Persistent homology and Mayer-Vietoris formulas

Barbara Di Fabio

Department of Mathematics, University of Bologna

barbara.difabio@unibo.it

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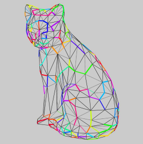
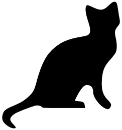
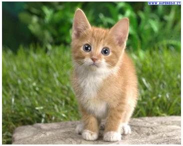
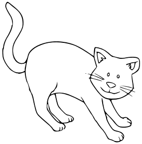
Shape recognition

- It is the task of finding a given object in a scene/dataset/ image/video sequence.
- Humans recognize a multitude of objects in images with little effort, despite variations in
  - different view points,
  - different sizes / scale
  - translations and rotations
  - partial obstructions from view.
- Increasing interest in automatic shape recognition
- This task is still a challenge for computer vision systems in general.

What is the shape of an object?

- Tentative definitions are generally based on *observers' perceptions*.
- Dependence on observers implies large subjectivity
  - changes due to object orientation and distance from the object
  - changes due to light conditions
- Human judgments focus on *persistent perceptions*
  - Non-persistent properties can be considered as noise.
  - Only *stable perceptions* concur to give a shape to objects.

The shape comparison problem

Similar shapes with respect to what?

How to model observations and perceptions?

- An observation can be modeled as a topological space X.
  - it depends on what the observer is observing.
- Observer's perceptions can be modeled as a function $f : X \to \mathbb{R}^n$.
  - it depends on the shape property the observer is perceiving.

Thus to model a shape we consider pairs (X, f) where

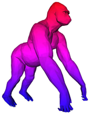
- X is a topological space
- $f : X \to \mathbb{R}$ is a (continuous) function.

$\dots$

X $\quad (X, \varphi)$ $\quad (X, \varsigma)$ $\quad (X, \xi)$ $\quad \dots$

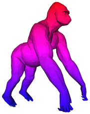
Comparing shapes

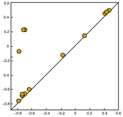
- How can we compare two pairs (X,φ), (Y,ψ)?

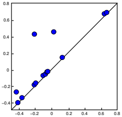


$$d\left((X,\varphi) , (Y,\psi) \right) = ?$$

- **Persistence** allows us to describe such a pair by means of suitable shape descriptors (persistence diagrams).

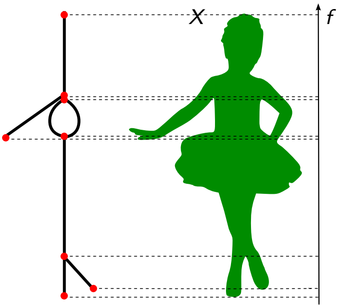




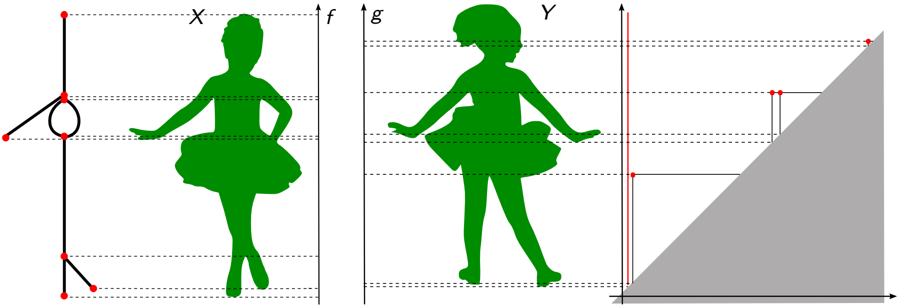


- Instead of comparing shapes, we can compare shape descriptors.

Reeb graphs and persistence diagrams



Reeb graphs and persistence diagrams



Size functions

- For every $u \in \mathbb{R}$, let us denote by X_u the **lower level set** $\{p \in X : f(p) \leq u\}$.

- Let Δ^+ be the open half plane $\{(u,v) \in \mathbb{R}^2 : u < v\}$.

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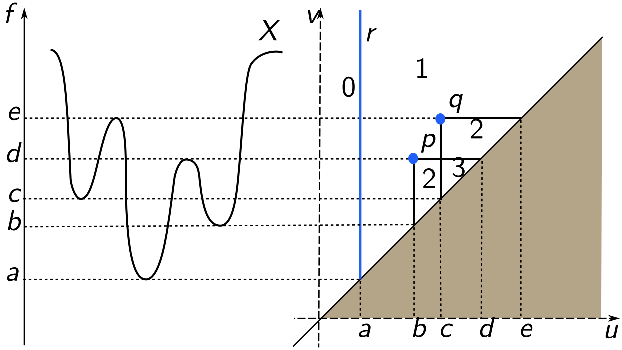
- Let Δ^+ be the open half plane $\{(u, v) \in \mathbb{R}^2 : u < v\}$.

The **size function** associated with the pair (X, f) is the function $D(X, f) : \Delta^+ \to \mathbb{N}$ that takes each $(u, v) \in \Delta^+$ to the number of connected components in X_v containing at least one point of X_u.

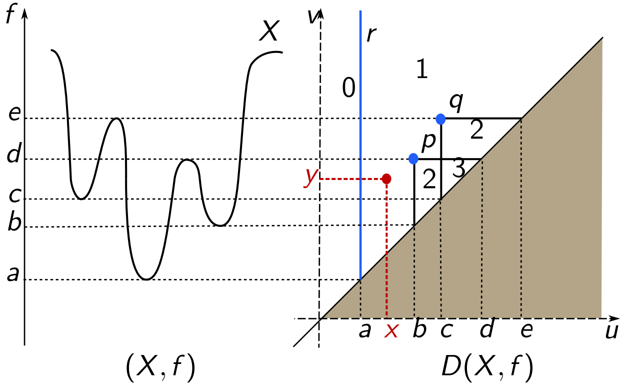
$$D(X, f)(u, v) = \dim H_0^{u,v}(X) = \dim \mathrm{im}(H_0(X_u) \to H_0(X_v)).$$

An example



(X, f) $D(X, f)$

An example



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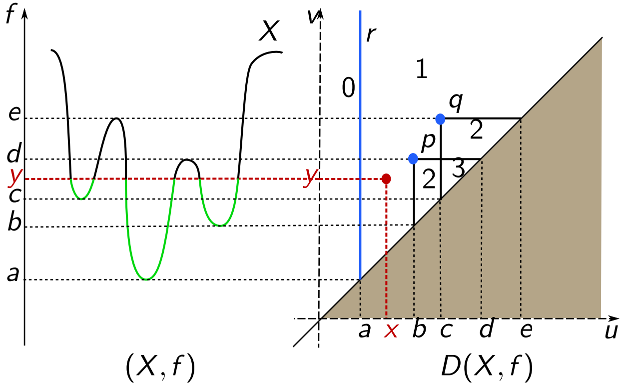

