Lecture 2
Sheaves and Functors

In this lecture we will introduce the basic concept of sheaf and we also will recall some of category theory.

1 Sheaves and locally ringed spaces

The definition of sheaf is central in algebraic and differential geometry since it provides a unified treatment. We start with two key examples.

Example 1.1. *Differentiable manifolds.* Let M be a differentiable manifold, whose topological space is Hausdorff and second countable. For each open set $U \subset M$, let $C^\infty(U)$ be the $\mathbf{R}$-algebra of smooth functions on U. The assignment:

$$U \longmapsto C^\infty(U)$$

satisfies the following two properties:

1. If $U \subset V$ are two open sets in M, we can define the *restriction* map:

$$\begin{aligned} \mathrm{r}_{V,U} : C^\infty(V) &\longrightarrow C^\infty(U) \\ f &\mapsto f|_U \end{aligned}$$

which is such that:
$i)$ $\mathrm{r}_{U,U} = \mathrm{id}$,
$ii)$ $\mathrm{r}_{W,U} = \mathrm{r}_{V,U} \circ \mathrm{r}_{W,V}$.

2. Let $\{U_i\}_{i \in I}$ be an open covering of U and let $\{f_i\}_{i \in I}$, $f_i \in C^\infty(U_i)$, be a family such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In other words the elements of the family $\{f_i\}_{i \in I}$ agree on the intersection of any two open sets $U_i \cap U_j$. Then there exists a unique $f \in C^\infty(U)$ such that $f|_{U_i} = f_i$.

Such an assignment is called a *sheaf.* The pair (M, C^∞), consisting of the topological space M and the sheaf of the C^∞ functions on M is an example of *locally ringed space* (the word "locally" refers to a local property of the sheaf of C^∞ functions).

Given two manifolds M and N and the respective sheaves of smooth functions C^∞_M and C^∞_N, a morphism $f : M \longrightarrow N$ as ringed spaces, consists of a morphism $|f| : M \longrightarrow N$ of the underlying topological spaces together with a morphism of algebras:

$$f^* : C^\infty_N(V) \longrightarrow C^\infty_M(f^{-1}(V)), \qquad f^*(\phi)(x) = \phi(|f|(x)), \quad V \quad \text{open in} \quad |N|$$

compatible with the restriction morphisms.

Notice that, as soon as we give the continuos map $|f|$ between the topological spaces, the morphism f^* is automatically assigned. This is a peculiarity of the sheaf of smooth functions on a manifold. Such property is no longer true for a generic ringed space and, in particular, as we shall see, it is not true for supermanifolds.

A morphism of differentiable manifolds gives rise to a unique (locally) ringed space morphism and vice-versa.

Moreover, given two manifolds, they are isomorphic as manifolds if and only if they are isomorphic as (locally) ringed spaces. In the language of categories, we say we have a fully faithfull functor from the category of manifolds to the category of locally ringed spaces.

Before going to the general treatment, let us consider another interesting example arising from classical algebraic geometry.

Example 1.2. *Algebraic varieties.* Let X be an affine algebraic variety in the affine space $\mathbf{A}^n$ over an algebraically closed field k and let $\mathcal{O}(X) = k[x_1 \dots x_n]/I$ be its coordinate ring, where the ideal I is prime. This corresponds topologically to the irreducibility of the variety X. We can think to the points of X as the zeros of the polynomials in the ideal I in $\mathbf{A}^n$. X is a topological space with respect to the Zariski topology, whose closed sets are the zeros of the polynomials in the ideals of $\mathcal{O}(X)$. For each open U in X, consider the assigment:

$$U \longmapsto \mathcal{O}_X(U)$$

where $\mathcal{O}_X(U)$ is the k-algebra of of algebraic functions on U. By definition, these are the functions $f : X \longrightarrow k$ that can be expressed as a quotient of two polynomials at each point of $U \subset X$. As in the case of differentiable manifolds, our assignment satisfies the properties (1) and (2) described above. The first property is clear, while for the second one, we leave it as a (hard) exercise.

Hence $\mathcal{O}_X$ is a sheaf called the *structure sheaf* of the variety X of the sheaf of regular functions and $(X, \mathcal{O}_X)$ is another example of (locally) ringed space.

