Lecture 3
Supergeometry

The concept of sheaf, which we have introduced in the previous lecture, allows us to provide a unified way to describe geometric objects. The notion of ringed space is perhaps the most general and useful for such purpose. A ringed space is of a pair, consisting of a topological space together with a sheaf of functions defined on it. For ordinary manifolds, for example, the sheaf of functions is the sheaf of the C^∞ functions. Our aim is to generalize this point of view, introducing supermanifolds as *superspaces*, which are the supergeometric counterpart of locally ringed spaces.

1 Superspaces

Definition 1.1. A *super ringed space* is a topological space $|S|$ endowed with a sheaf of commutative super rings, called the *structure sheaf* of S, which we denote by $\mathcal{O}_S$. Let S denote the super ringed space $(|S|, \mathcal{O}_S)$.

Notice that $S_0 = (|S|, \mathcal{O}_{S,0})$ is an ordinary ringed space where $\mathcal{O}_{S,0}(U) := \mathcal{O}_S(U)_0$ is a sheaf of ordinary rings on $|S|$. Notice also that $\mathcal{O}_{S,1}(U) := \mathcal{O}_S(U)_1$ defines a sheaf of $\mathcal{O}_{S,0}$-modules on $|S|$, i.e. for all open sets U in $|S|$, we have that $\mathcal{O}_{S,1}(U)$ is an $\mathcal{O}_{S,0}(U)$-module and this structure is compatible with the restriction morphisms.

Definition 1.2. A *superspace* is a super ringed space S with the property that the stalk $\mathcal{O}_{S,x}$ is a local super ring for all $x \in |S|$.

As in the ordinary setting a commutative super ring is *local* if it has a unique maximal ideal. Notice that any prime ideal in a commutative super ring must contain the whole odd part, since it contains all nilpotents.

Ordinary differentiable manifolds are examples of superspaces, where we think their sheaves of functions as sheaves of commutative super rings with trivial odd part.

Let us now see an example of a superspace with non trivial odd part.

Example 1.3. Let M be a differentiable manifold, $|M|$ its underlying topological space, C^∞_M the sheaf of ordinary C^∞ functions on M. We define the sheaf of supercommutative $\mathbf{R}$-algebras as: (for $V \subset M$ open)

$$V \longmapsto \mathcal{O}_M(V) := C^\infty_M(V)[\theta^1, \dots, \theta^q],$$

where $C^\infty_M(V)[\theta^1, \dots, \theta^q] = C^\infty_M(V) \otimes \wedge(\theta^1, \dots, \theta^q)$ and the θ^j are odd (anticommuting) indeterminates. As one can readily check, $(|M|, \mathcal{O}_M)$ is a super ringed space; moreover $(|M|, \mathcal{O}_M)$ is also a superspace. In fact $\mathcal{O}_{M,x}$ is a local ring, with maximal ideal $m_{M,x}$ generated by the maximal ideal of the local ring $C^\infty_{M,x}$ and the odd elements $\theta^1, \dots, \theta^q$. One can check immediately that all the elements in $\mathcal{O}_{M,x} \setminus m_{M,x}$ are invertible.

In the special case $M = \mathbf{R}^p$, we define the superspace

$$\mathbf{R}^{p|q} = (\mathbf{R}^p, C^\infty_{\mathbf{R}^p}[\theta^1 \dots \theta^q]).$$

From now on, with an abuse of notation, $\mathbf{R}^{p|q}$ denotes both the super vector space $\mathbf{R}^p \oplus \mathbf{R}^q$ and the superspace $(\mathbf{R}^p, C^\infty_{\mathbf{R}^p}[\theta^1 \dots \theta^q])$, the context making clear which one we mean. $\mathbf{R}^{p|q}$ plays a key role in the definition of supermanifold, since it is the local model. If $t^1 \dots t^p$ are global coordinates for $\mathbf{R}^p$ we shall speak of $t^1 \dots t^p, \theta^1 \dots \theta^q$ as a set of *global coordinates* for the superspace $\mathbf{R}^{p|q}$.

Definition 1.4. Let $S = (|S|, \mathcal{O}_S)$ be a superspace. Given an open subset $|U| \subset |S|$, the pair $(|U|, \mathcal{O}_S|_{|U|})$ is always a superspace, called the *open subspace* associated to $|U|$.

Example 1.5. Let $\mathrm{M}_{p|q} = \mathbf{R}^{p^2+q^2|2pq}$. This is the superspace corresponding to the super vector space of $p|q \times p|q$ matrices, the underlying topological space being $\mathrm{M}_p \times \mathrm{M}_q$, the direct product of $p \times p$ and $q \times q$ matrices. As super vector space

$$\mathrm{M}_{p|q} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}, \quad (\mathrm{M}_{p|q})_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad (\mathrm{M}_{p|q})_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$$

where A, B, C, D are respectively $p \times p$, $p \times q$, $q \times p$, $q \times q$ matrices with entries in $\mathbf{R}$.