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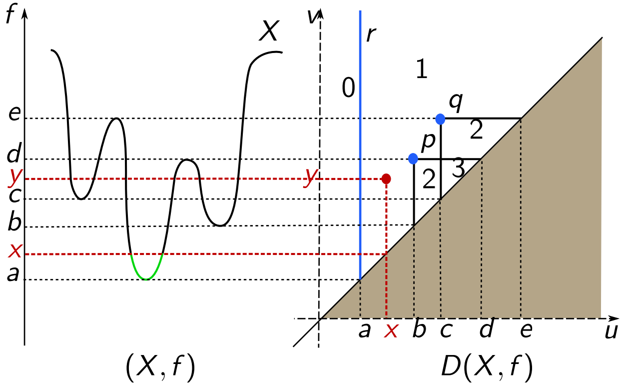
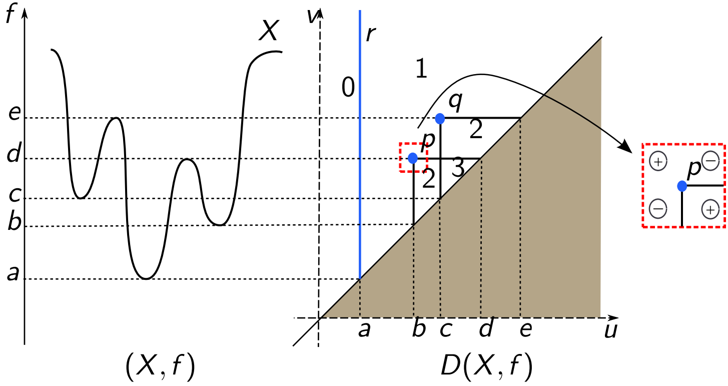


Points of a persistence diagram



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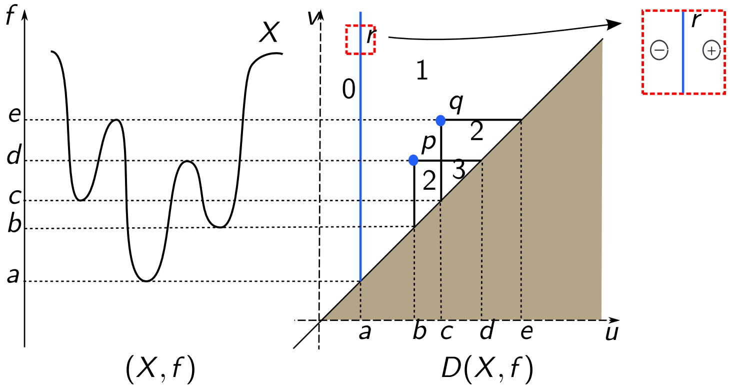
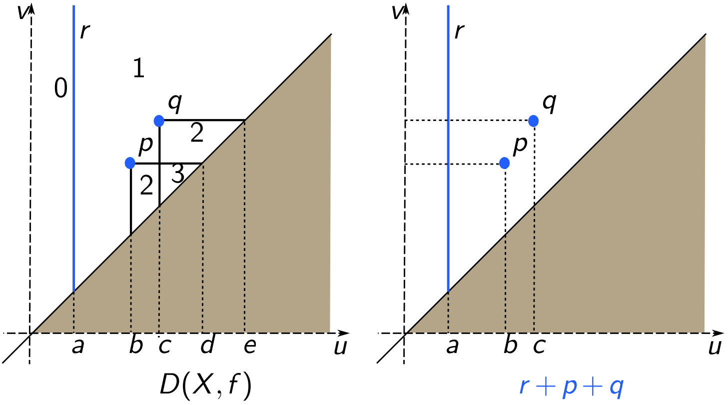
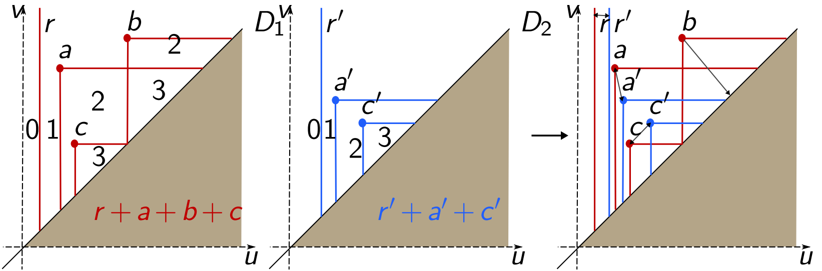


Points of a persistence diagram



$D(X, f)$ is equivalent to $r + p + q$.

Hausdorff distance



Let D_1 and D_2 be two persistence diagrams with the same number of points at infinity. Let A_1 (resp. A_2) be the set of all points for D_1 (resp. D_2), augmented by adding a countable infinity of points of the diagonal Δ. The **Hausdorff distance** between D_1 and D_2 is given by

$$d_H(D_1, D_2) = \max\{\max_{p \in A_1} \min_{q \in A_2} \|p - q\|_\infty, \max_{q \in A_2} \min_{p \in A_1} \|q - p\|_\infty\}.$$

Persistent homology groups

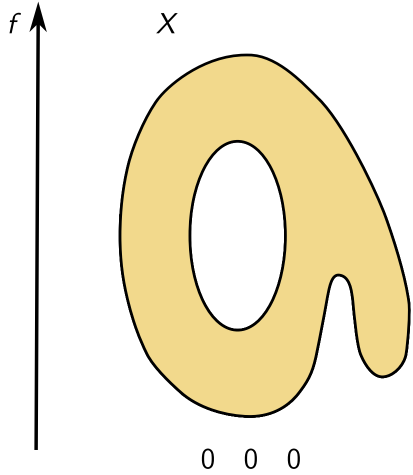
- For every $u \in \mathbb{R}$, let us denote by X_u the **lower level set** $\{p \in X : f(p) \leq u\}$.

Persistent homology groups

- For every $u \in \mathbb{R}$, let us denote by X_u the **lower level set** $\{p \in X : f(p) \leq u\}$.

Given a pair (X, φ), and $u, v \in \mathbb{R}$, with $u < v$, we shall denote by $\imath^{u,v}$ the inclusion of X_u into X_v. This mapping induces a homomorphism of homology groups $\imath_k^{u,v} : H_k(X_u) \to H_k(X_v)$ for each integer $k \geq 0$. The k**th persistent homology group** $H_k^{u,v}(X, f)$ is the image of the homomorphism $\imath_k^{u,v} : H_k(X_u) \to H_k(X_v)$, that is $H_k^{u,v}(X, f) = \mathrm{im}\,\imath_k^{u,v}$.

