Lecture 3
Supergeometry

1 Superspaces

As we have seen in the previous lecture, a unified way to look at the categories of ordinary differentiable manifolds or algebraic schemes, is to think of an object as a pair, consisting of a topological space together with a sheaf of functions defined on it. Such a pair is often referred to as a *ringed space*. For ordinary manifolds, for example, the sheaf of functions is the sheaf of the C^∞ functions, while for ordinary algebraic varieties, is the sheaf of regular functions, as we have seen in the previous chapter. Our aim is to generalize this point of view, introducing supermanifolds and superschemes as of *superspaces*, which are the supergeometric counterpart of locally ringed spaces.

Definition 1.1. A *super ringed space* is a topological space $|S|$ endowed with a sheaf of commutative super rings, called the *structure sheaf* of S, which we denote by $\mathcal{O}_S$. Let S denote the super ringed space $(|S|, \mathcal{O}_S)$.

Notice that $S_0 = (|S|, \mathcal{O}_{S,0})$ is an ordinary ringed space where $\mathcal{O}_{S,0}(U) := \mathcal{O}_S(U)_0$ is a sheaf of ordinary rings on $|S|$. Notice also that $\mathcal{O}_{S,1}(U) := \mathcal{O}_S(U)_1$ defines a sheaf of $\mathcal{O}_{S,0}$-modules on $|S|$, i.e. for all open sets U in $|S|$, we have that $\mathcal{O}_{S,1}(U)$ is an $\mathcal{O}_{S,0}(U)$-module and this structure is compatible with the restriction morphisms.

Definition 1.2. A *superspace* is a super ringed space S with the property that the stalk $\mathcal{O}_{S,x}$ is a local super ring for all $x \in |S|$.

As in the ordinary setting a commutative super ring is *local* if it has a unique maximal ideal. Notice that any prime ideal in a commutative super ring must contain the whole odd part, since it contains all nilpotents.

Ordinary differentiable manifolds and algebraic schemes are examples of superspaces, where we think their sheaves of functions as sheaves of commutative super rings with trivial odd part.

Let us now see an example of a superspace with non trivial odd part.

Example 1.3. Let M be a differentiable manifold, $|M|$ its underlying topological space, C^∞_M the sheaf of ordinary C^∞ functions on M. We define the sheaf of supercommutative $\mathbf{R}$-algebras as: (for $V \subset M$ open)

$$V \longmapsto \mathcal{O}_M(V) := C^\infty_M(V)[\theta^1, \dots, \theta^q],$$

where $C^\infty_M(V)[\theta^1, \dots, \theta^q] = C^\infty_M(V) \otimes \wedge(\theta^1, \dots, \theta^q)$ and the θ^j are odd (anticommuting) indeterminates. As one can readily check, $(|M|, \mathcal{O}_M)$ is a super ringed space; moreover $(|M|, \mathcal{O}_M)$ is also a superspace. In fact $\mathcal{O}_{M,x}$ is a local ring, with maximal ideal $m_{M,x}$ generated by the maximal ideal of the local ring $C^\infty_{M,x}$ and the odd elements $\theta^1, \dots, \theta^q$. One can check immediately that all the elements in $\mathcal{O}_{M,x} \setminus m_{M,x}$ are invertible.

In the special case $M = \mathbf{R}^p$, we define the superspace

$$\mathbf{R}^{p|q} = (\mathbf{R}^p, C^\infty_{\mathbf{R}^p}[\theta^1 \dots \theta^q]).$$

From now on, with an abuse of notation, $\mathbf{R}^{p|q}$ denotes both the super vector space $\mathbf{R}^p \oplus \mathbf{R}^q$ and the superspace $(\mathbf{R}^p, C^\infty_{\mathbf{R}^p}[\theta^1 \dots \theta^q])$, the context making clear which one we mean. $\mathbf{R}^{p|q}$ plays a key role in the definition of supermanifold, since it is the local model. If $t^1 \dots t^p$ are global coordinates for $\mathbf{R}^p$ we shall speak of $t^1 \dots t^p, \theta^1 \dots \theta^q$ as a set of *global coordinates* for the superspace $\mathbf{R}^{p|q}$.

