on the rotation axis without any change in the canonical equation, so one may assume $\nu = 0$, i.e., $O \equiv (0, \tau, 0)$ with $\tau < 0$ (S' coordinates). O' is then the orthogonal projection of O on the rotation axis.

PROPOSITION 6.3.2. *For a cylinder Q with s, $\mathbf{n}_1$, $\mathbf{n}_2$, O' as above, one has*

$$O' - O = \frac{\sqrt{-s}}{c+1}(\mathbf{n}_1 + \mathbf{n}_2).$$

A free vector of the rotation axis of Q is $\mathbf{n}_1 \wedge \mathbf{n}_2$.

Proof. The second part of the statement comes from the fact that the generatrices of Q in which $\mathscr{C}$ is tangent are parallel to each other: they must then be parallel to the intersection line of the two planes forming $\mathscr{C}$, and this has $\mathbf{n}_1 \wedge \mathbf{n}_2$ as a free vector. In order to prove the first part, observe that $O' - O$ is proportional (by a positive factor) to $\mathbf{n}_1 + \mathbf{n}_2$ by the conventions on the latter vector and on O'. It remains to compute the scalar factor. The matrix N has eigenvalues

$$\lambda_1 = -(s + \tau^2), \quad \lambda_2 = -s, \quad \lambda_3 = 0$$

and eigenvectors, respectively (in S' components),

$$\mathbf{v}_1 \equiv (1, 0, 0), \quad \mathbf{v}_2 \equiv (0, 1, 0), \quad \mathbf{v}_3 \equiv (0, 0, 1).$$

With a similar argument as for the cones ($O' - O$ is proportional both to $\mathbf{n}_1 + \mathbf{n}_2$ and to $\mathbf{v}_2$), one can match μ_2 with λ_2 and μ_3 with λ_1. By applying Lemma 5.1.2 one obtains

$$\tau = -\sqrt{\frac{-2s}{c+1}};$$

finally,

$$O' - O = -\tau(\mathbf{n}_1 + \mathbf{n}_2)/\|\mathbf{n}_1 + \mathbf{n}_2\| = \frac{\sqrt{-s}}{c+1}(\mathbf{n}_1 + \mathbf{n}_2). \quad ■$$

As an example, if the previously written image lines (used above as contours for the cone) are now interpreted as coming from the projection of a cylinder with $s =$