We are now going to formulate more generally the notion of sheaf and of ringed space.

Definition 1.3. Let $|M|$ be a topological space. A *presheaf* of commutative algebras $\mathcal{F}$ on X is an assignment

$$U \longmapsto \mathcal{F}(U), \quad U \quad \text{open in } |M|, \quad \mathcal{F}(U) \quad \text{a commutative algebra,}$$

such that:
1. If $U \subset V$ are two open sets in $|M|$, there exists a morphism
$\mathrm{r}_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$, called the *restriction morphism* and often denoted with $\mathrm{r}_{V,U}(f) = f|_U$, such that:
i) $\mathrm{r}_{U,U} = \mathrm{id}$,
ii) $\mathrm{r}_{W,U} = \mathrm{r}_{V,U} \circ \mathrm{r}_{W,V}$.
The presheaf $\mathcal{F}$ is called a *sheaf* if:
2. Given $\{U_i\}_{i \in I}$, an open covering of U and a family $\{f_i\}_{i \in I}$, $f_i \in \mathcal{F}(U_i)$, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$.

The elements in $\mathcal{F}(U)$ are called *sections* over U; when $U = |M|$ we call such elements *global sections*.

A most important object associated to a given presheaf is the *stalk* at a point.

Definition 1.4. Let $\mathcal{F}$ be a presheaf on the topological space $|M|$ and let x be a point in $|M|$. We define the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$, at the point x, as the inductive limit:

$$\lim_{\rightarrow} \mathcal{F}(U),$$

where the direct limit is taken for all the U open neighbourhoods of x in $|M|$. $\mathcal{F}_x$ consists of the disjoint union of all pairs (U, s) with U open in $|M|$, $x \in U$, and $s \in \mathcal{F}(U)$, modulo the equivalence relation: $(U, s) \cong (V, t)$ if and only if there exists a neighbourhood W of x, $W \subset U \cap V$, such that $s|_W = t|_W$.

The elements in $\mathcal{F}_x$ are called *germs of sections*.

Definition 1.5. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on $|M|$. A *morphism of presheaves* $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of morphisms $\phi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$, for each open set U in $|M|$, such that for all $V \subset U$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \mathrm{r}_{UV} \downarrow & & \downarrow \mathrm{r}_{UV} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

A *morphisms of sheaves* is just a morphism of the underlying presheaves.

Clearly any morphism of presheaves induces a morphism on the stalks: $\phi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$. The sheaf property, i.e. property (2) in the definition 1.3, ensures that, if we have two morphisms of sheaves ϕ and ψ such that $\phi_x = \psi_x$ for all x, then $\phi = \psi$.

We say that the morphism of sheaves ϕ is *injective (resp. surjective)* if ϕ_x is injective (resp. surjective).

On the notion of surjectivity, however, one should exert some care, since we can have a surjective sheaf morphism $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ such that $\phi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is *not* surjective for some open sets U. This strange phenomenon is a consequence of the following fact. While the assignment $U \mapsto ker(\phi(U))$ always defines a sheaf, the assignment $U \mapsto Im(\phi(U))$ defines in general only a presheaf. In order to define the image sheaf we need to do the *sheafification* of this presheaf. Intuitively, one may think to the sheafification as the sheaf that best "approximates" the given presheaf. We shall not pursue furtherly this point.

We are ready for the definition of locally ringed space. This definition is very important for us, since its supergeometric correspondent, that we shall introduce in the next lecture, is essential to define supermanifolds and superschemes, which are the basic ingredients of supergeometry.