Definition 1.4. Let $S = (|S|, \mathcal{O}_S)$ be a superspace. Given an open subset $|U| \subset |S|$, the pair $(|U|, \mathcal{O}_S|_{|U|})$ is always a superspace, called the *open subspace* associated to $|U|$.

Example 1.5. Let $\mathrm{M}_{p|q} = \mathbf{R}^{p^2+q^2|2pq}$. This is the superspace corresponding to the super vector space of $p|q \times p|q$ matrices, the underlying topological space being $\mathrm{M}_p \times \mathrm{M}_q$, the direct product of $p \times p$ and $q \times q$ matrices. As super vector space

$$\mathrm{M}_{p|q} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}, \quad (\mathrm{M}_{p|q})_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad (\mathrm{M}_{p|q})_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$$

where A, B, C, D are respectively $p \times p$, $p \times q$, $q \times p$, $q \times q$ matrices with entries in $\mathbf{R}$.

Hence as a superspace $\mathrm{M}_{p|q}$ has $p^2 + q^2$ even global coordinates t^{ij}, $1 \leq i, j \leq p$ or $p+1 \leq i, j \leq p+q$ and $2pq$ odd ones θ^{kl}, $1 \leq k \leq p$, $p+1 \leq l \leq p+q$ or $p+1 \leq k \leq p+q$, $1 \leq l \leq p$. The structure sheaf of $\mathrm{M}_{p|q}$ is the assignment

$$V \longmapsto C^\infty_{\mathrm{M}_p \times \mathrm{M}_q}(V)[\theta^{kl}], \qquad \text{for all} \quad V \quad \text{open in} \quad \mathrm{M}_p \times \mathrm{M}_q.$$

Now, let us consider in the topological space $\mathrm{M}_p \times \mathrm{M}_q = \mathbf{R}^{p^2+q^2}$, the open set U consisting of the points for which $\det(t_{ij})_{1\leq i,j\leq p} \neq 0$ and $\det(t_{ij})_{p+1\leq i,j\leq p+q} \neq 0$. We define the superspace $\mathrm{GL}_{p|q} := (U, \mathcal{O}_{\mathrm{M}_{p|q}}|_U)$, the open subspace of $\mathrm{M}_{p|q}$ associated to the open set U. As we shall see, this superspace has a Lie supergroup structure and it is called the *general linear supergroup*.

Next, we define a morphism of superspaces, so that we can talk about the category of superspaces.

Definition 1.6. Let S and T be superspaces. Then a morphism $\varphi : S \longrightarrow T$ is a continuous map $|\varphi| : |S| \longrightarrow |T|$ together with a sheaf morphism $\varphi^* : \mathcal{O}_T \longrightarrow \varphi_* \mathcal{O}_S$, so that $\varphi_x^*(\mathfrak{m}_{|\varphi|(x)}) \subset \mathfrak{m}_x$ where $\mathfrak{m}_{S,x}$ is the maximal ideal in $\mathcal{O}_{S,x}$, while $\mathfrak{m}_{T,\varphi(x)}$ is the maximal ideal in $\mathcal{O}_{T,\varphi(x)}$, and φ_x^* is the stalk map.

Remark 1.7. Recall from the previous chapter that the sheaf morphism $\varphi^* : \mathcal{O}_T \longrightarrow \varphi_* \mathcal{O}_S$ corresponds to the system of maps $\varphi_U^* : \mathcal{O}_T(U) \longrightarrow \mathcal{O}_S(|\varphi|^{-1}(U))$ for all open sets $U \subset |T|$. To ease notation, we also refer to the maps φ_U^* as φ^*.