Hence as a superspace $\mathrm{M}_{p|q}$ has $p^2 + q^2$ even global coordinates t^{ij}, $1 \leq i, j \leq p$ or $p+1 \leq i, j \leq p+q$ and $2pq$ odd ones θ^{kl}, $1 \leq k \leq p$, $p+1 \leq l \leq p+q$ or $p+1 \leq k \leq p+q$, $1 \leq l \leq p$. The structure sheaf of $\mathrm{M}_{p|q}$ is the assignment

$$V \longmapsto C^\infty_{\mathrm{M}_p \times \mathrm{M}_q}(V)[\theta^{kl}], \qquad \text{for all} \quad V \quad \text{open in} \quad \mathrm{M}_p \times \mathrm{M}_q.$$

Now, let us consider in the topological space $\mathrm{M}_p \times \mathrm{M}_q = \mathbf{R}^{p^2+q^2}$, the open set U consisting of the points for which $\det(t_{ij})_{1\leq i,j\leq p} \neq 0$ and $\det(t_{ij})_{p+1\leq i,j\leq p+q} \neq 0$. We define the superspace $\mathrm{GL}_{p|q} := (U, \mathcal{O}_{\mathrm{M}_{p|q}}|_U)$, the open subspace of $\mathrm{M}_{p|q}$ associated to the open set U. As we shall see, this superspace has a Lie supergroup structure and it is called the *general linear supergroup*.

Next, we define a morphism of superspaces, so that we can talk about the category of superspaces.

Definition 1.6. Let S and T be superspaces. Then a morphism $\varphi : S \longrightarrow T$ is a continuous map $|\varphi| : |S| \longrightarrow |T|$ together with a sheaf morphism $\varphi^* : \mathcal{O}_T \longrightarrow \varphi_*\mathcal{O}_S$, so that $\varphi_x^*(\mathfrak{m}_{|\varphi|(x)}) \subset \mathfrak{m}_x$ where $\mathfrak{m}_{S,x}$ is the maximal ideal in $\mathcal{O}_{S,x}$, while $\mathfrak{m}_{T,\varphi(x)}$ is the maximal ideal in $\mathcal{O}_{T,\varphi(x)}$, and φ_x^* is the stalk map.

Remark 1.7. Recall from the previous chapter that the sheaf morphism $\varphi^* : \mathcal{O}_T \longrightarrow \varphi_*\mathcal{O}_S$ corresponds to the system of maps $\varphi_U^* : \mathcal{O}_T(U) \longrightarrow \mathcal{O}_S(|\varphi|^{-1}(U))$ for all open sets $U \subset |T|$. To ease notation, we also refer to the maps φ_U^* as φ^*.

Essentially the condition $\varphi_x^*(\mathfrak{m}_{|\varphi|(x)}) \subset \mathfrak{m}_x$ means that the sheaf homomorphism is local. Note also that φ^* is a morphism of supersheaves, so, as usual, it preserves the parity. The main point to make here is that, when we are giving a morphism of superspaces, the sheaf morphism must be specified along with the continuous topological map, since sections are not necessarily genuine functions on the topological space as in ordinary differential geometry. An arbitrary section cannot be viewed as a function because commutative super rings have many nilpotent elements, and nilpotent sections are identically zero as functions on the underlying topological space. Therefore we employ the methods of algebraic geometry to study such objects. We will address this in more detail later.

2 Supermanifolds

A supermanifold is a specific type of superspace, which we describe via a local model, namely it is locally isomorphic to the superspace $\mathbf{R}^{p|q}$ introduced previously. Let us now see in detail the definition.

Let C_U^∞ be the sheaf of C^∞-functions on the domain $U \subset \mathbf{R}^p$. We define the *superdomain* $U^{p|q}$ to be the superspace $(U, C_U^\infty[\theta^1, \ldots, \theta^q])$ where

$C^\infty_U[\theta^1,\dots,\theta^q] = C^\infty_{\mathbf{R}^p}|_U \otimes \wedge(\theta^1,\dots,\theta^q)$. Most immediately, the superspaces $\mathbf{R}^{p|q}$ are superdomains with sheaf $C^\infty_{\mathbf{R}^p}[\theta^1,\dots,\theta^q]$.

Definition 2.1. A *supermanifold* $M = (|M|, \mathcal{O}_M)$ of dimension $p|q$ is a superspace which is locally isomorphic to $\mathbf{R}^{p|q}$. In other words, given any point $x \in |M|$, there exists a neighborhood $V \subset |M|$ of x with q odd indeterminates θ^j so that

$$V \cong V_0 \quad \text{open in } \mathbf{R}^p, \qquad \mathcal{O}_M|_V \cong \underbrace{C^\infty(t^1,\dots,t^p)}_{C^\infty_{\mathbf{R}^p}|_{V_0}}[\theta^1,\dots,\theta^q]. \tag{1}$$

We call $t^1 \dots t^p, \theta^1 \dots \theta^q$ the *local coordinates* of M in V and $p|q$ the *superdimension* of the supermanifold M.