Definition 1.6. We define *ringed space* a pair $M = (|M|, \mathcal{F})$ consisting of a topological space $|M|$ and a sheaf of commutative rings $\mathcal{F}$ on $|M|$. We say that the ringed space $(|M|, \mathcal{F})$ is a *locally ringed space*, if the stalk $\mathcal{F}_x$ is a local ring for all $x \in |M|$. A *morphism* of ringed spaces $\phi : M = (|M|, \mathcal{F}) \longrightarrow N = (|N|, \mathcal{G})$ consists of a morphism $|\phi| : |M| \longrightarrow |N|$ of the topological spaces (in other words, $|\phi|$ is a continuous map) and a sheaf morphism $\phi^* : \mathcal{O}_N \longrightarrow \phi_* \mathcal{O}_M$ where $\phi_* \mathcal{O}_M$ is the sheaf on $|N|$ defined as follows: $(\phi_* \mathcal{O}_M)(U) = \mathcal{O}_M(\phi^{-1}(U))$ for all U open in $|N|$. A morphism of ringed spaces induces a morphism on the stalks for each $x \in |M|$: $\phi_x : \mathcal{O}_{N,|\phi|(x)} \longrightarrow \mathcal{O}_{M,x}$. If M and N are locally ringed spaces, we say that the morphism of ringed spaces ϕ is a *morphism of locally ringed spaces* if ϕ_x is local, i.e. $\phi_x^{-1}(m_{M,x}) = m_{N,|\phi|(x)}$ where $m_{N,|\phi|(x)}$ and $m_{M,x}$ are the maximal ideals in the local rings $\mathcal{O}_{N,|\phi|(x)}$ and $\mathcal{O}_{M,x}$ respectively.

Observation 1.7. In the previous section we have seen differentiable manifolds and algebraic varieties as examples of ringed spaces. Actually both are also examples of locally ringed spaces, as one can readily verify. Moreover, one can also check that their morphisms, in the differential or in the algebraic setting respectively, correspond precisely to morphisms as locally ringed spaces.

At this point it is not hard to convince ourselves that we can take a different point of view on the definition of differentiable manifold. Namely we can equivalently define a differentiable manifold as a ringed space $M = (|M|, \mathcal{O}_M)$ as follows.

Definition 1.8. *Alternative definition of differentiable manifold.* Let M be a topological space, Hausdorff and second countable, and let $\mathcal{O}_M$ be a sheaf of commutative algebras on M, so that $(M, \mathcal{O}_M)$ is a locally ringed space. We say that $(M, \mathcal{O}_M)$ is a *real differentiable manifold* if it is locally isomorphic to the locally ringed space $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$, where $C^\infty_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbf{R}^n$.

In the same way we can define analytic real or complex manifolds as locally ringed spaces locally isomorphic to $(\mathbf{R}^n, \mathcal{H}_{\mathbf{R}^n})$ or $(\mathbf{C}^n, \mathcal{H}_{\mathbf{C}^n})$, where $\mathcal{H}_{\mathbf{R}^n}$ and $\mathcal{H}_{\mathbf{C}^n}$ denote the sheaves of analytic functions on $\mathbf{R}^n$ or $\mathbf{C}^n$ respectively (we leave to the reader as an exercise their definition, see [2] for more details).

2 Schemes

The concept of scheme is a step towards a further abstraction. We shall start by defining affine schemes and then we procede to the definition of schemes in general.

Let us start by associating to any commutative ring A its *spectrum*, that is the topological space Spec$A$. As a set, Spec$A$ consists of all the prime ideals in A. For each subset $S \subset A$ we define as *closed sets* in Spec$A$:

$$V(S) := \{\mathfrak{p} \in \mathrm{Spec}A \quad | \quad S \subset \mathfrak{p}\} \subset \mathrm{Spec}A.$$

One can check that this actually defines a topology on Spec$A$ called the *Zariski topology.*

If X is an affine variety, defined over an algebraically closed field, and $\mathcal{O}(X)$ is its coordinate ring, we have that the points of the topological space underlying X are in one to one correspondence with the maximal ideals in

$\mathcal{O}(X)$. So we notice immediately that Spec$\mathcal{O}(X)$ contains far more than just the points of the topological space of X; in fact it contains also all the subvarieties of X, whose information is encoded by the prime ideals in $\mathcal{O}(X)$. This tells us that the notion of scheme, we are about to introduce, is not just a generalization of the concept of algebraic variety, but it is something deeper, containing more information about the geometric objects we are interested in.

We also the *basic open sets* in Spec$A$ as:

$$U_f := \mathrm{Spec}A \setminus V(f) = \mathrm{Spec}A_f, \qquad \text{with} \quad f \in A,$$

where $A_f = A[f^{-1}]$ is the localization of A obtained by inverting the element f. The collection of the basic open sets U_f, for all $f \in A$, forms a base for the Zariski topology.