Essentially the condition $\varphi_x^*(\mathfrak{m}_{|\varphi|(x)}) \subset \mathfrak{m}_x$ means that the sheaf homomorphism is local. Note also that φ^* is a morphism of supersheaves, so, as usual, it preserves the parity. The main point to make here is that, when we are giving a morphism of superspaces, the sheaf morphism must be specified along with the continuous topological map, since sections are not necessarily genuine functions on the topological space as in ordinary differential geometry. An arbitrary section cannot be viewed as a function because commutative super rings have many nilpotent elements, and nilpotent sections are identically zero as functions on the underlying topological space. Therefore we employ the methods of algebraic geometry to study such objects. We will address this in more detail later. Now we introduce two types of superspaces that we examine in detail in the forthcoming chapters: supermanifolds and superschemes.

2 Supermanifolds

A supermanifold is a specific type of superspace, which we describe via a local model, namely it is locally isomorphic to the superspace $\mathbf{R}^{p|q}$ introduced previously. Let us now see in detail the definition.

Let C^∞_U be the sheaf of C^∞-functions on the domain $U \subset \mathbf{R}^p$. We define the *superdomain* $U^{p|q}$ to be the superspace $(U, C^\infty_U[\theta^1, \dots, \theta^q])$ where $C^\infty_U[\theta^1, \dots, \theta^q] = C^\infty_{\mathbf{R}^p}|_U \otimes \wedge(\theta^1, \dots, \theta^q)$. Most immediately, the superspaces $\mathbf{R}^{p|q}$ are superdomains with sheaf $C^\infty_{\mathbf{R}^p}[\theta^1, \dots, \theta^q]$.

Definition 2.1. A *supermanifold* $M = (|M|, \mathcal{O}_M)$ of dimension $p|q$ is a superspace which is locally isomorphic to $\mathbf{R}^{p|q}$. In other words, given any point $x \in |M|$, there exists a neighborhood $V \subset |M|$ of x with q odd indeterminates θ^j so that

$$V \cong V_0 \quad \text{open in } \mathbf{R}^p, \qquad \mathcal{O}_M|_V \cong \underbrace{C^\infty(t^1, \dots, t^p)}_{C^\infty_{\mathbf{R}^p}|_{V_0}}[\theta^1, \dots, \theta^q]. \tag{1}$$

We call $t^1 \dots t^p, \theta^1 \dots \theta^q$ the *local coordinates* of M in V and $p|q$ the *superdimension* of the supermanifold M.

Morphisms of supermanifolds are morphisms of the underlying superspaces. For M, N supermanifolds, a morphism $\varphi : M \longrightarrow N$ is a continuous map $|\varphi| : |M| \longrightarrow |N|$ together with a local morphism of sheaves of superalgebras $\varphi^* : \mathcal{O}_N \longrightarrow \varphi_* \mathcal{O}_M$, where *local* means that $\varphi_x^{-1}(m_{M,x}) = m_{N,|\varphi|(x)}$, where $\varphi_x : \mathcal{O}_{N,|\varphi|(x)} \longrightarrow \mathcal{O}_{M,x}$ is the stalk morphism, for a point $x \in |M|$, and $m_{M,x}$, $m_{N,|\varphi|(x)}$ are the maximal ideals in the stalks. Note that in the purely even case of ordinary C^∞-manifolds, the above notion of a morphism agrees with the ordinary one. We may therefore talk about the category of supermanifolds. The difficulty in dealing with C^∞-supermanifolds arises when one tries to think of "points" or "functions" in the traditional sense. The ordinary points only account for the topological space and the underlying sheaf of ordinary C^∞-functions, and one may truly only talk about the "value" of a section $f \in \mathcal{O}_M(U)$ for $U \subset |M|$ an open subset; the value of f at $x \in U$ is the unique real number c so that $f - c$ is not invertible in any neighborhood of x. For concreteness, let us consider the example of $M = \mathbf{R}^{1|1}$, with global coordinates t and θ. Let us take the global section $f = t\theta \in \mathcal{O}_M(\mathbf{R})$. For any non zero real number c, we have that $t\theta - c$ is always invertible, since $t\theta$ is nilpotent, its inverse being $-c^{-2}t\theta - c^{-1}$. Hence the value of $t\theta$ at all points $x \in \mathbf{R} = |\mathbf{R}^{1|1}|$ is zero. What this says is that we cannot reconstruct a section by knowing only its values at topological points. Now that we understand this point we shall follow the established notation and call the sections *"functions on U"*.

