Lecture 1
Linear Superalgebra

In this lecture we want to introduce some basic concepts of linear supergeometry, as super vector spaces and Lie superalgebras. We shall work over a field, though most of our definitions hold in a more general setting.

Let $k$ be a field, $\text{char} k \neq 2, 3$.

1 Super Vector Spaces

Definition 1.1. A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$V = V_0 \oplus V_1$$

where elements of $V_0$ are called “even” and elements of $V_1$ are called “odd”. The parity of $v \in V$, denoted $\text{p}(v)$ or $|v|$, is defined only on nonzero homogeneous elements, that is elements of either $V_0$ or $V_1$:

$$\text{p}(v) = |v| = \begin{cases} 
0 & \text{if } v \in V_0 \\
1 & \text{if } v \in V_1 
\end{cases}$$

Since any element may be expressed as the sum of homogeneous elements, it suffices to consider only homogeneous elements in the statement of definitions, theorems, and proofs. The super dimension of a super vector space $V$ is the pair $(p, q)$ where $\dim(V_0)=p$ and $\dim(V_1)=q$ as ordinary vector spaces. We simply write $\dim(V) = p|q$.

The most important example of super space is $k^{p|q} = k^p \oplus k^q$, where $k^{p|0} = k^p$ and $k^{1|q} = k^q$.

More in general if we have a finite dimensional super vector space $V$ with $\dim(V) = p|q$, then $V$ always find admits an homogeneous basis $\{e_1, \ldots, e_p, \epsilon_1, \ldots, \epsilon_q\}$, in other words a basis such that $\{e_1, \ldots, e_p\}$ is a basis of $V_0$ and $\{\epsilon_1, \ldots, \epsilon_q\}$ is a basis of $V_1$. In this way $V$ becomes canonically identified with $k^{p|q}$.

Definition 1.2. A morphism from a super vector space $V$ to a super vector space $W$ is a linear map from $V$ to $W$ preserving the $\mathbb{Z}/2\mathbb{Z}$-grading. Let $\text{Hom}(V, W)$ denote the vector space of morphisms $V \longrightarrow W$. 

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Thus we have formed the category of super vector spaces that we denote with \((\text{smod})\). It is important to note that the category of super vector spaces also admits an “inner Hom”, which we denote \(\text{Hom}(V, W)\); for super vector spaces \(V, W\), \(\text{Hom}(V, W)\) consists of all linear maps from \(V\) to \(W\); it is made into a super vector space itself by the following definitions:

\[
\text{Hom}(V, W)_0 = \{ T : V \to W \mid T \text{ preserves parity} \} \quad (= \text{Hom}(V, W)) ;
\]

\[
\text{Hom}(V, W)_1 = \{ T : V \to W \mid T \text{ reverses parity} \} .
\]

For example if \(V = k^{1|1}\) and we fix the canonical basis, we have that

\[
\text{Hom}(k^{1|1}, k^{1|1}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in k \right\}, \quad \text{Hom}(k^{1|1}, k^{1|1}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in k \right\}
\]

The category of super vector spaces admits tensor products: Given the super vector spaces \(V, W\), \(V \otimes W\) has the following natural \(\mathbb{Z}/2\mathbb{Z}\)-grading:

\[
(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)
\]

\[
(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).
\]

The assignment \(V, W \mapsto V \otimes W\) is additive and exact in each variable as in the ordinary vector space category. The object \(k\) functions as a unit element with respect to tensor multiplication \(\otimes\); and tensor multiplication is associative, i.e. the two products \(U \otimes (V \otimes W)\) and \((U \otimes V) \otimes W\) are naturally isomorphic. Moreover, \(V \otimes W \cong W \otimes V\) by the commutativity map

\[
e_{V,W} : V \otimes W \to W \otimes V
\]

where \(v \otimes w \mapsto (-1)^{|v||w|} w \otimes v\).

The significance of this definition is as follows. If we are working with the category of vector spaces, the commutativity isomorphism takes \(v \otimes w\) to \(w \otimes v\). In super linear algebra we have to add the sign factor in front. This is a special case of the general principle called the “sign rule” that one finds in some physics and math literature. The principle says that in making definitions and proving theorems, the transition from the usual theory to the super theory is often made by just simply following this principle, which introduces a sign factor whenever one reverses the order of two odd elements.
The functoriality underlying the constructions makes sure that the definitions are all consistent.