Morphisms of supermanifolds are morphisms of the underlying superspaces. For M, N supermanifolds, a morphism $\varphi : M \longrightarrow N$ is a continuous map $|\varphi| : |M| \longrightarrow |N|$ together with a local morphism of sheaves of superalgebras $\varphi^* : \mathcal{O}_N \longrightarrow \varphi_*\mathcal{O}_M$, where *local* means that $\varphi_x^{-1}(m_{M,x}) = m_{N,|\varphi|(x)}$, where $\varphi_x : \mathcal{O}_{N,|\varphi|(x)} \longrightarrow \mathcal{O}_{M,x}$ is the stalk morphism, for a point $x \in |M|$, and $m_{M,x}$, $m_{N,|\varphi|(x)}$ are the maximal ideals in the stalks. Note that in the purely even case of ordinary C^∞-manifolds, the above notion of a morphism agrees with the ordinary one. We may therefore talk about the category of supermanifolds. The difficulty in dealing with C^∞-supermanifolds arises when one tries to think of "points" or "functions" in the traditional sense. The ordinary points only account for the topological space and the underlying sheaf of ordinary C^∞-functions, and one may truly only talk about the "value" of a section $f \in \mathcal{O}_M(U)$ for $U \subset |M|$ an open subset; the value of f at $x \in U$ is the unique real number c so that $f - c$ is not invertible in any neighborhood of x. For concreteness, let us consider the example of $M = \mathbf{R}^{1|1}$, with global coordinates t and θ. Let us take the global section $f = t\theta \in \mathcal{O}_M(\mathbf{R})$. For any non zero real number c, we have that $t\theta - c$ is always invertible, since $t\theta$ is nilpotent, its inverse being $-c^{-2}t\theta - c^{-1}$. Hence the value of $t\theta$ at all points $x \in \mathbf{R} = |\mathbf{R}^{1|1}|$ is zero. What this says is that we cannot reconstruct a section by knowing only its values at topological points. Now that we understand this point we shall follow the established notation and call the sections *"functions on U"*.

Remark 2.2. Let M be a supermanifold, U an open subset in $|M|$, and f a function on U. If $\mathcal{O}_M(U) = C^\infty(t^1,\dots,t^p)[\theta^1,\dots,\theta^q]$ as in (1), there exist even functions $f_I \in C^\infty(t)$ $(t = t^1,\dots t^p))$ so that

$$f(t,\theta) = f_0(t) + \sum_i f_i(t)\theta^i + \sum_{i<j} f_{ij}(t)\theta^i\theta^j + \ldots = f_0(t) + \sum_{|I|=1}^{q} f_I(t)\theta^I \quad (2)$$

where $I = \{i_1 < i_2 < \ldots < i_r\}_{r=1}^q$.

So in some sense, we can expand $f(t,\theta)$ in power series, with respect to the odd coordinates θ^j's.

3 The functor of points

The presence of odd coordinates steals some of the geometric intuition away from the language of supergeometry. For instance, we cannot see an "odd point" – they are invisible both topologically and as classical functions on the underlying topological space. We see the odd points only as sections of the structure sheaf. To bring some of the intuition back, we turn to the functor of points approach.

Definition 3.1. Let S and T be supermanifolds. A *T-point* of S is a morphism $T \longrightarrow S$. We denote the set of all T-points by $S(T)$. Equivalently,

$$S(T) = \mathrm{Hom}(T,S).$$

We define the *functor of points* of the superspace S the functor:

$$S : (\text{smflds})^o \longrightarrow (\text{sets}), \qquad T \longmapsto S(T)$$

where (smflds) denotes the category of supermanifolds.

By a common abuse of notation the supermanifolds S and the functor of points of S are denoted with the same letter. Whenever is necessary to make a distinction, we shall write h_S for the functor of points of the supermanifold S.

Yoneda's lemma allows us to replace a supermanifold S with its set of T-points, $S(T)$. We can now think of S as a representable functor from the category of supermanifolds to the category of sets. In fact, when constructing a supermanifold, it is often most convenient to construct the functor of points and then prove that the functor is *representable* in the appropriate category.

The following proposition is very useful when we want to explicitly describe the functor of points of a supermanifold.

Remark 3.2. To ease the notation we write $\mathcal{O}_T(T)$ or simply $\mathcal{O}(T)$ for the global sections of a superspace T.

Proposition 3.3. *Let* $M = (|M|, \mathcal{O}_M)$ *and* $T = (|T|, \mathcal{O}_T)$ *be supermanifolds. Then*

$$Hom(T, M) = Hom(\mathcal{O}(T), \mathcal{O}(M)).$$

Let us give some examples of T-points.

Example 3.4. (i) Let T be just an ordinary topological point viewed as supermanifold, i.e. $T = \mathbf{R}^{0|0} = (\mathbf{R}^0, \mathbf{R})$. By definition a T-point of a manifold M is a morphism $\phi : \mathbf{R}^{0|0} \longrightarrow M$. ϕ consists of a continuous map $|\phi| : \mathbf{R}^0 \longrightarrow |M|$, which corresponds to the choice of a point x in the topological space $|M|$, and a sheaf morphism $\phi^* : \mathcal{O}_M \longrightarrow \phi_*(\mathbf{R})$. Then a T-point of M is an ordinary topological point of $|M|$.