Ordinary persistence



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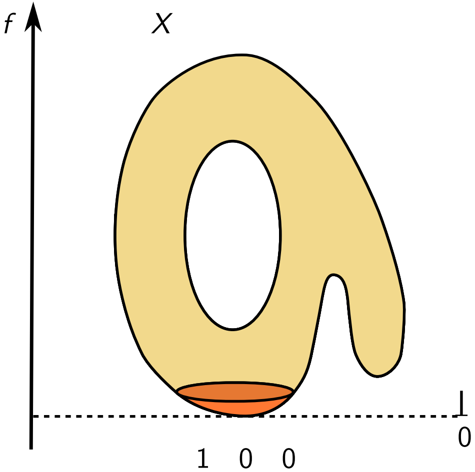

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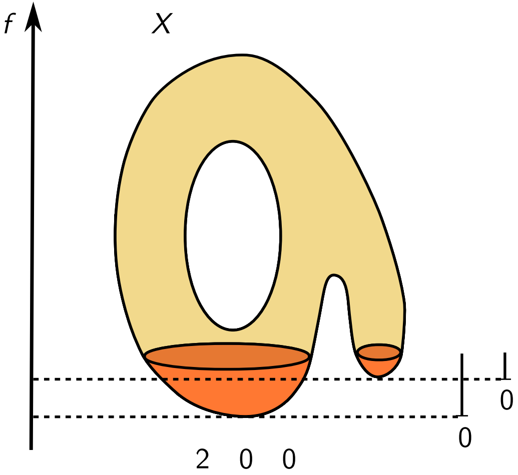

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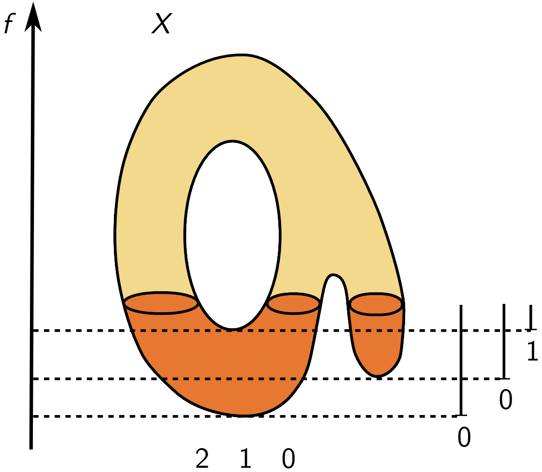

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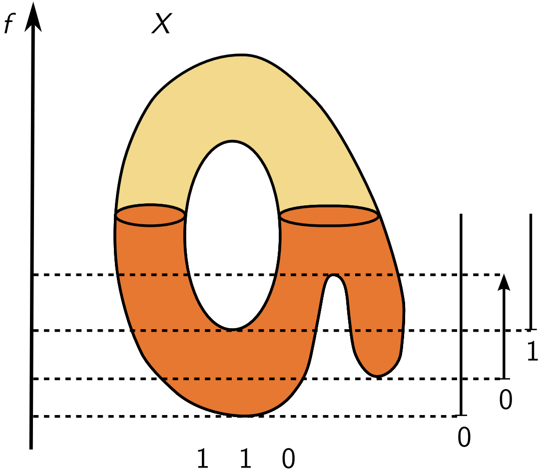
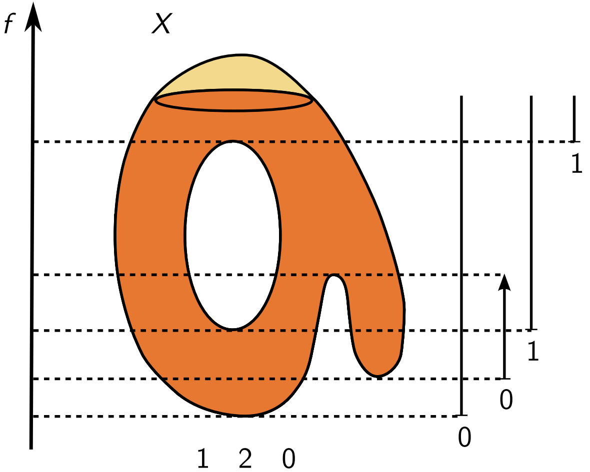
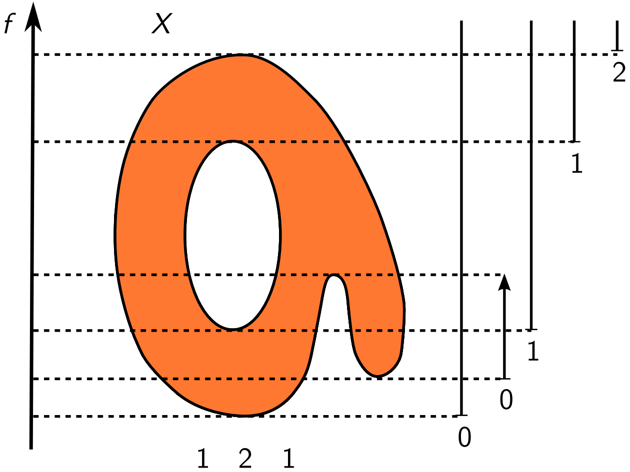


Ordinary persistence

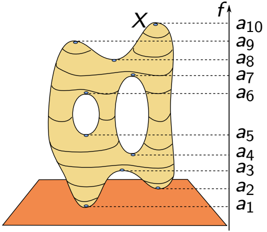


Ordinary persistence



Sub- and super-level set filtrations

- Let X be a triangulable subspace of $\mathbb{R}^n$.
- Let $f : X \to \mathbb{R}$ be a continuous function with a finite number of homological critical values $a_1 < a_2 < \ldots < a_r$.
- For $s_0 < s_1 < \ldots < s_r$ such that $s_{i-1} < a_i < s_i$ we have
    - sub-level sets $X_i = f^{-1}((-\infty, s_i])$,
    - super-level sets $X^i = f^{-1}([s_i, +\infty))$.



- We obtain two sequences of inclusions:

$$\emptyset = X_0 \hookrightarrow \ldots \hookrightarrow X_i \hookrightarrow \ldots \hookrightarrow X_j \hookrightarrow \ldots X_r = X,$$

$$(X, \emptyset) = (X, X^r) \hookrightarrow \ldots \hookrightarrow (X, X^j) \hookrightarrow \ldots \hookrightarrow (X, X^i) \hookrightarrow \ldots (X, X^0) = (X, X).$$

Persistent homology

- Apply the homology functor:

$$\begin{array}{ccccc} H_k(X_0) \to \dots \to H_k(X_i) \to \dots \to H_k(X_j) \to \dots \to & H_k(X) & & & 0 \\ \shortparallel & \shortparallel & & & \shortparallel \\ 0 & H_k(X,X^r) & \to \dots \to H_k(X,X^j) \to \dots \to H_k(X,X^i) \to \dots \to & & H_k(X,X^0) \end{array}$$