The commutativity isomorphism satisfies the so called *hexagon diagram*:

\[
\begin{array}{ccc}
U \otimes V \otimes W & \xrightarrow{c_{U,V} \otimes W} & V \otimes W \otimes U \\
\downarrow c_{U,V} & & \downarrow c_{U,W} \\
V \otimes U \otimes W
\end{array}
\]

where, if we had not suppressed the arrows of the associativity morphisms, the diagram would have the shape of a hexagon.

The definition of the commutativity isomorphism, also informally referred to as the sign rule, has the following very important consequence. If \(V_1, \ldots, V_n\) are super vector spaces and \(\sigma\) and \(\tau\) are two permutations of \(n\) elements, no matter how we compose associativity and commutativity morphisms, we always obtain the same isomorphism from \(V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}\) to \(V_{\tau(1)} \otimes \cdots \otimes V_{\tau(n)}\) namely:

\[
\begin{align*}
V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)} & \longrightarrow V_{\tau(1)} \otimes \cdots \otimes V_{\tau(n)} \\
v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} & \mapsto (-1)^N v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}
\end{align*}
\]

where \(N\) is the number of pairs of indices \(i, j\) such that \(v_i\) and \(v_j\) are odd and \(\sigma^{-1}(i) < \sigma^{-1}(j)\) with \(\tau^{-1}(i) < \tau^{-1}(j)\).

The *dual*, \(V^*\), of \(V\) is defined as

\[
V^* = \text{def } \text{Hom}(V, k).
\]

Notice that, if \(V\) is even, that is \(V = V_0\), we have that \(V^*\) is the ordinary dual of \(V\), consisting of all even morphisms \(V \longrightarrow k\). If \(V\) is odd, that is \(V = V_1\), then \(V^*\) is also an odd vector space and consists of all odd morphisms \(V^1 \longrightarrow k\). This is because any morphism from \(V_1\) to \(k = k^{10}\) is necessarily odd since it sends odd vectors into even ones.

## 2 Superalgebras

In the ordinary setting, an algebra is a vector space \(A\) with a multiplication which is bilinear. We may therefore think of it as a vector space \(A\) together with a linear map \(A \otimes A \longrightarrow A\), which comes from the multiplication. We now define a superalgebra in the same way.
Definition 2.1. A superalgebra is a super vector space $A$ together with a multiplication morphism $\tau : A \otimes A \rightarrow A$. which is associative and admits a unit element.

We say that a superalgebra $A$ is (super)commutative if

$$\tau \circ c_{A,A} = \tau.$$ 

One can check right away that this implies:

$$ab = (-1)^{|a||b|}ba.$$ 

We shall denote the category of commutative superalgebras with (salg).

This is an example of the sign rule mentioned earlier. The signs do not appear in the definition: this is a clear advantage of the categorical viewpoint and we are going to see this again later.

If $A$ and $B$ are two superalgebras we can form the tensor product $A \otimes B$. This is also a superalgebra and the product obeys the rule:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} (ac \otimes bd).$$

We are now going to see the most important examples of superalgebras, namely the tensor and the polynomial superalgebras.

Definition 2.2. Let $V$ be a super vector space, we define tensor superalgebra the super vector space $T(V) = \bigoplus_{n \geq 0} V^\otimes n$, $T(V)_0 = \bigoplus_{n \text{ even}} V^\otimes n$, $T(V)_1 = \bigoplus_{n \text{ odd}} V^\otimes n$, together with the product defined, as usual, via the ordinary bilinear map $\phi_{r,s} : V^\otimes r \otimes V^\otimes s \rightarrow V^\otimes (r+s)$,

$$\phi_{r,s}(v_{i_1} \otimes \cdots \otimes v_{i_r}, w_{j_1} \otimes \cdots \otimes w_{j_s}) = v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes w_{j_1} \cdots \otimes w_{j_s}.$$ 

Example 2.3. Polynomial superalgebra. Let

$$A = k[t_1, \ldots, t_p, \theta_1, \ldots, \theta_q]$$
where the \( t_1, \ldots, t_p \) are ordinary indeterminates and the \( \theta_1, \ldots, \theta_q \) are odd indeterminates, i.e. they behave like Grassmannian coordinates:
\[
\theta_i \theta_j = -\theta_j \theta_i.
\]
(This of course implies that \( \theta_i^2 = 0 \) for all \( i \).) In other words we can view \( A \) as the ordinary tensor product \( k[t_1 \ldots t_p] \otimes \wedge(\theta_1 \ldots \theta_q) \), where \( \wedge(\theta_1 \ldots \theta_q) \) is the exterior algebra generated by \( \theta_1 \ldots \theta_q \).