Remark 2.2. Let M be a supermanifold, U an open subset in $|M|$, and f a function on U. If $\mathcal{O}_M(U) = C^\infty(t^1, \dots, t^p)[\theta^1, \dots, \theta^q]$ as in (1), there exist even functions $f_I \in C^\infty(t)$ $(t = t^1, \dots t^p))$ so that

$$f(t,\theta) = f_0(t) + \sum_i f_i(t)\theta^i + \sum_{i<j} f_{ij}(t)\theta^i\theta^j + \dots = f_0(t) + \sum_{|I|=1}^{q} f_I(t)\theta^I \quad (2)$$

where $I = \{i_1 < i_2 < \dots < i_r\}_{r=1}^q$.

So in some sense, we can expand $f(t,\theta)$ in power series, with respect to the odd coordinates θ^j's.

3 Superschemes

The other most important example of superspace is given by the *spectrum of a superalgebra A*. It consists of the spectrum of the even part A_0 together with a certain sheaf of superalgebras on it. Such superspaces will be our building blocks for superschemes. Let us see this construction in detail.

All superalgebras are assumed to be commutative and over the ground field k. Their category is denoted by (salg).

Definition 3.1. The superspace $\underline{\mathrm{Spec}}A$.

Let A be an object of (salg).

Let us consider $\mathcal{O}_{A_0}$ the structure sheaf of $\mathrm{Spec}(A_0)$. The stalk of this sheaf at the prime $\mathfrak{p} \in \mathrm{Spec}(A_0)$ is the localization of A_0 at $\mathfrak{p}$. As for any superalgebra, A is a module over A_0. So, according to the classical construction detailed in the previous lecture, we have a sheaf, that we shall customarily denote with $\mathcal{O}_A$, of $\mathcal{O}_{A_0}$-modules over $\mathrm{Spec}A_0$, with stalk $A_\mathfrak{p}$, the localization of the A_0-module A over each prime $\mathfrak{p} \in \mathrm{Spec}(A_0)$.

$$A_\mathfrak{p} = \left\{ \frac{f}{g} \quad | \quad f \in A, g \in A_0 - \mathfrak{p} \right\}.$$

The localization $A_\mathfrak{p}$ has a unique two-sided maximal ideal which consists of the maximal ideal in the local ring $(A_\mathfrak{p})_0$ and the generators of $(A_\mathfrak{p})_1$ as A_0-module. For more details on this construction see ch. 2 [4].

Hence the pair $(\mathrm{Spec}A_0, \mathcal{O}_A)$ is a superspace that we will denote with $\underline{\mathrm{Spec}}A$.

A *superscheme* is an object in the category of superspaces which generalizes the notion of an ordinary scheme.

Definition 3.2. An *affine superscheme* is a superspace that is isomorphic to $\underline{\mathrm{Spec}}A$ for some superalgebra A in (salg). Such superalgebra A, by the very construction of $\underline{\mathrm{Spec}}A$ is isomorphic to the algebra of the global sections of the structure sheaf of X, that we shall denote with $\mathcal{O}(X)$ (instead of the more cumbersome notation $\mathcal{O}_X(|X|)$). A *superscheme* is a superspace that is locally isomorphic to an affine superscheme (the affine superscheme may vary with the point).

Example 3.3. 1. *Affine superscheme* $\mathbf{A}^{m|n}$.

Consider the polynomial superalgebra $k[x_1 \dots x_m, \xi_1 \dots \xi_n]$ over an algebraically closed field k where $x_1 \dots x_m$ are even indeterminates and $\xi_1 \dots \xi_n$ are odd indeterminates. We will call $\underline{\mathrm{Spec}}k[x_1 \dots x_m, \xi_1 \dots \xi_n]$ the affine superscheme of superdimension $m|n$ and we denote it by $\mathbf{A}^{m|n}$.