Persistent homology

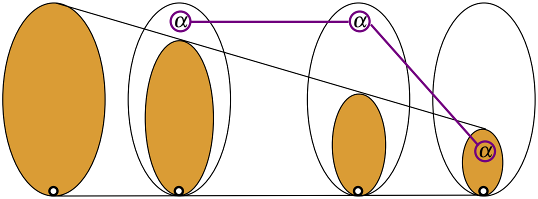
- Apply the homology functor:

$$H_k(X_0) \to \dots \to H_k(X_i) \to \dots \to H_k(X_j) \to \dots \to H_k(X) \qquad 0$$

$$\| \qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$

$$0 \qquad H_k(X, X^r) \to \dots \to H_k(X, X^j) \to \dots \to H_k(X, X^i) \to \dots \to H_k(X, X^0)$$

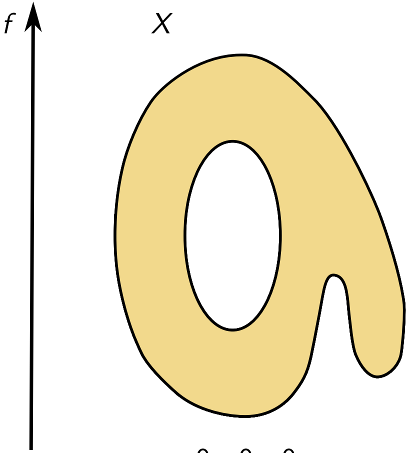
- analyze the scale at which a homological feature is created, and when it is annihilated along the filtration:



$$\dots \to H_k(X_{i-1}) \to H_k(X_i) \to \dots \to H_k(X_{j-1}) \to H_k(X_j) \to \dots$$

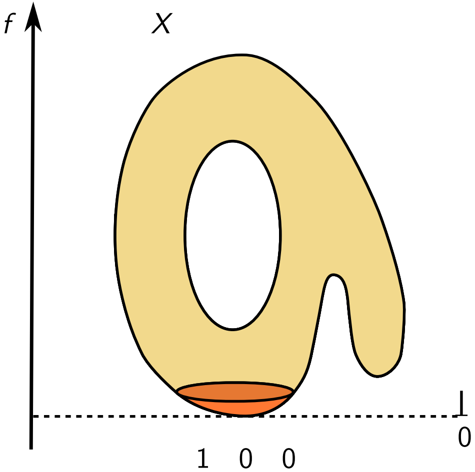
α is born at X_i and dies at X_j

Ordinary persistence



0 0 0

Ordinary persistence



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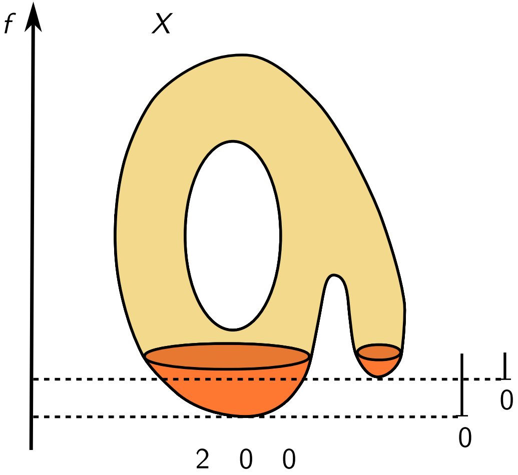

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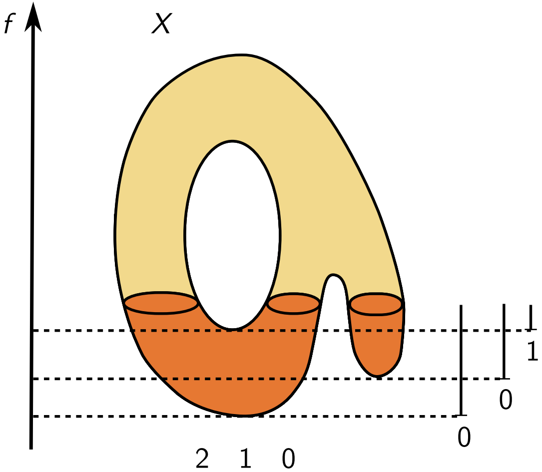

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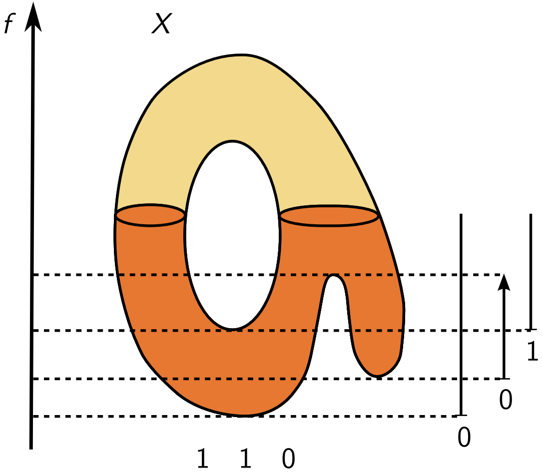

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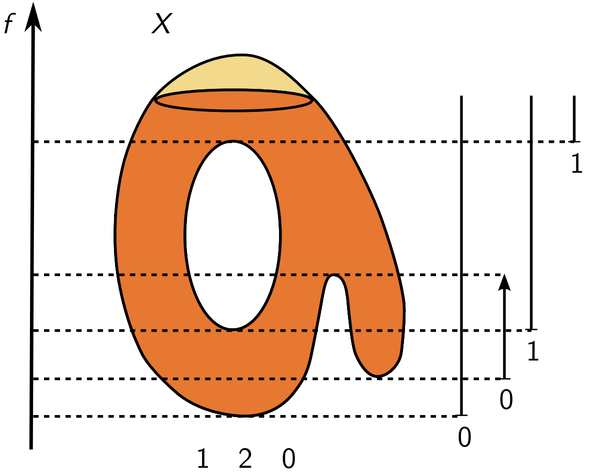

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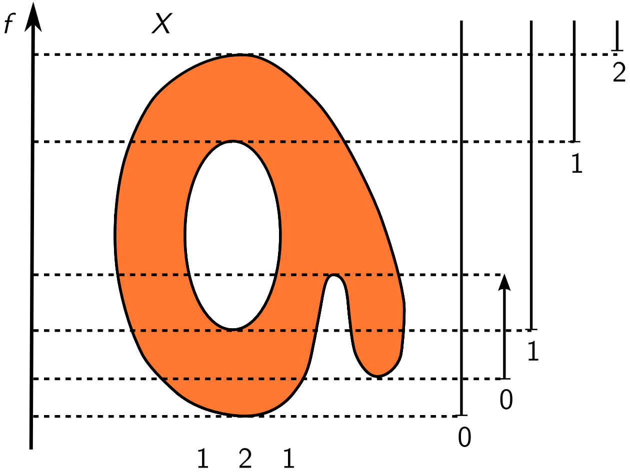
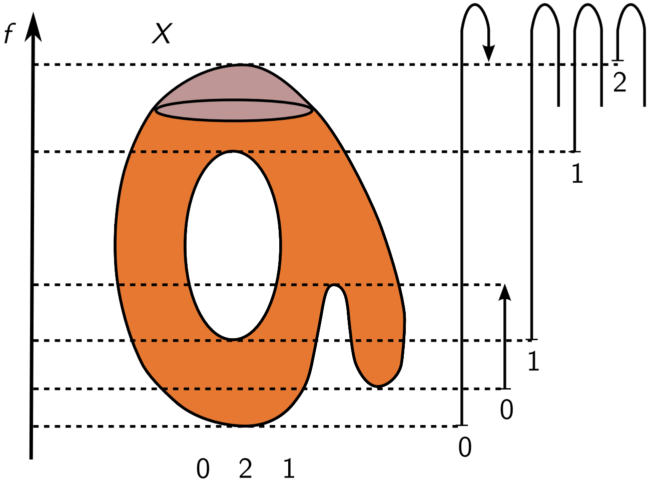