We claim that \( A \) is a supercommutative algebra. In fact,
\[
A_0 = \{ f_0 + \sum_{|I| \text{ even}} f_I \theta_I | I = \{ i_1 < \ldots < i_r \} \}
\]
where \( \theta_I = \theta_{i_1} \theta_{i_2} \ldots \theta_{i_r} \) and \( f_0, f_I \in k[t_1, \ldots, t_p] \), and
\[
A_1 = \{ \sum_{|J| \text{ odd}} f_J \theta_J | J = \{ j_1 < \ldots < j_s \} \}.
\]
Note that although the \( \{ \theta_j \} \in A_1 \), there are plenty of nilpotents in \( A_0 \); take for example \( \theta_1 \theta_2 \in A_0 \).

This example is important since any finitely generated commutative superalgebra is isomorphic to a quotient of the algebra \( A \) by a homogeneous ideal.

## 3 Super Lie Algebras

An important object in supersymmetry is the super Lie algebra.

**Definition 3.1.** A super Lie algebra \( L \) is an object in the category of super vector spaces together with a morphism \([,] : L \otimes L \rightarrow L\), often called the super bracket, or simply, the bracket, which satisfies the following conditions.

1. Anti-symmetry
\[
[,] + [,] \circ c_{L,L} = 0
\]
which is the same as \([x, y] + (-1)^{|x||y|}[y, x] = 0\) for \( x, y \in L \) homogeneous.

2. The Jacobi identity
\[
[,] + [,] \circ \sigma + [,] \circ \sigma^2 = 0
\]

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where \( \sigma \in S_3 \) is a three-cycle, i.e. it takes the first entry of \([, [\,]]\) to the second, the second to the third, and the third to the first. So for \( x, y, z \in L \) homogeneous, this reads:

\[
[x, [y, z]] + (-1)^{|x||y|+|x||z|} [y, [z, x]] + (-1)^{|y||z|+|x||z|} [z, [x, y]] = 0.
\]

It is important to note that in the super category, these conditions are modifications of the properties of the bracket in a Lie algebra, designed to accommodate the odd variables.

We shall often use also the term super Lie algebra instead of Lie super-algebra, since both are present in the literature.

**Example 3.2.** Let \( \text{End}(V) \) denote the endomorphisms of a super vector space \( V \). \( \text{End}(V) \) is a super vector space itself, where \( \text{End}(V)_0 \) are the endomorphisms preserving parity, while \( \text{End}(V)_1 \) are those reversing it. If \( \text{dim}(V) = p|q \) and if we choose an homogeneous basis for \( V \), then \( \text{End}(V)_0 \) consists of block diagonal matrices, while \( \text{End}(V)_1 \) of off-diagonal ones.

\( \text{End}(V) \) is an associative superalgebra with the composition as product and it is a Lie superalgebra with bracket:

\[
[X, Y] = XY - (-1)^{|X||Y|} YX,
\]

where the bracket as usual is defined only on homogeneous elements and then extended by linearity.

**Remark 3.3.** It is important to notice that we can make any associative superalgebra \( A \) into a Lie superalgebra by taking the bracket to be

\[
[a, b] = ab - (-1)^{|a||b|} ba,
\]

**4 Modules for superalgebras**

Let \( A \) be a commutative superalgebra.

**Definition 4.1.** A left \( A \)-module is a super vector space \( M \) with a morphism \( A \otimes M \to M, a \otimes m \mapsto am \) of super vector spaces obeying the usual identities i.e. for all \( a, b \in A \) and \( x, y \in M \) we have:

1. \( a(x + y) = ax + ay \),
2. \( (a + b)x = ax + bx \),
3. \( (ab)x = a(bx) \),
4. \( 1x = x \).
A right \( A \)-module is defined similarly. Note that since \( A \) is commutative, a left \( A \)-module is also a right \( A \)-module if we define (the sign rule)

\[
m \cdot a = (-1)^{|m||a|} a \cdot m
\]

for \( m \in M, a \in A \). Morphisms of \( A \)-modules are defined in the obvious manner: there are super vector space morphisms \( \phi : M \rightarrow N \) such that \( \phi(am) = a\phi(m) \) for all \( a \in A, m \in M \). So we have the category of \( A \)-modules. For \( A \) commutative, the category of \( A \)-modules admits tensor products: for \( M_1, M_2 \) \( A \)-modules, \( M_1 \otimes M_2 \) is taken as the tensor of \( M_1 \) as a right module with \( M_2 \) as a left module.