Next, we define the *structure sheaf* $\mathcal{O}_A$ on the topological space Spec$A$. In order to do this, it is enough to give an assignment $U \mapsto \mathcal{O}_A(U)$ for each basic open set $U = U_f$ in Spec$A$.

Proposition 2.1. *Let the notation be as above. The assignment:*

$$U_f \longmapsto A_f$$

extends uniquely to a sheaf of commutative rings on Spec$A$*, called the structure sheaf and denoted by* $\mathcal{O}_A$*. Moreover the stalk in a point* $\mathbf{p} \in \mathrm{Spec}A$*,* $\mathcal{O}_{A,\mathbf{p}}$ *is the localization* $A_{\mathbf{p}}$ *of the ring* A *at the prime* $\mathbf{p}$*.*

Proof. Direct check. See also [1] I-18. ■

Hence, given a commutative ring A, proposition 2.1 tells us that the pair $(\mathrm{Spec}A, \mathcal{O}_A)$ is a locally ringed space, that we call $\underline{\mathrm{Spec}}A$ the *spectrum of the ring* A. By an abuse of notation we shall use the word *spectrum* to mean both the topological space Spec$A$ and the locally ringed space $\underline{\mathrm{Spec}}A$, the context making clear which one we are talking about.

We are finally ready for the definition of scheme. While the differentiable manifolds are locally modelled, as ringed spaces, by $(\mathbf{R}^n, C^\infty_{\mathbf{R}^n})$, the schemes are geometric objects modelled by the spectrums of commutative rings.

Definition 2.2. We define *affine scheme* a locally ringed space isomorphic to $\underline{\mathrm{Spec}}A$ for some commutative ring A. We say that X is a *scheme* if

$X = (|X|, \mathcal{O}_X)$ is a locally ringed space, which is locally isomorphic to affine schemes. In other words, for each $x \in |X|$, there exists an open set $U_x \subset |X|$ such that $(U_x, \mathcal{O}_X|_{U_x})$ is an affine scheme. A *morphism* of schemes is just a morphism of locally ringed spaces.

Observation 2.3. 1. There is an equivalence of categories between the category of affine schemes (aschemes) and the category of commutative rings (rings). This equivalence is defined on the objects by:

$$\begin{array}{ccc} (\text{rings})^o & \longrightarrow & (\text{aschemes}) \\ A & \mapsto & \underline{\text{Spec}} A \end{array}$$

In particular a morphism of commutative rings $A \longrightarrow B$ corresponds contravariantly to a morphism $\underline{\text{Spec}} B \longrightarrow \underline{\text{Spec}} A$ of the corresponding affine superschemes. For more details see proposition 2.3, ch. II, [3] and [1] ch. I, theorem I-40.

2. Since any affine variety X is completely described by the knowledge of its coordinate ring $\mathcal{O}(X)$, (the ring of regular functions on the whole variety), we can associate uniquely to an affine variety X the affine scheme $\underline{\text{Spec}} \mathcal{O}(X)$. As we noted previously, the two notions of X as algebraic variety or as a scheme are different, however they describe the same geometrical object from two different points of view. Similarly to any algebraic variety (not necessarily affine) we can associate uniquely a scheme. Moreover a morphism between algebraic varieties determines uniquely a morphism between the corresponding schemes. In the language of categories, we say we have a fully faithful functor from the category of algebraic varieties to the category of schemes. For more details see [3] proposition 2.3, ch. 2 and [1] ch. I.

3 The functor of points

When we are dealing with classical manifolds and algebraic varieties, we can altogether avoid the use of their functor of points. In fact, both differentiable manifolds and algebraic varieties are well understood just by looking at their underlying topological spaces and the regular functions on the open sets.

However, if we go to generality of schemes, the extra structure overshadows the topological points and leaves out crucial details so that we have little information, without the full knowledge of the sheaf.

The functor of points is a categorical device to bring back the attention to the points of a scheme; however the notion of *point* needs to be suitably generalized to go beyond the points of the topological space underlying the scheme.

Grothendieck idea behind the definition of the functor of points associated to a scheme is the following. If X is a scheme, for each commutative ring A, we can define the set of the A-points of X in analogy to the way the classical geometers used to define the rational or integral points on a variety. The crucial difference is that we do not focus on just one commutative ring A, but we consider the A-points for all commutative rings A. In fact, the scheme we start from, is completely recaptured only by the collection of the A-points for *every* commutative ring A, together with the admissible morphisms.