Extended persistence



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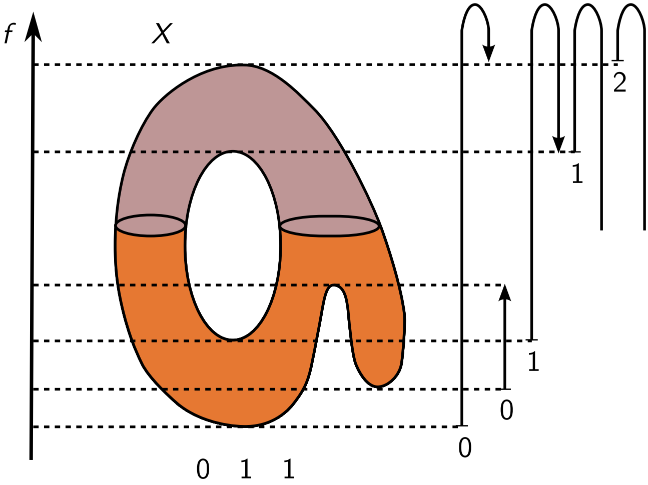
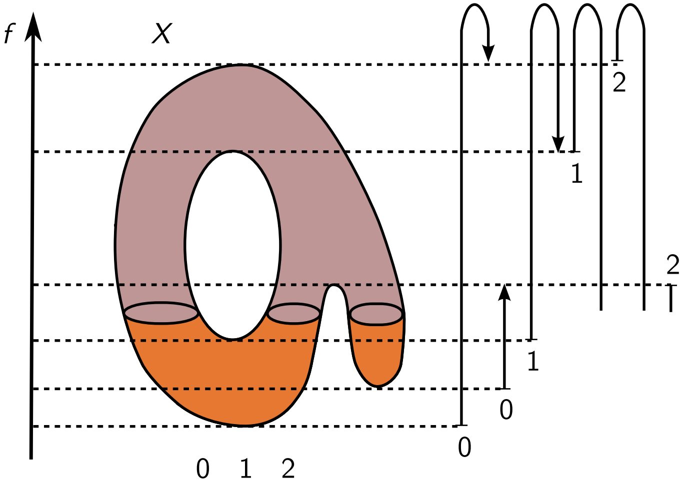


Extended persistence



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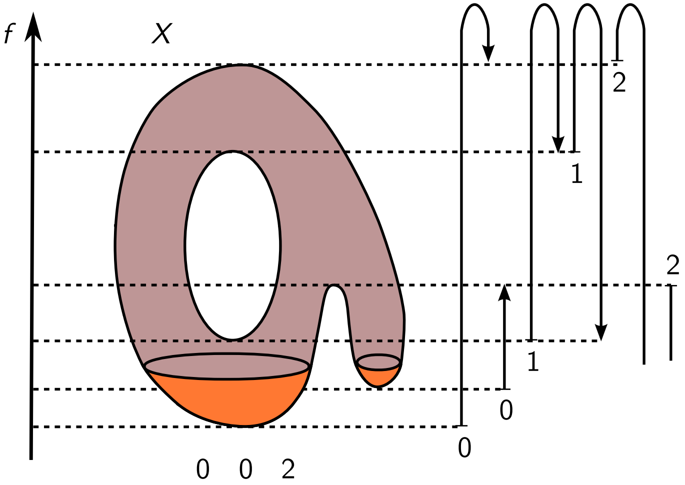

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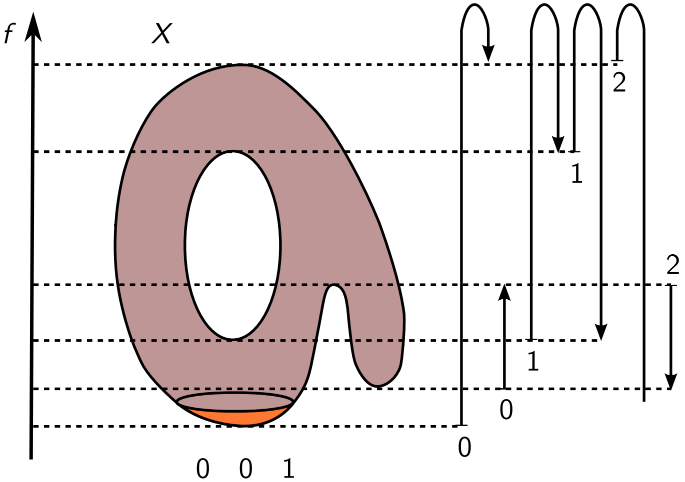

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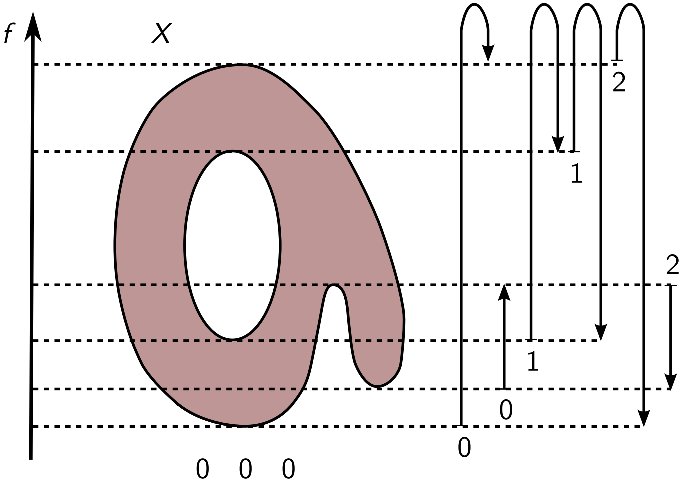
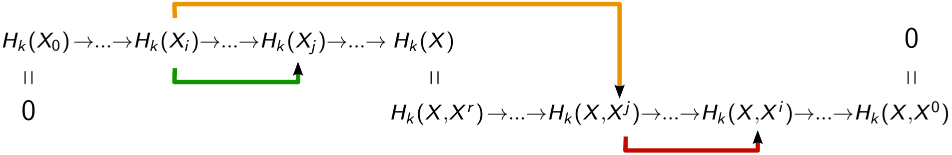