Let us now turn our attention to free \( A \)-modules. We already have the notion of the vector space \( k^{p|q} \) over \( k \), and so we define \( A^{p|q} := A \otimes_k k^{p|q} \)

where

\[
(A^{p|q})_0 = A_0 \otimes (k^{p|q})_0 \oplus A_1 \otimes (k^{p|q})_1 \\
(A^{p|q})_1 = A_1 \otimes (k^{p|q})_0 \oplus A_0 \otimes (k^{p|q})_1.
\]

**Definition 4.2.** We say that an \( A \)-module \( M \) is free if it is isomorphic (in the category of \( A \)-modules) to \( A^{p|q} \) for some \((p, q)\).

We now want to represent morphisms of free \( A \)-modules using matrices with entries in \( A \).

Let \( T : A^{p|q} \rightarrow A^{r|s} \) be a morphism of free \( A \)-modules and let \( e_1 \ldots e_n, e_{p+1}, \ldots, e_{p+q} \) be the canonical basis (we write \( e_{p+1}, \ldots, e_{p+q} \) for the odd basis elements \( e_1, \ldots, e_q \). Then \( T \) is defined on the basis elements \( \{e_1, \ldots, e_{p+q}\} \) by

\[
T(e_j) = \sum_{i=1}^{p+q} e_i t_{ij}.
\]

(1)

[Notice: we have written the coefficients \( t_{ij} \in A \) on the right for a reason which will be clear soon].

For example if \( T : A^{1|1} \rightarrow A^{1|1} \) we have that \( T(e_1) = e_1 a + e_2 \alpha, T(e_2) = e_1 \beta + e_2 b \), in other words we can represent \( T \) with a \( 2 \times 2 \) matrix:

\[
T = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}
\]
where $a, b \in A_0$ and $\alpha, \beta \in A_1$ (the parities are forced because of the parity preserving requirements on $T$).

Hence in general $T$ can be represented as a matrix of size $(r+s) \times (p+q)$:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$ (2)

where $T_1$ is an $r \times p$ matrix consisting of even elements of $A$, $T_2$ is an $r \times q$ matrix of odd elements, $T_3$ is an $s \times p$ matrix of odd elements, and $T_4$ is an $s \times q$ matrix of even elements. We say that $T_1$ and $T_4$ are even blocks and that $T_2$ and $T_3$ are odd blocks. Note that the fact that $T$ is a morphism of super $A$-modules means that it must preserve parity, and therefore the parity of the blocks is determined. Note also that when we define $T$ on the basis elements, in the expression (1) the basis element precedes the coordinates $t^j_i$. This is important to keep the signs in order and comes naturally from composing morphisms. In other words if the module is written as a right module with $T$ acting from the left, composition becomes matrix product in the usual manner:

$$(S \cdot T)(e_j) = S \left( \sum_i e_i t^i_j \right) = \sum_{i,k} e_k s^k_i t^i_j.$$

Hence for any $x \in A^{p|q}$, we can express $x$ as the column vector $x = \sum e_i x^i$ and so $T(x)$ is given by the matrix product $Tx$.

5 The Supertrace and the Berezinian

We now turn to the supersymmetric extensions of the trace and determinant. Let $T : A^{p|q} \longrightarrow A^{p|q}$ be a morphism (i.e. $T \in (\text{Mat}(A^{p|q}))(0)$) with block form (2).

**Definition 5.1.** We define the super trace of $T$ to be:

$$\text{str}(T) := \text{tr}(T_1) - \text{tr}(T_4)$$ (3)

where “tr” denotes the ordinary trace.

This negative sign is actually forced upon us when we take a categorical view of the trace. We will not discuss this here, but we later motivate this
definition when we explore the supersymmetric extension of the determinant called the \textit{berezinian}. Most strikingly, as we shall see, while the supertrace is defined for all matrices, the berezinian is defined only for the \textit{invertible} ones. This forces us to introduce first the \textit{general linear supergroup}.