Let (rings) denote the category of commutative rings and (schemes) the category of schemes.

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a scheme and let $A \in$ (rings). We call the *A-points* of X, the set of all scheme morphisms $\{\underline{\mathrm{Spec}} A \longrightarrow X\}$, that we denote with $\mathrm{Hom}(\underline{\mathrm{Spec}} A, X)$. We define the *functor of points* h_X of the scheme X as the representable functor:

$$h_X : (\text{schemes})^o \longrightarrow (\text{sets}), \qquad h_X(T) = \mathrm{Hom}(T, X).$$

$h_X(T)$ are called the T-points of the scheme X. The restriction of h_X to affine schemes is not in general representable. However, since, as we noticed in observation 2.3, the category of affine schemes is equivalent to the the category of commutative rings we have that such restriction gives a new functor h^a_X:

$$h^a_X : (\text{rings}) \longrightarrow (\text{sets}), \quad h^a_X(A) = \mathrm{Hom}(\underline{\mathrm{Spec}} A, X) = A\text{-points of } X.$$

Notice that when X is affine, $X \cong \underline{\mathrm{Spec}} \mathcal{O}(X)$ and we have:

$$h^a_X(A) = \mathrm{Hom}(\underline{\mathrm{Spec}} A, \mathcal{O}(X)) = \mathrm{Hom}(\mathcal{O}(X), A)$$

and in this case the functor h^a_X is again representable.

This is a consequence of the following proposition, which comes from 2.3.

Proposition 3.2. *Consider the affine schemes $X = \underline{\mathrm{Spec}} \mathcal{O}(X)$ and $Y = \mathcal{O}(Y)$. There is a one to one correspondence between the scheme morphisms $X \longrightarrow Y$ and the ring morphisms $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$.*

Both h_X and h^a_X are defined on morphisms in the natural way. If a $\phi : T \longrightarrow S$ is a morphism and $f \in \mathrm{Hom}(S, X)$ we define $h_X(\phi)(f) = f \cdot \phi$.

The next proposition tells us that the functors h_X and h^a_X, for a given scheme X, are not really different, but carry the same information.

Proposition 3.3. *The functor of points h_X of a scheme X is completely determined by its restriction to the category of affine schemes or equivalently by the functor*

$$h^a_X : (\text{rings}) \longrightarrow (\text{sets}), \qquad h^a_X(A) = Hom(\underline{\mathrm{Spec}}A, X).$$

Proof. See [1] ch. VI. ■

Example 3.4. *The affine space.* Let $\mathbf{A}^n$ be the affine space over a field k; its coordinate ring is $k[x_1 \dots x_n]$, the ring of polynomials. Its functor of points is by definition $h^a_{\mathbf{A}^n} : (\text{rings}) \longrightarrow (\text{sets})$, $h^a_{\mathbf{A}^n}(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}A, \mathbf{A}^n)$. Since a morphism of two affine varieties corresponds contravariantly to a morphisms of their coordinate rings (see observation 2.3), we have that $\mathrm{Hom}(\underline{\mathrm{Spec}}A, \mathbf{A}^n)$ $= \mathrm{Hom}(k[x_1 \dots x_n], A)$. Any morphism $\phi : k[x_1 \dots x_n] \longrightarrow A$ is determined by the knowledge of $\phi(x_1) = a_1, \dots \phi(x_n) = a_n$, $a_i \in A$, hence the choice of such morphism ϕ corresponds to the choice of an n-uple $(a_1 \dots a_n)$, $a_i \in A$. So, we can identify $h_{\mathbf{A}^n}(A)$ with the set of n-uples $(a_1 \dots a_n)$ with entries in A. If $A = \mathbb{Z}$ or $A = \mathbb{Q}$, this is the notion we already encounter in classical algebraic geometry.

The functor of points, originally introduced as a tool in algebraic geometry, can actually be employed in a much wider context.