Persistent homology

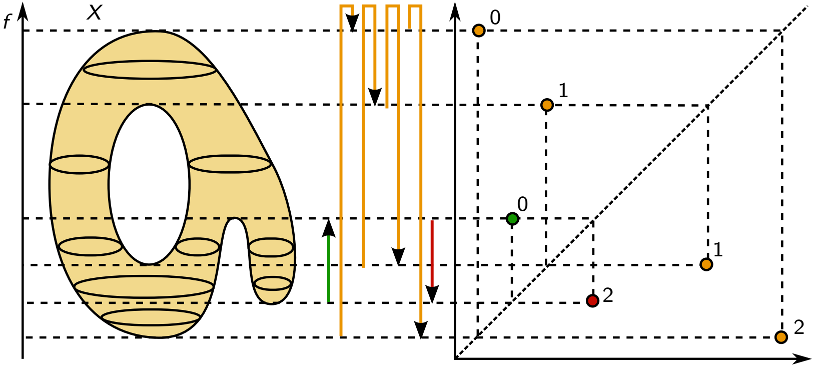
Define the ordinary, relative ed extended persistent Betti numbers as follows:



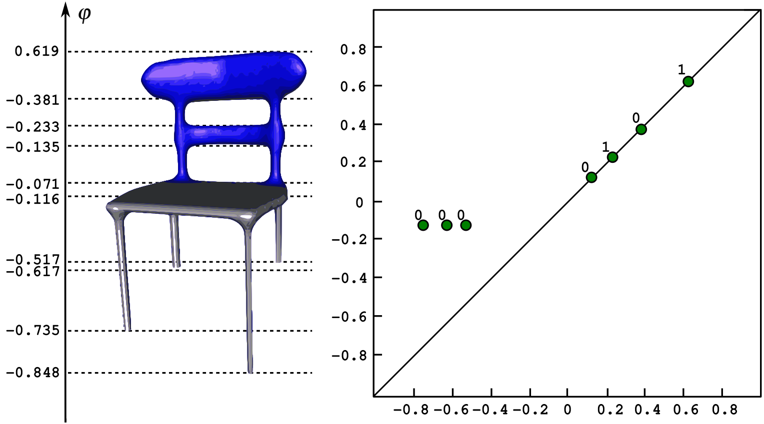
- $Ord_k^{i,j}(X) := \mathrm{rk} H_k(X_i \hookrightarrow X_j)$, per $i < j$
- $Rel_k^{j,i}(X) := \mathrm{rk} H_k((X, X^j) \hookrightarrow (X, X^i))$, per $i \leq j$
- $Ext_k^{i,j}(X) := \mathrm{rk} H_k(X_i \hookrightarrow (X, X^j))$.

Persistence diagrams

$D^{Ord}(X, f)$, $D^{Rel}(X, f)$, $D^{Ext}(X, f)$

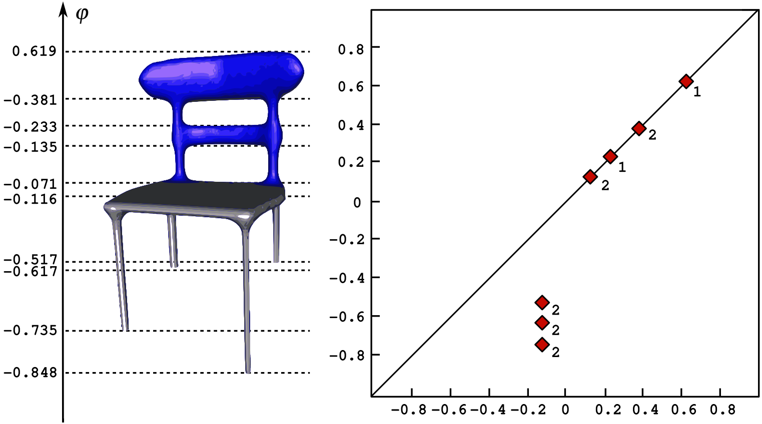


Persistence diagrams



$$D(X,\varphi) = D^{Ord}(X,\varphi) \cup D^{Rel}(X,\varphi) \cup D^{Ext}(X,\varphi)$$

Persistence diagrams



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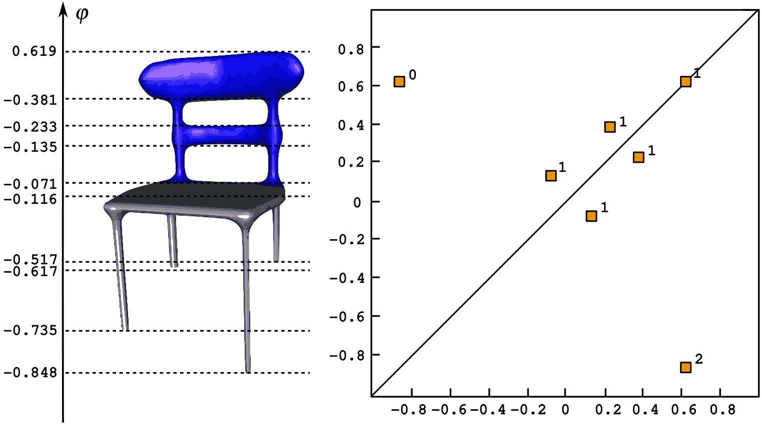


$$D(X,\varphi)=D^{Ord}(X,\varphi)\cup D^{Rel}(X,\varphi)\cup D^{Ext}(X,\varphi)$$

Shape recognition from partial information

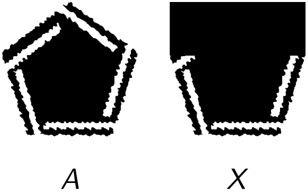
It is the task of detecting sub-parts similarities between objects possibly having different overall shape.

- Recognition/classification of 2D content has to handle occlusions

- 3D content is based on the full representation of the shape, nevertheless recognition of partial similarities may help in the handling of non-rigid and articulated objects.

Recognition under occlusion: how do we model the problem?

- A is the object of interest;
- B the occluding pattern;
- $X = A \cup B$ the occluded object;
- $f : X \to \mathbb{R}$ the measuring function.



Assuming X, A, B compact, locally connected Hausdorff spaces, what is the relation among

$$D(X, f), D(A, f|_A), D(B, f|_B)?$$

Classical Mayer-Vietoris Formula

Given a triad (X, A, B) with $X = A \cup B$, a Mayer-Vietoris formula is a relation between the Betti numbers of X, A, B, and $C = A \cap B$:

$$\mathrm{rk} H_k(X) = \mathrm{rk} H_k(A) + \mathrm{rk} H_k(B) - \mathrm{rk} H_k(C) + \mathrm{rk}\delta_k + \mathrm{rk}\delta_{k-1}.$$

It is obtained from the Mayer-Vietoris sequence

$$\cdots \to H_{k+1}(X) \stackrel{\delta_k}{\to} H_k(C) \to H_k(A) \oplus H_k(B) \to H_k(X) \stackrel{\delta_{k-1}}{\to} \cdots,$$

when this is exact.