(ii) Let M be the supermanifold $\mathbf{R}^{p|q}$ and let T be a supermanifold. By the previous proposition we have that a T-point of M corresponds to a morphism:

$$\mathcal{O}(M) = C^\infty(t^1 \dots t^p)[\theta^1 \dots \theta^q] \longrightarrow \mathcal{O}(T).$$

Then, in this case, a T-point of M is a choice of p even and q odd global sections on T. Thus $\mathbf{R}^{p|q}(T) = \mathcal{O}_{T,0}^p(T) \oplus \mathcal{O}_{T,1}^q(T) = (\mathcal{O}_T^{p|q}(T))_0$.

(iii) Let T be a supermanifold. In lecture 1 and 2 we have described the super vector space of matrices $\mathrm{M}_{m|n} \cong \mathbf{R}^{m^2+n^2|2mn}$. Reasoning as in (ii), we have that $\mathrm{M}_{m|n}(T)$ can be identified with the endomorphisms of the $\mathcal{O}(T)_0$-module $\mathbf{R}^{m|n}(T) \cong (\mathcal{O}(T) \otimes \mathbf{R}^{m|n})_0$, by rearranging the $m^2 + n^2$ even sections and the the $2mn$ odd ones in a matrix form with diagonal block matrices with even entries and off diagonal matrices with odd entries:

$$\mathrm{M}_{m|n}(T) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$$

where $A = (a_{ij})$, $B = (\beta_{il})$, $C = (\gamma_{kj})$, $D = (d_{kl})$ with $a_{ij}, d_{kl} \in \mathcal{O}(T)_0$, and $\beta_{il}, \gamma_{kj} \in \mathcal{O}(T)_1$. Notice that in doing this, we reobtain $\mathrm{M}_{m|n}(A)$ as discussed in lecture 1 with $A = \mathcal{O}(T)$.

Define $F(T)$ as the group of automorphisms of $\mathbf{R}^{m|n}(T)$. Clearly $F(T) \subset \mathrm{M}_{m|n}(T)$. We want to show that $F = \mathrm{GL}_{m|n}$, i.e. F is the functor of points of the *general linear supergroup*, the supermanifold defined previously. In that example, $\mathrm{GL}_{m|n}$ is defined as the open submanifold of the supermanifold $\mathrm{M}_{m|n}$, whose reduced space consists of diagonal block $m+n \times m+n$ real matrices with non zero determinant:

$$|\mathrm{GL}_{m|n}| = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | \det(A) \neq 0, \quad \det(D) \neq 0 \right\}$$

Hence one can check that the functor of points if $\mathrm{GL}_{m|n}$ is given by:

$$\mathrm{GL}_{m|n}(T) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A, D \text{ invertible} \right\} \subset \mathrm{M}_{m|n}(T).$$

This is what we have defined as $\mathrm{GL}_{m|n}(A)$ in lecture 1.

We already see the power of T-points in these examples. The first example ($T = \mathbf{R}^{0|0}$) gives us complete topological information, while the second ($M = \mathbf{R}^{p|q}$) allow us to talk about coordinates on supermanifolds.

4 Superdomains

In this section we collect few fundamental results on superdomains, that we shall need in the sequel to develop the theory of smooth supermanifolds. The main result here is the *Chart's Theorem*, which allows us to identify a morphism between superdomains of dimension $m|n$ and $p|q$ with p even and q odd functions in m even and n odd inderminates. This is a very natural generalization of the similar result for ordinary differentiable manifolds.

Definition 4.1. We call smooth *superdomain* of dimension $(p|q)$ the super ringed space $U^{p|q} = (U, \mathcal{O}_{U^{p|q}})$ where U is an open subset of $\mathbf{R}^p$ and $\mathcal{O}_{U^{p|q}}(V) = \mathcal{C}^\infty(V) \otimes \wedge^q$ for each V open subset of U, where $\wedge^q = \wedge(\theta^1 \dots \theta^q)$. $p|q$ is called the *superdimension* of the superdomain. We denote with $\mathcal{O}(U^{p|q})$ the global sections of the sheaf $\mathcal{O}_{U^{p|q}}$.

A *morphism* between two superdomains $V^{n|m}$ and $U^{p|q}$ is a morphism of super ringed spaces.

If $\{ t^i \}_{i=1}^p$ are coordinates on $\mathcal{C}^\infty(U)$, and $\{ \theta^j \}_{j=1}^q$ is a system of generators of $\wedge^q$, then the set $\{ t^i, \theta^j \}$ is called a *system of (super) coordinates* on $U^{p|q}$. The assignment of a superdomain $U^{p|q}$ together with a system of super coordinates is called a *superchart* or *chart* for short. We notice that on $U^{p|q}$ there is a *canonical chart*, consisting of the canonical coordinates inherited from $\mathbf{R}^{m|n}$.

We now want to discuss morphisms between superdomains in more detail. We start with an example, that will lead us to the formulation of the Chart's Theorem.

Example 4.2. Consider the supermanifold $\mathbf{R}^{1|2}$ with a morphism $\phi : \mathbf{R}^{1|2} \longrightarrow \mathbf{R}^{1|2}$.