Definition 3.5. Let $M = (|M|, \mathcal{O}_M)$ be a locally ringed space and let (rspaces) denote the category of locally ringed spaces. We define the *functor of points of the locally ringed space M* as the representable functor:

$$h_M : (\text{rspaces})^o \longrightarrow (\text{sets}), \quad h_M(T) = \mathrm{Hom}(T, M).$$

If the locally ringed space M is a differentiable manifold, we have the following important characterization of morphisms.

Proposition 3.6. *Let M and N be differentiable manifolds. Then:*

$$Hom(M, N) \cong Hom(C^\infty(N), C^\infty(M)).$$

We are now going to state Yoneda's Lemma, a basic categorical result. As an immediate consequence, we have that the functor of points of a scheme (resp. differentiable manifold) does determine the scheme (resp. differentiable manifold) itself.

Theorem 3.7. *Yoneda's Lemma.*
Let $\mathcal{C}$ be a category and let X, Y be objects in $\mathcal{C}$ and let $h_X : \mathcal{C}^o \longrightarrow (\text{sets})$ be the representable functor defined on the objects as $h_X(T) = Hom(T, X)$ and, as usual, on the arrows as $h_X(\phi)(f) = f \cdot \phi$, for $\phi : T \longrightarrow S$, $f \in Hom(T, S)$.

1. If $F : \mathcal{C}^o \longrightarrow (\text{sets})$, then we have a one to one correspondence between the sets:

$$\{h_X \longrightarrow F\} \quad \Longleftrightarrow \quad F(X)$$

2. The functor

$$h : \mathcal{C} \longrightarrow Fun(\mathcal{C}^o, (\text{sets}))$$

$$X \mapsto h_X$$

is an equivalence of $\mathcal{C}$ with a full subcategory of functors. In particular, $h_X \cong h_Y$ if and only if $X \cong Y$ and the natural transformations $h_X \longrightarrow h_Y$ are in one to one correspondence with the morphisms $X \longrightarrow Y$.

Proof. We briefly sketch it, leaving the details to the reader. Let $\alpha : h_X \longrightarrow F$. We can associate to α, $\alpha_X(\text{id}_X) \in F(X)$. Vice-versa, if $p \in F(X)$, we associate to p, $\alpha : h_X \longrightarrow F$ such that

$$\alpha_Y : \text{Hom}(Y, X) \longrightarrow F(Y), \qquad f \longmapsto F(f)p.$$

■

Corollary 3.8. *Two schemes (resp. manifolds) are isomorphic if and only if their functor of points are isomorphic.*

The advantages of using the functorial language are many. Morphisms of schemes are just maps between the sets of their A-points, respecting functorial properties. This often simplifies matters, allowing to leave the sheaves machinery on the background.

4 Coherent sheaves

In this section we briefly give the definition and some basic properties of coherent sheaves. This notion is introduced in ordinary algebraic geometry in order to characterize sheaves which have good properties and in general are well behaved. As we shall see, it is one of the building blocks for our definition of superscheme.

Let A be a commutative ring and M an A-module. We want to define a sheaf $\widetilde{M}$ on $\underline{\mathrm{Spec}}A$, which has an $\mathcal{O}_A$-module structure, i. e. for all open sets U in $\underline{\mathrm{Spec}}A$, we want $\widetilde{M}(U)$ to have an $\mathcal{O}_A(U)$-module structure compatibible with respect to the restriction morphisms. We are going to define the sheaf on the basic open sets U_f introduced in the previous section.

Let us consider the assignment:

$$U_f \longmapsto M_f$$

where $M_f = M[f^{-1}]$ is the A_f-module obtained by M by inverting just the element $f \in A$. This assignment extends uniquely to a sheaf on $\underline{\mathrm{Spec}}A$, that we denote with $\widetilde{M}$. Next proposition summarizes the properties of $\widetilde{M}$.

Proposition 4.1. *Let M be a module for a commutative algebra A. The sheaf $\widetilde{M}$ defined above has the following properties:*
1. $\widetilde{M}$ is a $\mathcal{O}_A$-module;
2. $(\widetilde{M})_\mathfrak{p} \cong M_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}A$, i.e. the stalk at any prime $\mathfrak{p}$ of the sheaf $\widetilde{M}$ coincides with the localization of M at $\mathfrak{p}$;
3. $(\widetilde{M})(\mathrm{Spec}A) = M$, i.e. the global sections of the sheaf coincide with the A-module M.