Mayer-Vietoris formula for persistent homology

We use

- Čech homology
- homology coefficients in a vector space

and assume that

- $X = A \cup B$, A, B, $C = A \cap B$ are triangulable subspaces of some $\mathbb{R}^n$,
- the homology groups of the sub- and super-level sets of f, $f|_A$, $f|_B$, $f|_C$ are finitely generated.

Mayer-Vietoris formula for ordinary persistence

- For every $u < v \in \mathbb{R}$, and $k \in \mathbb{Z}$

$$Ord_k^{u,v}(X) = Ord_k^{u,v}(A) + Ord_k^{u,v}(B) - Ord_k^{u,v}(C) + \mathrm{rk}\,\delta'_k - \mathrm{rk}\,\delta''_k + \mathrm{rk}\,\delta_{k-1}$$

with δ'_k, δ''_k, and δ_{k-1} as in the following diagram:

$$\begin{array}{ccccccc}
H_{k+1}(X_u) & \overset{\delta_k}{\rightarrow} & H_k(C_u) & \rightarrow & H_k(A_u)\oplus H_k(B_u) & \rightarrow & H_k(X_u) \\
\downarrow h_{k+1} & & \downarrow f_k & & \downarrow g_k & & \downarrow h_k \\
H_{k+1}(X_v) & \overset{\delta'_k}{\rightarrow} & H_k(C_v) & \rightarrow & H_k(A_v)\oplus H_k(B_v) & \rightarrow & H_k(X_v) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{k+1}(X_v,X_u) & \overset{\delta''_k}{\rightarrow} & H_k(C_v,C_u) & \rightarrow & H_k(A_v,A_u)\oplus H_k(B_v,B_u) & \rightarrow & H_k(X_v,X_u) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_k(X_u) & \overset{\delta_{k-1}}{\rightarrow} & H_{k-1}(C_u) & \rightarrow & H_{k-1}(A_u)\oplus H_{k-1}(B_u) & \rightarrow & H_{k-1}(X_u)
\end{array}$$

Mayer-Vietoris formula for relative persistence

- For every $v \leq u \in \mathbb{R}$, and $k \in \mathbb{Z}$

$$Rel_k^{u,v}(X) = Rel_k^{u,v}(A) + Rel_k^{u,v}(B) - Rel_k^{u,v}(C) + \mathrm{rk}\bar{\delta}'_k - \mathrm{rk}\bar{\delta}''_{k-1} + \mathrm{rk}\bar{\delta}_{k-1}$$

with $\bar{\delta}'_k$, $\bar{\delta}''_{k-1}$, and $\bar{\delta}_{k-1}$ as in the following diagram:

$$\begin{array}{ccccccc}
H_{k+1}(X,X^u) & \xrightarrow{\bar{\delta}_k} & H_k(C,C^u) & \rightarrow & H_k(A,A^u)\oplus H_k(B,B^u) & \rightarrow & H_k(X,X^u) \\
\downarrow \bar{h}_{k+1} & & \downarrow \bar{f}_k & & \downarrow \bar{g}_k & & \downarrow \bar{h}_k \\
H_{k+1}(X,X^v) & \xrightarrow{\bar{\delta}'_k} & H_k(C,C^v) & \rightarrow & H_k(A,A^v)\oplus H_k(B,B^v) & \rightarrow & H_k(X,X^v) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_k(X^v,X^u) & \xrightarrow{\bar{\delta}''_{k-1}} & H_{k-1}(C^v,C^u) & \rightarrow & H_{k-1}(A^v,A^u)\oplus H_{k-1}(B^v,B^u) & \rightarrow & H_{k-1}(X^v,X^u) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_k(X,X^u) & \xrightarrow{\bar{\delta}_{k-1}} & H_{k-1}(C,C^u) & \rightarrow & H_{k-1}(A,A^u)\oplus H_{k-1}(B,B^u) & \rightarrow & H_{k-1}(X,X^u)
\end{array}$$

Mayer-Vietoris formula for extended persistence ($u<v$)

- For every $u < v \in \mathbb{R}$, and $k \in \mathbb{Z}$

$$Ext_k^{u,v}(X) = Ext_k^{u,v}(A) + Ext_k^{u,v}(B) - Ext_k^{u,v}(C) + \mathrm{rk}\hat{\delta}'_k - \mathrm{rk}\hat{\delta}''_k + \mathrm{rk}\hat{\delta}_{k-1}$$

with $\hat{\delta}'_k$, $\hat{\delta}''_k$, and $\hat{\delta}_{k-1}$ as in the following diagram:

$$\begin{array}{ccccccc}
H_{k+1}(X_u) & \stackrel{\hat{\delta}_k}{\rightarrow} & H_k(C_u) & \rightarrow & H_k(A_u)\oplus H_k(B_u) & \rightarrow & H_k(X_u) \\
\downarrow \hat{h}_{k+1} & & \downarrow \hat{f}_k & & \downarrow \hat{g}_k & & \downarrow \hat{h}_k \\
H_{k+1}(X,X^v) & \stackrel{\hat{\delta}'_k}{\rightarrow} & H_k(C,C^v) & \rightarrow & H_k(A,A^v)\oplus H_k(B,B^v) & \rightarrow & H_k(X,X^v) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{k+1}(X,^{\cup}X_u^v) & \stackrel{\hat{\delta}''_k}{\rightarrow} & H_k(C,^{\cup}C_u^v) & \rightarrow & H_k(A,^{\cup}A_u^v)\oplus H_k(B,^{\cup}B_u^v) & \rightarrow & H_k(X,^{\cup}X_u^v) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_k(X_u) & \stackrel{\hat{\delta}_{k-1}}{\rightarrow} & H_{k-1}(C_u) & \rightarrow & H_{k-1}(A_u)\oplus H_{k-1}(B_u) & \rightarrow & H_{k-1}(X_u)
\end{array}$$

Mayer-Vietoris formula for extended persistence ($v \leq u$)