On $\mathbf{R}^{1|2}$ we have global coordinates t, θ^1, θ^2 and so we may express any function f as in (2):

$$f = f(t, \theta^1, \theta^2) = f_0(t) + f_1(t)\theta^1 + f_2(t)\theta^2 + f_{12}(t)\theta^1\theta^2.$$

Then $f_0(t) \in C^\infty(\mathbf{R})$ sits as a function on the C^∞-manifold $\mathbf{R}$. By definition the morphism ϕ is described by a continuous map $|\phi|$ and a sheaf morphism ϕ^*.

Let us first prescribe the images of the global coordinates under ϕ^*:

$$\begin{array}{cccl} t & \mapsto & t^* & := & t + \theta^1\theta^2 \\ \theta^1 & \mapsto & \theta^{1*} & := & \theta^1 \\ \theta^2 & \mapsto & \theta^{2*} & := & \theta^2. \end{array} \tag{3}$$

We claim that knowing ϕ^* on only these global coordinates is enough to completely describe ϕ. Indeed, we first see that $t \mapsto t + \theta^1\theta^2$ tells us that $|\phi|$ is just the identity map. Next, let $f \in C^\infty(t)[\theta^1, \theta^2]$ be as above. Then $f \mapsto \phi^*(f) := f^*$ can be written formally

$$\begin{aligned} f^* &= f(t^*, \theta^{1*}, \theta^{2*}) = f_0(t + \theta^1\theta^2) + f_1(t + \theta^1\theta^2)\theta^1 + f_2(t + \theta^1\theta^2)\theta^2 \\ &\quad + f_{12}(t + \theta^1\theta^2)\theta^1\theta^2. \end{aligned} \tag{4}$$

And so we must only make sense of $f_I(t + \theta^1\theta^2)$. The key is that we take a formal *Taylor series expansion*; the series of course terminates thanks to the nilpotence of the the odd coordinates:

$$f_I(t + \theta^1\theta^2) = f_I(t) + \theta^1\theta^2 f_I'(t). \tag{5}$$

It is easy to check that this in fact gives a homomorphism of superalgebras. For $g, h \in C^\infty(\mathbf{R})$, $(gh)^* = gh + \theta^1\theta^2(gh)' = g^*h^*$. Notice moreover that in order to determine the sheaf morphism, it is enough to specify the images of the global sections, since the full sheaf map is determined by the restrictions of the global coordinates as we shall see in the complete and detailed generality later. So in this example, in fact, the morphism induced by the equations (3) is unique via the above construction.

This fact is indeed true in general. As we shall see, the Chart Theorem states that a morphism ϕ between superdomains is determined by the images of the local coordinates under the sheaf morphism ϕ^*.

Theorem 4.3 (Chart theorem)**.** *Let $U \subset \mathbf{R}^{p|q}$ and $V \subset \mathbf{R}^{m|n}$ be open superdomains (with canonical charts). There is a bijection between*

(i) the set of morphisms $\phi : V \longrightarrow U$ and

(ii) the set of systems of p even functions t^{i*} and q odd functions θ^{j*} in $\mathcal{O}(V)$, such that ${t^{i*}}_0(m) \in |U|$ for all $m \in |V|$.

Proof. We briefly sketch it. It is clear that if we have a morphism ϕ we can uniquely associate to it a set of p even and q odd functions in $\mathcal{O}(V)$. In fact, we take $t^{i*} := \phi^*(t^i)$, $\theta^{j*} := \phi^*(\theta^j)$, where $\{t^i, \theta^j\}$ is the canonical chart in $\mathcal{O}(U)$.

Assume now (ii) holds. We denote with $\{\, t^i, \theta^j \,\}$ and $\{\, x^r, \xi^s \,\}$ coordinates on U and V respectively. It is clear that we have a continuous map $|\phi| : |V| \to |U|$. We now need to show that there exists a unique superalgebra morphism $\phi^* : \mathcal{C}^\infty(|U|) \otimes \Lambda^p \to \mathcal{C}^\infty(|V|) \otimes \Lambda^n$ such that $\phi^*(t^i) = t^{i*}$ and $\phi^*(\theta^j) = \theta^{j*}$. Let us start with the existence of such ϕ^*. It is enough to show that there exists a superalgebra morphism $\phi^* : \mathcal{C}^\infty(|U|) \to \mathcal{C}^\infty(|V|) \otimes \Lambda^n$, since we know the image of the polynomial odd generators θ^j. Let $t^{*i} = \widetilde{t^{*i}} + n^i$ with $\widetilde{t^{*i}} \in \mathcal{C}^\infty(|V|)$, $n^i := \sum_{|I|>1} t^{*i}_I \xi^I$ and define the pullback, through a formal Taylor expansion, as follows

$$\phi^*(f) := \sum_\gamma \frac{1}{\gamma!} \frac{\partial^\gamma f}{\partial t^\gamma} n^\gamma. \tag{6}$$