Proof. See proposition 5.1 ch. II in [3]. ∎

Definition 4.2. Let $X = (|X|, \mathcal{O}_X)$ be a scheme, $\mathcal{F}$ a sheaf on $|X|$. We say $\mathcal{F}$ is *quasi-coherent*, if there exists an open affine cover $\{U_i = \underline{\mathrm{Spec}}A_i\}_{i\in I}$ of X such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for a suitable A_i-module M_i. $\mathcal{F}$ is *coherent* if the affine cover can be chosen so that the M_i's are finitely generated A_i-modules.

We are going to give an example of quasi-coherent sheaf, which is most important for our algebraic supergeometry applications.

Example 4.3. Let R be a commutative super ring. R_0 is a commutative ring in the ordinary sense. Since R_1 is an R_0-module, the whole R is also an R_0-module, hence we can construct the quasi-coherent sheaf $\widetilde{R}$ on the topological space SpecR_0. One can easily check that $(\mathrm{Spec}R_0, \widetilde{R})$ is a locally ringed space. We shall see that this is the local model for our definition of superscheme.

The next proposition establishes an important equivalence of categories.

Proposition 4.4. *Let A be a commutative ring. The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_A$-modules. The inverse of this functor is the functor: $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}A)$. If A is noetherian, the same functor gives an equivalence of categories between the category of finitely generated A-modules and the coherent sheaves which are $\mathcal{O}_A$-modules.*

Proof. See corollary 5.5 ch. II in [3]. ■

5 Appendix: Categories and functors

We want to make a brief summary of formal properties and definitions relative to categories.

Definition 5.1. A *category* $\mathcal{C}$ consists of a collection of objects, denoted by $Ob(\mathcal{C})$, and sets of *morphisms* between objects. For all pairs $A, B \in Ob(\mathcal{C})$, we denote the set of morphisms from A to B by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ so that for all $A, B, C \in \mathcal{C}$, there exists an association

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$

called the "composition law" $((f, g) \to f \circ g)$ which satisfies the properties
(i) the law "$\circ$" is associative,
(ii) for all $A, B \in Ob(\mathcal{C})$, there exists $id_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ so that we get $f \circ id_A = f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $id_A \circ g = g$ for all $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$,
(iii) $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{C}}(A', B')$ are disjoint unless $A = A'$, $B = B'$ in which case they are equal.

If a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is invertible, in other words there exist another morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $f \circ g$ and $g \circ f$ are the identities respectively in $\mathrm{Hom}_{\mathcal{C}}(B, B)$ and $\mathrm{Hom}_{\mathcal{C}}(A, A)$, we say that f is an *isomorphism*.

Once the category is understood, it is conventional to write $A \in \mathcal{C}$ instead of $A \in Ob(\mathcal{C})$ for objects. We may also suppress the "$\mathcal{C}$" from $\mathrm{Hom}_{\mathcal{C}}$ and just write Hom whenever there is no danger of confusion.

Essentially a category is a collection of objects which share some basic structure, along with maps between objects which preserve that structure.

Example 5.2. 1. Let (sets) denote the category of sets. The objects are the sets, and for any two sets $A, B \in Ob((\text{sets}))$, the morphisms are the maps from A to B.

2. Let $\mathcal{G}$ denote the category of groups. Any object $G \in \mathcal{G}$ is a group, and for any two groups $G, H \in Ob(\mathcal{G})$, the set $\mathrm{Hom}_{\mathcal{G}}(G, H)$ is the set of group homomorphisms from G to H.

Definition 5.3. A category $\mathcal{C}'$ is a *subcategory* of category $\mathcal{C}$ if $Ob(\mathcal{C}') \subset Ob(\mathcal{C})$ and if for all $A, B \in \mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B)$ so that the composition law "$\circ$" on $\mathcal{C}'$ is induced by that on $\mathcal{C}$.

Example 5.4. The category $\mathcal{A}$ of *abelian groups* is a subcategory of the category of groups $\mathcal{G}$.