\textbf{Definition 5.2.} If $M$ is an $A$-module, then $\text{GL}(M)$ is defined as the group of automorphisms of $M$ and we call it the \textit{super general linear group of automorphisms of $M$}. If $M = A^{p|q}$ the free $A$-module generated by $p$ even and $q$ odd variables, then we write $\text{GL}(M) = \text{GL}_{p|q}(A)$. We may also use the notation $\text{GL}_{p|q}(A) = \text{GL}(A^{p|q})$.

We now turn to the definition of the \textit{Berezinian}, on elements of $\text{GL}(A^{p|q})$. We may say that this is the point where linear supergeometry differs most dramatically from the ordinary theory.

\textbf{Proposition 5.3.} Let $T : A^{p|q} \to A^{p|q}$ be a morphism with the usual block form (2). Then $T$ is invertible if and only if $T_1$ and $T_4$ are invertible.

\textit{Proof.} Let $J_A \subset A$ be the ideal generated by odd elements and let $\bar{A} = A/J_A$. There is a natural map $M_{p|q}(A) \to M_{p|q}(\bar{A})$, $T \mapsto \bar{T}$, where $\bar{T}$ is obtained from the matrix $T$ by applying to its entries the map $A \to \bar{A}$. We claim that $T$ is invertible if and only if $\bar{T}$ is invertible. One direction is obvious, namely the case in which $T$ is invertible. Now assume that $\bar{T}$ is invertible. This implies that there exists $\bar{S} \in M_{p|q}(\bar{A})$, such that $\bar{T}\bar{S} = \bar{S}\bar{T} = I$, where $I$ denotes the identity (both in $M_{p|q}(\bar{A})$ and in $M_{p|q}(A)$). Hence there exists $S \in M_{p|q}(A)$ such that $TS = I + N$ (we consider only the case of a right inverse since the left inverse is the same). To prove $T$ is invertible it is enough to show $N$ is nilpotent, i.e. $N^r = 0$ for some $r$. Since the entries of $N^m$ are in $A_1^m$ for $m$ sufficiently large they are all zero. \hfill \blacksquare

\textbf{Definition 5.4.} Let $T$ be an invertible element in $M_{p|q}(A)$ i.e. $T \in \text{GL}(A^{p|q})$ with the standard block form (2) from above. Then we formulate $\text{Ber}$:

$$\text{Ber}(T) = \det(T_1 - T_3 T_4^{-1} T_3) \det(T_4)^{-1}$$

where “det” is the usual determinant.

The Berezinian is named after Berezin, who was one of the pioneers of superalgebra and superanalysis.
Remark 5.5. The first thing we notice is that in the super category, we only define the Berezinian for invertible transformations. This marks an important difference with the determinant, which is defined in ordinary linear algebra for all endomorphisms of a vector space. We immediately see that it is necessary that the block $T_4$ be invertible for the formula (4) to make sense, however one can actually define the Berezinian on all matrices with only the $T_4$ block invertible (i.e. the matrix itself may not be invertible, but the $T_4$ block is). There is a similar formulation of the Berezinian which requires that only the $T_1$ block be invertible:

$$\text{Ber}(T) = \det(T_4 - T_3 T_1^{-1} T_2)^{-1} \det(T_1)$$

So we can actually define the Berezinian on all matrices with either the $T_1$ or the $T_4$ block invertible. Note that in the case where both blocks are invertible (i.e. when the matrix $T$ is invertible), both formulae of the Berezinian give the same answer as we shall see after the next proposition.

Proposition 5.6. The Berezinian is multiplicative: For $S, T \in \text{GL}(A_{p|q})$,

$$\text{Ber}(ST) = \text{Ber}(S)\text{Ber}(T).$$

Proof. We will only briefly sketch the proof here and leave the details to the reader. First note that any $T \in \text{GL}(A_{p|q})$ with block form (2) may be written as the product of the following “elementary matrices”:

$$T_+ = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}, \quad T_0 = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}, \quad T_- = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix}. \quad (5)$$

If we equate $T = T_+ T_0 T_-$, we get a system of equations which lead to the solution

$$X = T_2 T_4^{-1},$$
$$Y_1 = T_1 - T_2 T_4^{-1} T_3,$$
$$Y_2 = T_3,$$
$$Z = T_4^{-1} T_3$$

It is also easy to verify that $\text{Ber}(ST) = \text{Ber}(S)\text{Ber}(T)$ for $S$ of type $\{T_+, T_0\}$ for all $T$ or $T$ of type $\{T_-, T_0\}$ for all $S$. Let $G$ is the subgroup of $\text{GL}(A_{p|q})$ consisting of all elements $S$ such that $\text{Ber}(ST) = \text{Ber}(S)\text{Ber}(T)$ for all $T$. It is enough to show that elements of type $T_0, T_+, T_-$ are in $G$, since
they generate $\text{GL}(A^{p|q})$. We leave to the reader the check for the first two types. For type $T_-$, the only non trivial case to verify is for

$$S = \begin{pmatrix} 1 & 0 \\ Z & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}. $$