The a topological space underlying $\mathbf{A}^{m|n}$ is $\mathrm{Spec}k[x_1 \dots x_m, \xi_1 \dots \xi_n]_0$ and consists of the even maximal ideals

$$(x_i - a_i, \xi_j \xi_k), \qquad i = 1 \dots m, \quad j, k = 1 \dots n$$

and the even prime ideals

$$(p_1 \dots p_r, \xi_j \xi_k), \qquad i = 1 \dots m, \quad j, k = 1 \dots n$$

where $(p_1 \dots p_r)$ is a prime ideal in $k[x_1 \dots x_m]$. In other words the prime ideals in $k[x_1 \dots x_m, \xi_1 \dots \xi_n]_0$ are generated by the prime ideals in $k[x_1 \dots x_m]$ and the even nilpotent ideal $(\xi_i \xi_j, i \leq j)$.

The stalk of the structure sheaf of $\mathbf{A}^{m|n}$ at the prime ideal $\mathfrak{p} \in \mathrm{Spec}k[x_1 \dots x_m, \xi_1 \dots \xi_n]_0$ is:

$$\mathcal{O}_{\mathbf{A}^{m|n},\mathfrak{p}} = \left\{ \frac{f}{g} \quad | \quad f \in \mathcal{O}_{\mathbf{A}^{m|n}}(\mathbf{A}^m), \quad g \in \mathcal{O}_{\mathbf{A}^{m|n}}(\mathbf{A}^m)_0, \quad g \notin \mathfrak{p} \right\}.$$

2. *Superscheme over the sphere* S^2.

Consider the polynomial superalgebra generated over an algebraically closed field k $k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]$, and the ideal

$$I = (x_1^2 + x_2^2 + x_3^2 - 1,\ x_1 \cdot \xi_1 + x_2 \cdot \xi_2 + x_3 \cdot \xi_3).$$

Let $k[X] = k[x_1, x_2, x_3, \xi_1, \xi_2, \xi_3]/I$ and $X = \underline{\mathrm{Spec}} k[X]$. A maximal ideal in $k[X]_0$ is given by:

$$\mathbf{m} = (x_i - a_i,\ \xi_i \xi_j) \quad \text{with} \ \ i, j = 1, 2, 3, \ \ a_i \in k \ \text{ and } \ a_1^2 + a_2^2 + a_3^2 = 1.$$

the local ring of $k[X]_0$ at the maximal ideal $\mathbf{m}_0$ is the ring of fractions:

$$(k[X]_0)_{\mathbf{m}} = \{\frac{f}{g} \ / \ f, g \in k[X]_0, \ g \notin \mathbf{m}\}.$$

The stalk of the structure sheaf at $\mathbf{m}$ is the localization of $k[X]$ as a $k[X]_0$-module, that is

$$k[X]_{\mathbf{m}} = \{\frac{m}{g} \ / \ m \in k[X], \ g \in k[X]_0, \ g \notin \mathbf{m}\}.$$

Notice that if $a_1 \neq 0$ (not all a_i are zero simultaneously), then x_1 is invertible in the localization and we have

$$\xi_1 = \frac{1}{x_1}(x_2\xi_2 + x_3\xi_3),$$

so $\{\xi_2, \xi_3\}$ generate $k[X]_{\mathbf{m}}$ as an $\mathcal{O}_{k[X]_0}$-module.