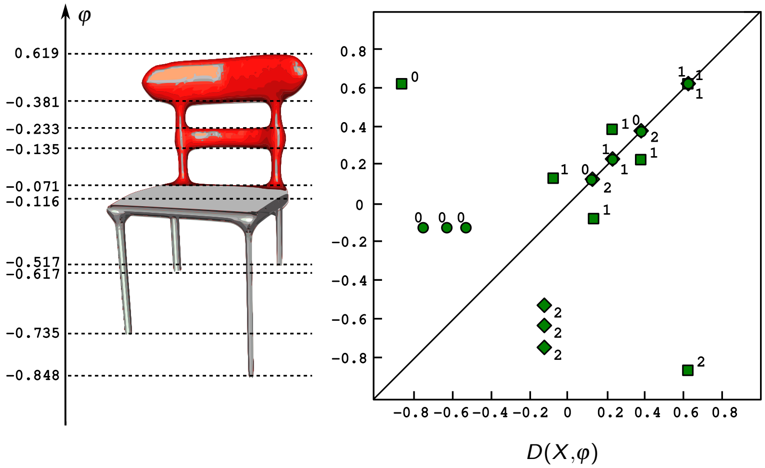
- For every $v \leq u \in \mathbb{R}$, and $k \in \mathbb{Z}$

$$Ext_k^{u,v}(X) = Ext_k^{u,v}(A) + Ext_k^{u,v}(B) - Ext_k^{u,v}(C) + \mathrm{rk}\widetilde{\delta}'_k - \mathrm{rk}\widetilde{\delta}''_{k-1} + \mathrm{rk}\widetilde{\delta}_{k-1}$$

with $\widetilde{\delta}'_k$, $\widetilde{\delta}''_{k-1}$, and $\widetilde{\delta}_{k-1}$ as in the following diagram:

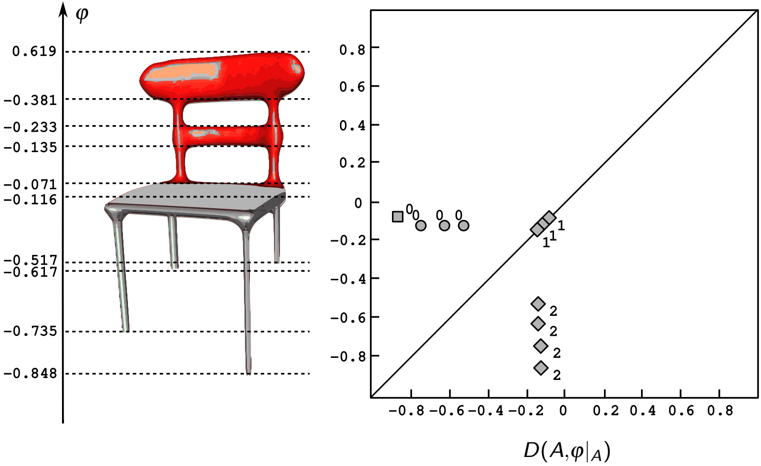
$$\begin{array}{ccccccc}
H_{k+1}(X_u) & \overset{\widetilde{\delta}_k}{\longrightarrow} & H_k(C_u) & \longrightarrow & H_k(A_u) \oplus H_k(B_u) & \longrightarrow & H_k(X_u) \\
\downarrow \widetilde{h}_{k+1} & & \downarrow \widetilde{f}_k & & \downarrow \widetilde{g}_k & & \downarrow \widetilde{h}_k \\
H_{k+1}(X_u, {}^{\cap}X_u^v) & \overset{\widetilde{\delta}'_k}{\longrightarrow} & H_k(C_u, {}^{\cap}C_u^v) & \longrightarrow & H_k(A_u, {}^{\cap}A_u^v) \oplus H_k(B_u, {}^{\cap}B_u^v) & \longrightarrow & H_k(X_u, {}^{\cap}X_u^v) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_k({}^{\cap}X_u^v) & \overset{\widetilde{\delta}''_{k-1}}{\longrightarrow} & H_{k-1}({}^{\cap}C_u^v) & \longrightarrow & H_{k-1}({}^{\cap}A_u^v) \oplus H_{k-1}({}^{\cap}B_u^v) & \longrightarrow & H_{k-1}({}^{\cap}X_u^v) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_k(X_u) & \overset{\widetilde{\delta}_{k-1}}{\longrightarrow} & H_{k-1}(C_u) & \longrightarrow & H_{k-1}(A_u) \oplus H_{k-1}(B_u) & \longrightarrow & H_{k-1}(X_u)
\end{array}$$

An example



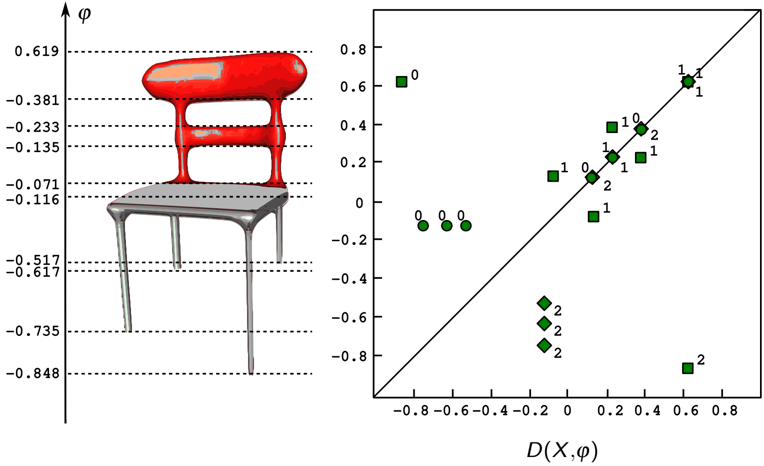
φ
0.619
−0.381
−0.233
−0.135
−0.071
−0.116
−0.517
−0.617
−0.735
−0.848
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8
D(X,φ)

An example

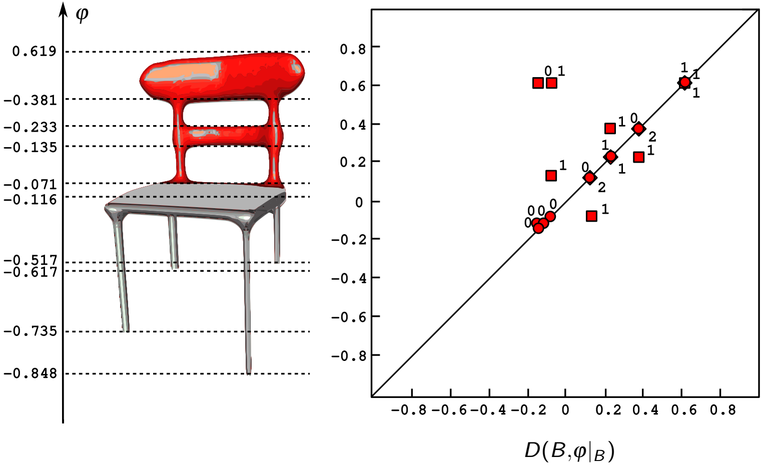


φ
0.619
−0.381
−0.233
−0.135
−0.071
−0.116
−0.517
−0.617
−0.735
−0.848
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
0
0
0
1
1
1
2
2
2
2
−0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8
D(A,φ|A)

An example


φ
0.619
−0.381
−0.233
−0.135
−0.071
−0.116
−0.517
−0.617
−0.735
−0.848
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8
D(X,φ)

An example



φ
0.619
−0.381
−0.233
−0.135
−0.071
−0.116
−0.517
−0.617
−0.735
−0.848
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.8 −0.6 −0.4 −0.2 0 0.2 0.4 0.6 0.8
$D(B,\varphi|_B)$

Experimental results