We may assume that both $X$ and $Z$ each have only one non-zero entry since the product of two matrices of type $T_+$ results in the sum of the upper right blocks, and likewise with the product of two type $T_-$ matrices. Let $x_{ij}, z_{kl} \neq 0$. Then

$$ST = \begin{pmatrix} 1 & X \\ Z & 1 + ZX \end{pmatrix}$$

and $\text{Ber}(ST) = \det(1 - X(1 + ZX)^{-1}Z)\det(1 + ZX)^{-1}$. Because all the values within the determinants are either upper triangular or contain an entire column of zeros ($X,Z$ have at most one non-zero entry), the values $x_{ij}, z_{kl}$ contribute to the determinant only when the product $ZX$ has its non-zero term on the diagonal, i.e. only when $i = j = k = l$. But then $\det(1 - X(1 + ZX)^{-1}Z) = 1 + x_{ii}z_{ii}$, and it is clear that $\text{Ber}(ST) = 1$. A direct calculation shows that $\text{Ber}(S) = \text{Ber}(T) = 1$.

**Corollary 5.7.** Let $T \in \text{GL}_{p|q}(A)$. Then

$$\text{Ber}(T) = \det(T_4 - T_3T_1^{-1}T_2)^{-1} \det(T_1).$$

**Proof.** Consider the decomposition:

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ T_3T_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_4 - T_3T_1^{-1}T_2 \end{pmatrix} \begin{pmatrix} 1 & T_1^{-1}T_2 \\ 0 & 1 \end{pmatrix}. $$

By multiplicativity of the Berezinian, we obtain the result.

**Corollary 5.8.** The Berezinian is a homomorphism

$$\text{Ber} : \text{GL}(A^{p|q}) \longrightarrow \text{GL}_{1|0}(A) = A^\times_0$$

into the invertible elements of $A$.

**Proof.** This follows immediately from the multiplicativity property.
Remark 5.9. In the course of the proof of 5.6 we have seen the decomposition:

$$GL_{p|q}(A) = UHV$$

where

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \right\}, \quad V = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \right\}. \tag{1}$$

This very much resembles the big cell decomposition in the theory of ordinary algebraic groups, however here the decomposition holds globally.

The usual determinant on the general linear group $GL_n$ induces the trace on its Lie algebra, namely the matrices $M_n$. Euristically we have that the extension to the Berezinian gives

$$\text{Ber}(I + \epsilon T) = 1 + \epsilon \text{str}(T)$$

where $I$ is the $p|q \times p|q$ identity matrix (ones down the diagonal, zeros elsewhere) and $\epsilon^2 = 0$. An easy calculation then exposes the super trace formula with the negative sign.

Using this calculation one can easily show that:

$$\text{str}(ST) = \text{str}(TS)$$

and we shall leave this as an exercise to the reader.

This of course leads to the question of how the formula for the Berezinian arises. The answer lies in the SUSY-version of integral forms on supermanifolds called densities. In F.A. Berezin’s pioneering work in superanalysis, Berezin calculated the change of variables formula for densities on isomorphic open submanifolds of $\mathbb{R}^{p|q}$. This lead to an extension of the Jacobian in ordinary differential geometry; the Berezinian is so named after him.

We are ready for the formula for the inverse of a supermatrix.

Proposition 5.10. Let

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \in GL_{p|q}(A)$$

(hence $T_1$ and $T_4$ are invertible ordinary matrices). Then

$$T^{-1} = \begin{pmatrix} (T_1 - T_2 T_4^{-1} T_3)^{-1} & -T_4^{-1}T_2(T_4 - T_3 T_1^{-1} T_2)^{-1} \\ -T_4^{-1}T_3(T_1 - T_2 T_4^{-1} T_3)^{-1} & (T_4 - T_3 T_1^{-1} T_2)^{-1} \end{pmatrix}. \tag{2}$$

Proof. Direct check. \hfill \blacksquare
References