We show that it is a superalgebra morphism. Indeed

$$\begin{aligned}\phi^*(f\cdot g) &= \sum_{\gamma}\frac{1}{\gamma!}\frac{\partial^\gamma(f\cdot g)}{\partial t^\gamma}n^\gamma =\\ &= \sum_{\gamma,\alpha}\frac{1}{\gamma!}\left(\begin{array}{c}\gamma\\ \alpha\end{array}\right)\frac{\partial^\gamma f}{\partial t^\gamma}\frac{\partial^{\gamma-\alpha}g}{\partial t^{\gamma-\alpha}}n^\gamma =\\ &= \sum_{\alpha,\beta}\frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial t^\alpha}n^\alpha\frac{1}{\beta!}\frac{\partial^\beta g}{\partial t^\beta}n^\beta\end{aligned}$$

We now come to uniqueness. Suppose ϕ_1^* and ϕ_2^* are two morphisms $\mathcal{C}^\infty(|U|)\otimes\Lambda^q\to\mathcal{C}^\infty(|V|)\otimes\Lambda^n$ such that $\phi_1^*(t^i)=\phi_2^*(t^i)=t^{i*}$, $\phi_1^*(\theta^i)=\phi_2^*(\theta^i)=\theta^{i*}$. They clearly coincide on polynomial sections, hence one can check they coincide (using also the classical result).

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Remark 4.4. Eq. 6 in the proof of the previous proposition gives the recipe for calculating the pull back of a generic section from the pull back of the super coordinates.
Notice also that because the expansion (6) involves an arbitrary number of derivatives, there is no way to make sense of C^k-supermanifolds.

5 The Local structure of morphisms

In this section we want to state the main results characterizing the local behaviour of the morphisms of supermanifolds. First we need to define tangent spaces.

Let $M=(|M|,\mathcal{O}_M)$ be a supermanifold.

Definition 5.1. We call *tangent vector* at $x\in|M|$ a (super) derivation of the stalk $\mathcal{O}_{M,x}$, *i. e.* a linear map:

$$v:\,\mathcal{O}_{M,x}\longrightarrow\mathbf{R}$$

such that

$$v(f\cdot g)=v(f)g(x)+(-1)^{|v||f|}f(x)v(g)$$

Remark 5.2. Notice that the sign $(-1)^{|v||f|}f$ is immaterial. In fact the signs appears only when f and v are odd, but in this case $f(x)=0$.

Definition 5.3. The supervector space of all tangent vectors at a point $x \in |M|$ is called the *super tangent space* at x and is denoted with $T_x(M)$.

Any supermanifold morphism $\phi : M \longrightarrow N$ induces a stalk morphism: $\phi_x^* : \mathcal{O}_{N,|\phi|(x)} \longrightarrow \mathcal{O}_{M,x}$ which in turn defines a linear morphism of the tangent spaces:

$$\begin{array}{rccc} (\mathrm{d}\phi)_x : & T_xM & \longrightarrow & T_{\phi(x)}N \\ & v & \mapsto & v \circ \phi_x^* \end{array}$$

The linear map $(\mathrm{d}\phi)_x$ is called the *differential* of ϕ at x.

The differential $(\mathrm{d}\phi)_x$ is an even linear map of super vector spaces and for this reason, once we fix homogeneous bases for such vector spaces, it corresponds to a diagonal blocks matrix, in other words it will not contain many information about the behaviour of the odd variables. For example, if ϕ is the morphism discussed in 4.2, one readily checks that $(\mathrm{d}\phi)_x$ is the identity for all x. In order to better study infinitesimally the odd directions, we need the concept of *jacobian* that we shall introduce later

In the next two propositions we provide very useful characterizations of the tangent space to a supermanifold, we leave them as exercises, sending the reader to [7] for more details.

Proposition 5.4. *Let $x \in |M|$ Suppose $v \in T_x(M)$ and t^i, θ^j are a super coordinate system around x. Then*

1. *v is completely determined by $v([t^i]), v([\theta^j])$*
2. *the derivations $\{\left(\frac{\partial}{\partial t^i}\right)_x\}_{i=1}^m, \{\left(\frac{\partial}{\partial \theta^j}\right)_x\}_{j=1}^n$ form a basis of T_xM, hence $\dim M = \dim T_xM$, where such derivations are defined as:*

$$\frac{\partial}{\partial t^i}([t^k]) = \delta_{ik}, \quad \frac{\partial}{\partial t^i}([\theta^j]) = 0, \qquad \frac{\partial}{\partial \theta^j}([t^k]) = 0, \quad \frac{\partial}{\partial \theta^j}([\theta^l]) = \delta_{jl}.$$

Let $x \in |M|$. Define the supervector space

$$\mathrm{Der}_x\left(\mathcal{O}(M), \mathbf{R}\right) = \{v : \mathcal{O}(M) \longrightarrow \mathbf{R} \mid v(fg) = v(f)g(x) + (-1)^{|v||f|} f(x)v(g)\}.$$

Proposition 5.5. *The linear morphism*

$$\begin{gathered} \alpha : T_xM \longrightarrow \mathrm{Der}_x\left(\mathcal{O}(M), \mathbf{R}\right) \\ v \longrightarrow v \circ \epsilon_x \end{gathered}$$

is an isomorphism, where $\epsilon_x : \mathcal{O}(M) \longrightarrow \mathcal{O}_{M,x}$ is the natural map.