3. *The matrix superscheme and the general linear supergroup scheme* Let $\mathrm{M}_{p|q}^{alg} = \mathbf{A}^{p^2+q^2|2pq}$ be the affine superspace corresponding to the super vector space of $p|q \times p|q$ matrices with entries in the field k. The underlying topological space of $\mathrm{M}_{p|q}^{alg}$ is the product $\mathrm{M}_p^{alg} \times \mathrm{M}_q^{alg}$, where M_p^{alg} denotes the set of $p \times p$ matrices with entries in k, with the Zariski topology. The super ring of global sections of $\mathrm{M}_{p|q}^{alg}$ is $k[t^{ij}, \theta^{kl}]$, with $1 \leq i, j \leq p$ or $p+1 \leq i, j \leq p+q$ and $1 \leq k \leq p$, $p+1 \leq l \leq p+q$ or $p+1 \leq k \leq p+q$, $1 \leq l \leq p$. The conditions $\det(t_{ij})_{1 \leq i,j \leq p} \neq 0$ and $\det(t_{ij})_{p+1 \leq i,j \leq p+q} \neq 0$ define a Zariski open set U in $\mathrm{M}_p^{alg} \times \mathrm{M}_q^{alg}$, hence we have a superspace $\mathrm{GL}_{p|q}^{alg} = (U, \mathcal{O}_{\mathrm{M}_{p|q}}|_U)$. One can check immediately that this is a superscheme. From now on we shall drop the suffix *alg* to improve readability, the context making clear if we are considering the general linear supergroup $\mathrm{GL}_{p|q}$ in the algebraic or in the differential context.

As in the ordinary setting, the category of affine superschemes is equivalent to the category of superalgebras, in other words, the superscheme $X = \underline{\mathrm{Spec}} A$ and the superalgebra A contain the same information. This is the generalization of the equivalence between the category of affine schemes and the category of algebras and the proof is the same as in the classical setting (see [3]).

Proposition 3.4. *There exists an equivalence of categories between the category of commutative superalgebras and the category of affine superschemes.*

Morphisms of superschemes are just morphisms of superspaces, so we may talk about the subcategory of superschemes. The category of superschemes is larger than the category of schemes; any scheme is a superscheme if we take a trivial odd component in the structure sheaf.

4 The functor of points

The presence of odd coordinates steals some of the geometric intuition away from the language of supergeometry. For instance, we cannot see an "odd point" – they are invisible both topologically and as classical functions on the underlying topological space. We see the odd points only as sections of the structure sheaf. To bring some of the intuition back, we turn to the functor of points approach from algebraic geometry.

Definition 4.1. Let S and T be superspaces. A *T-point* of S is a morphism $T \longrightarrow S$. We denote the set of all T-points by $S(T)$. Equivalently,

$$S(T) = \mathrm{Hom}(T, S).$$

We define the *functor of points* of the superspace S the functor:

$$S : (\text{sspaces})^o \longrightarrow (\text{sets}), \qquad T \longmapsto S(T)$$

where (sspaces) denotes the category of superspaces.

By a common abuse of notation the superspace S and the functor of points of S are denoted with the same letter. Whenever is necessary to make a distinction, we shall write h_S for the functor of points of the superspace S.

We have defined the functor of points of a superspace; clearly we can also define the functor of points of a supermanifold or a superscheme, just by changing the category we start from.

Definition 4.2. Let (smflds) and (sschemes) denote respectively the categories of supermanifolds and superschemes introduced above. We define the *functor of points of the supermanifold M* the functor:

$$M : (\text{smflds})^o \longrightarrow (\text{sets}), \qquad T \longmapsto M(T).$$

Similarly we define the *functor of points of the superscheme* X the functor:

$$X : (\text{sschemes})^o \longrightarrow (\text{sets}), \qquad T \longmapsto X(T).$$

The importance of the functor of points is a consequence of the following lemma, which is one of the many versions of Yoneda's Lemma.

Lemma 4.3. *(Yoneda's Lemma) Let M and N be two superspaces (resp. supermanifolds or superschemes). There is a bijection from the set of morphisms $\varphi : M \longrightarrow N$ to the set of maps $\varphi_T : M(T) \longrightarrow N(T)$, functorial in T. In particular M and N are isomorphic if and only if their functor of points are isomorphic.*

Proof. Given a map $\varphi : M \longrightarrow N$, for any morphism $t : T \longrightarrow M$, $\varphi \circ t$ is a morphism $T \longrightarrow N$. Conversely, we attach to the system (φ_T) the image of the identity map from $\varphi_M : M(M) \longrightarrow N(M)$. For more details see the appendix.