Definition 5.5. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. Then a *covariant [contravariant] functor* $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ consists of
(1) a map $F : Ob(\mathcal{C}_1) \longrightarrow Ob(\mathcal{C}_2)$ and
(2) a map (denoted by the same F) $F : \mathrm{Hom}_{\mathcal{C}_1}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}_2}(F(A), F(B))$ $[F : \mathrm{Hom}_{\mathcal{C}_1}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}_2}(F(B), F(A))]$ so that
(i) $F(id_A) = id_{F(A)}$ and
(ii) $F(f \circ g) = F(f) \circ F(g)$ $[F(f \circ g) = F(g) \circ F(f)]$
for all $A, B \in Ob(\mathcal{C}_1)$.

When we say "functor" we mean covariant functor. A contravariant functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is the same as a covariant functor from $\mathcal{C}_1^o \longrightarrow \mathcal{C}_2$ where $\mathcal{C}_1^o$ denotes the *opposite* category i. e. the category where all morphism arrows are reversed.

Definition 5.6. Let F_1, F_2 be two functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. We say that there is a *natural transformation* of functors $\varphi : F_1 \longrightarrow F_2$ if for all $A \in \mathcal{C}_1$ there is a set of morphisms $\varphi_A : F_1(A) \longrightarrow F_2(A)$ so that for any $f \in \text{Hom}_{\mathcal{C}_1}(A, B)$ ($B \in \mathcal{C}_1$), the following diagram commutes:

$$\begin{array}{ccc} F_1(A) & \xrightarrow{\varphi_A} & F_2(A) \\ {\scriptstyle F_1(f)} \downarrow & & \downarrow {\scriptstyle F_2(f)} \\ F_1(B) & \xrightarrow{\varphi_B} & F_2(B). \end{array} \tag{1}$$

We say that the family of functions φ_A is *functorial* in A.

We say that two functors $F, G : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are *isomorphic* if there exist two natural transformations $\phi : F \longrightarrow G$ and $\psi : G \longrightarrow F$ such that $\phi \circ \psi = \text{id}$ and $\psi \circ \phi = \text{id}$.

The functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ for any two given categories together with their natural transformations form a category.

The notion of equivalence of categories is important since it allows to identify two categories which are apparently different.

Definition 5.7. We say that two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there exists two functors $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ and $G : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ such that $FG \cong id_{\mathcal{C}_2}$, $GF \cong id_{\mathcal{C}_1}$ (where $id_{\mathcal{C}}$ denotes the identity functor of a given category, defined in the obvious way).

If F is a functor from the category $\mathcal{C}_1$ to the category $\mathcal{C}_2$, for any two objects A, $B \in \mathcal{C}_1$, by its very definition F induces a function (that we denoted previously with F):

$$F_{A,B} : \text{Hom}_{\mathcal{C}_1}(A, B) \longrightarrow \text{Hom}_{\mathcal{C}_2}(F(A), F(B)).$$

Definition 5.8. Let F be a functor. We say that F is *faithful* if $F_{A,B}$ is injective, we say F is *full* if $F_{A,B}$ is surjective and we say that F is *fully faithful* if $F_{A,B}$ is bijective.

Next we want to formally define what it means for a functor to be *representable*. Let us first define the representation functors.

Definition 5.9. Let $\mathcal{C}$ be a category, A a fixed object in $\mathcal{C}$. We define the two *representation functors* Hom^A, Hom_A as

$$\begin{array}{ll} \text{Hom}^A : \mathcal{C}^o \longrightarrow (\text{sets}), & B \mapsto \text{Hom}_{\mathcal{C}}(B, A) \\ \text{Hom}_A : \mathcal{C} \longrightarrow (\text{sets}), & B \mapsto \text{Hom}_{\mathcal{C}}(A, B) \end{array}$$

where (sets) denotes the category of sets. On the arrow $f \in \mathrm{Hom}(B, C)$ we have:

$$\mathrm{Hom}^A(f)\phi = \phi \circ f, \quad \phi \in \mathrm{Hom}^A(B) \qquad \mathrm{Hom}_A(f)\psi = f \circ \psi, \quad \psi \in \mathrm{Hom}_A(B).$$

Definition 5.10. Let F be a functor from the category $\mathcal{C}$ to the category of sets. We say that F is *representable by* $X \in \mathcal{C}$ if for all $A \in \mathcal{C}$, $F \cong \mathrm{Hom}_A$ or $F \cong \mathrm{Hom}^A$.

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