Observation 5.6. From the propositions 5.5, 5.6 we immediately have the following facts:
1. For any tangent vector $v \in T_x M$ and any neighbourhood U of x, there exists a unique derivation, that we still denote with $v : \mathcal{O}_M(U) \longrightarrow \mathbf{R}$.
2. If (t^i, θ^j) are local coordinates in U, any derivation $v : \mathcal{O}_M(U) \longrightarrow \mathbf{R}$ is determined once we know $v(t^i)$ and $v(\theta^j)$.

We are now ready to examine the local structure of morphisms. We start with the inverse function theorem. We will not provide proofs for the results, the reader can work them out as exercises, they rely heavily on the ordinary proofs (see [7]).

Proposition 5.7. *The inverse function Theorem. Let* $\phi : M \to N$ *be a supermanifold morphism and let* $m \in |M|$ *such that* $(\mathrm{d}\phi)_m$ *is bijective. Then there exist charts* U *and* V *around* m *and* $|\phi|(m)$ *respectively such that* $|\phi|(U) \subseteq V$ *and* $\phi_{|_U} : U \to V$ *is an isomorphism of* U *onto* V*.*

Proposition 5.8. Immersions

Let $\phi : M \longrightarrow N$ *be a supermanifold morphism, with* $\dim(M) = m|n \leq \dim(N) = m + p|n + q$*. The following facts are equivalent:*

1. $\phi : M \to N$ *is an immersion at* x*, that is* $(\mathrm{d}\phi)_x$ *is injective;*
2. $(\mathrm{d}\phi)_x$ *has rank* $(m|n)$*;*
3. *There exist charts* U, $\{ t^i \}_{i=1}^n$, $\{ \theta^j \}_{j=1}^m$ *around* x *and,* $V = V_1 \times V_2$, $\{ t^i \}_{i=1}^n$, $\{ \overline{t}^a \}_{a=1}^p$, $\{ \theta^j \}_{j=1}^m$, $\{ \overline{\theta}^b \}_{b=1}^q$ *around* $|\phi|(x)$ *such that the restriction of the map to* U *and* V *has the form*

$$\begin{aligned} t^i &\longrightarrow t^i \\ \theta^j &\longrightarrow \theta^j \\ \overline{t}^a, \overline{\theta}^b &\longrightarrow 0. \end{aligned}$$

Proposition 5.9. Submersions

Let $\phi : M \longrightarrow N$ *be a supermanifold morphism, with* $\dim(M) = m + p|n + q \geq \dim(N) = p|q$*. The following facts are equivalent*

1. $\phi : M \to N$ *is a submersion at* x*, that is* $(\mathrm{d}\psi)_x$ *is surjective;*
2. $(\mathrm{d}\phi)_x$ *has rank* $(m|n)$*;*

3. *There exist charts* $U = U_1 \times U_2, \{\, t^i \,\}_{i=1}^{m}, \{\, \overline{t}^a \,\}_{a=1}^{p}, \{\, \theta^j \,\}_{j=1}^{n}, \{\, \overline{\theta}^b \,\}_{b=1}^{q}$ *around* x *and* $V, \{\, t^i \,\}_{i=1}^{n}, \{\, \theta^j \,\}_{j=1}^{m}$ *around* $|\phi|\,(x)$ *such that the restriction of the map to* U *and* V *has the form*

$$\begin{aligned} t^i &\longrightarrow t^i \\ \theta^j &\longrightarrow \theta^j \end{aligned}$$

Similarly to the ordinary setting, the constant rank morphisms are very important, however in supergeometry there is an extra difficulty due to the fact that there are different notions of rank of a supermatrix. We shall not describe constant rank morphisms in these notes, however we want to give the definition and point out the differences in the ordinary setting.

Definition 5.10. Let $Z = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right) \in \mathrm{M}_{p|q \times m|n}(A)$ for a commutative superalgebra A, in other words Z is a $p|q \times m|n$ matrix with diagonal blocks entries in A_0 and off diagonal block entries in A_1 (our prototype for A is $\mathcal{O}_M(U)$).

We define the rank of Z as the rank of the reduced matrix.