■

Yoneda's lemma allows us to replace a superspace (resp. a supermanifold or a superscheme) S with its set of T-points, $S(T)$. We can now think of S as a representable functor from the category of superspaces (resp. supermanifolds or superschemes) to the category of sets. In fact, when constructing a superspace, it is often most convenient to construct the functor of points and then prove that the functor is *representable* in the appropriate category.

The following proposition is very useful when we want to explicitly describe the functor of points of a supermanifold or a superscheme. It's very formulation shows how the functorial treatment allows us to deal at once with both the differential and algebraic categories.

Remark 4.4. To ease the notation we write $\mathcal{O}_T(T)$ or simply $\mathcal{O}(T)$ for the global sections of a superspace T.

Proposition 4.5. *Let $M = (|M|, \mathcal{O}_M)$ and $T = (|T|, \mathcal{O}_T)$ be supermanifolds or affine superschemes. Then*

$$Hom(T, M) = Hom(\mathcal{O}(T), \mathcal{O}(M)).$$

Let us give some examples of T-points.

Example 4.6. (i) Let T be just an ordinary topological point viewed as supermanifold, i.e. $T = \mathbf{R}^{0|0} = (\mathbf{R}^0, \mathbf{R})$. By definition a T-point of a manifold M is a morphism $\phi : \mathbf{R}^{0|0} \longrightarrow M$. ϕ consists of a continuous map $|\phi| : \mathbf{R}^0 \longrightarrow |M|$, which corresponds to the choice of a point x in the topological space $|M|$, and a sheaf morphism $\phi^* : \mathcal{O}_M \longrightarrow \phi_*(\mathbf{R})$. Then a T-point of M is an ordinary topological point of $|M|$.

(ii) Let M be the supermanifold $\mathbf{R}^{p|q}$ and let T be a supermanifold. By the previous proposition we have that a T-point of M corresponds to a morphism:

$$\mathcal{O}(M) = C^\infty(t^1 \dots t^p)[\theta^1 \dots \theta^q] \longrightarrow \mathcal{O}(T).$$

Then, in this case, a T-point of M is a choice of p even and q odd global sections on T. Thus $\mathbf{R}^{p|q}(T) = \mathcal{O}_{T,0}^p(T) \oplus \mathcal{O}_{T,1}^q(T) = (\mathcal{O}_T^{p|q}(T))_0$.

(iii) Let X be the superscheme $\mathbf{A}^{m|n}$ and let T be an affine superscheme. By definition, a T-point of X is a morphism of schemes $\phi : T \longrightarrow \mathbf{A}^{m|n}$, which again, by the previous proposition, corresponds to a super ring morphism $\psi : \mathcal{O}(\mathbf{A}^{m|n}) \longrightarrow \mathcal{O}(T)$, that is $\psi : k[x_1 \dots x_m, \xi_1 \dots \xi_n] \longrightarrow \mathcal{O}(T)$, where $k[x_1 \dots x_m, \xi_1 \dots \xi_n]$ denotes the polynomial algebra in the even (commuting) indeterminates $x_1 \dots x_m$ and in the odd (anticommuting) ones $\xi_1 \dots \xi_n$. Hence, as before, ψ amounts to a choice of m even global sections in $\mathcal{O}(T)$ and n odd ones:

$$\begin{aligned} \mathbf{A}^{m|n}(T) &= \mathcal{O}(T)_0^m \oplus \mathcal{O}(T)_1^n = \\ &= \{(a_1 \dots a_m, \alpha_1 \dots \alpha_n) \quad | \quad a_i \in \mathcal{O}(T)_0, \quad \alpha_j \in \mathcal{O}(T)_1\}. \end{aligned}$$

We already see the power of T-points in these examples. The first example ($T = \mathbf{R}^{0|0}$) gives us complete topological information, while the second ($M = \mathbf{R}^{p|q}$) allow us to talk about coordinates on supermanifolds. The third example shows how remarkably the differential and the algebraic categories resemble each other under the functorial treatment.

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