We say instead that Z has *constant rank* $r|s$ if there exist $G_1 \in \mathrm{GL}_{p|q}(A)$ and $G_2 \in \mathrm{GL}_{m|n}(A)$ such that $G_1 Z G_2$ has the form:

$$G_1 Z G_2 = \begin{pmatrix} \mathrm{id}_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_s & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We say that the supermanifold morphism $\psi : M \longrightarrow N$ has *constant rank* at $m \in |M|$ if its *jacobian*

$$\begin{pmatrix} \frac{\partial \psi^*(s^k)}{\partial t^i} & -\frac{\partial \psi^*(s^k)}{\partial \theta^j} \\ \frac{\partial \psi^*(\eta^l)}{\partial t^i} & \frac{\partial \psi^*(\eta^l)}{\partial \theta^j} \end{pmatrix}$$

has constant rank in a neighbourhood of m

Notice that the minus sign appears in the definition of jacobian since as one can check we have:

$$J_{\psi \circ \phi} = J_\psi \circ J_\phi.$$

Remark 5.11. It is important to notice that the constant rank is not defined for all supermatrices. For example, for the supermatrix $Z = \left(\begin{smallmatrix} 0 & 0 \\ \xi & 0 \end{smallmatrix}\right) \in \mathrm{M}_{1|1 \times 1|1}$ we cannot define any constant rank; in fact, as one can readily check, it is not possible with a basis change to transform Z into a diagonal matrix with 1 or 0 on the diagonal, as the definition of constant rank requires. However, once the constant rank is defined, it coincides with the rank.

6 Submanifolds

As in the classical theory submanifolds of a given supermanifold M are defined as pairs (N, j) where N is a supermanifold and $j : N \to M$ is an injective morphism with some regularity property. We will distinguish two kinds of submanifolds according to the properties of the morphism j.

Definition 6.1. We say that (N, j) is an *immersed submanifold* if $j : N \to M$ is an injective immersion, in other words if $|j| : |N| \longrightarrow |M|$ is injective and $(\mathrm{d}j)_m$ in injective for all $m \in |M|$.

As in the ordinary setting, we can strengthen this notion by introducing the notion of *embedding*.

Definition 6.2. We say that $j : N \to M$ is an *embedding* if it is an immersion and if $|j| : |N| \to |M|$ is a homeomorphism onto its image. We say that (N, j) is an *embedded submanifold* if j is an embedding. We say that (N, j) is a *closed embedded submanifold* if it is an embedded submanifold and $|j|\,(|N|)$ is a closed subset of $|M|$.

We shall now give the main result that allows us to single out submanifolds of a given supermanifold. We shall not provide a proof for this result, sending the reader to [5] for more details.

Proposition 6.3. *Let $\phi : L \to M$ be a supermanifold morphism and suppose $m \in |M|$. Suppose that for each x in $|L'| = |\phi|^{-1}(m)$ there exists a neighborhood where ϕ is a constant rank morphism.*

i) then there exists a supermanifold $L' = (|L'|, \mathcal{O}_{L'})$ of L, whose superalgebra of global sections is $\mathcal{O}(L)/\mathcal{J}$ for a suitable ideal $\mathcal{J}$ such that $\phi_{L'}$ is a closed embedding;

ii) if (L', j) *denotes the closed embedded submanifold distinguished by* $\mathcal{J}$ *then, for each* $x \in |L'| = |\phi|^{-1}(m)$

$$T_x L' \simeq \ker(\mathrm{d}\phi)_x.$$

Example 6.4. Consider the morphism $\phi_T : \mathrm{GL}_{m|n}(T) \longrightarrow \mathbf{R}^{1|0}(T)$, $\phi(X) = \mathrm{Ber}(X)$, for T a supermanifold. This natural transformation between the functor of points of these two supermanifolds correspond to the morphism on the superalgebras of global sections: $\phi^* : \mathcal{O}(\mathbf{R}^{1|0}) \longrightarrow \mathcal{O}(\mathrm{GL}_{m|n})$, $\phi^*(t) = Ber$, where t is the global canonical coordinate in $\mathbf{R}^{1|0}$ and Ber is the Berezinian function, that is, if x_{ij} and ξ_{kl} are the usual global canonical coordinates in $\mathcal{O}(\mathrm{GL}_{m|n})$,

$$\mathrm{Ber} = \det(x_{kl})^{-1} \det(x_{ij} - \sum_{kl} \xi_{ik} x^{kl} \xi_{lj}), \quad i, j = 1 \ldots m, k, l = m+1 \ldots m+n.$$

and x^{kl} denotes the element of the inverse of the matrix (x_{kl}). One can readily check that this is a submersion, hence a constant rank morphism. By proposition 6.3 we have that $|\phi|^{-1}(\mathrm{id})$ has a supermanifold structure and one can readily check that this is $\mathrm{SL}_{m|n}$, the supermanifold whose functor of points at T consists of the matrices in $\mathrm{GL}_{m|n}(T)$ with berezinian equal to 1.

The differential at the identity $(\mathrm{d}\phi)_{\mathrm{id}}$ can be calculated and it is the morphism:

$$\begin{array}{rccc} (\mathrm{d}\phi)_{\mathrm{id}} : & T_{\mathrm{id}}(\mathrm{GL}_{m|n}) \cong \mathrm{M}_{m|n} & \longrightarrow & T_1\mathbf{R}^{1|0} \cong \mathbf{R} \\ & M & \longrightarrow & \mathrm{str}(M). \end{array}$$

where str denotes the supertrace of a matrix, i.e. $\mathrm{str}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \mathrm{tr}(A) - \mathrm{tr}(B)$. Hence the tangent space to $\mathrm{SL}_{m|n}$ at the identity, consists of those matrices with supertrace zero. We shall see in the next chapter, that this super vector space has the extra structure of a super Lie algebra.

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