Lecture 4
Supergroups

Let k be a field, $char k \neq 2, 3$.

Throughout this lecture we assume all superalgebras are associative, commutative (i.e. $xy = (-1)^{p(x)p(y)}\ yx$) with unit and over k unless otherwise specified.

1 Supergroups

A *supergroup scheme* is a superscheme whose functor of points is group valued, that is to say, valued in the category of groups. It associates functorially a group to each superscheme or equivalently to each superalgebra. Let us see this in more detail.

Definition 1.1. A *supergroup functor* is a group valued functor:

$$G : (\text{salg}) \longrightarrow (\text{sets})$$

This is equivalent to have the following natural transformations:

1. Multiplication $\mu : G \times G \longrightarrow G$, such that $\mu \circ (\mu \times \text{id}) = (\mu \times \text{id}) \circ \mu$, i. e.

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu \times \text{id}"] \arrow[d, "\text{id} \times \mu"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

2. Unit $e : e_k \longrightarrow G$, where $e_k : (\text{salg}) \longrightarrow (\text{sets})$, $e_k(A) = 1_A$, such that $\mu \circ (\text{id} \otimes e) = \mu \circ (e \times \text{id})$, i. e.

$$\begin{tikzcd}
G \times e_k \arrow[r, "\text{id} \times e"] \arrow[dr] & G \times G \arrow[d, "\mu"'] & e_k \times G \arrow[l, "e \times id"'] \arrow[dl] \\
 & G &
\end{tikzcd}$$

3. Inverse $i : G \longrightarrow G$, such that $\mu \circ (\mathrm{id}, i) = e \circ \mathrm{id}$, i. e.

$$\begin{tikzcd}
G \arrow[r, "(\mathrm{id},i)"] \arrow[d] & G \times G \arrow[d, "\mu"] \\
e_k \arrow[r, "e"] & G
\end{tikzcd}$$

The supergroup functors together with their morphisms, that is the natural transformations that preserve μ, e and i, form a category.

If G is the functor of points of a superscheme X, i.e. $G = h_X$, in other words $G(A) = \mathrm{Hom}(\underline{\mathrm{Spec}}A, X)$, we say that X is a *supergroup scheme.* An *affine supergroup scheme* X is a supergroup scheme which is an affine superscheme, that is $X = \underline{\mathrm{Spec}}\mathcal{O}(X)$ for some superalgebra $\mathcal{O}(X)$. To make the terminology easier we will drop the word "scheme" when speaking of supergroup schemes.

As we shall presently see, the functor of points of an affine supergroup is represented by a superalgebra, which has the additional structure of a *Hopf superalgebra*.

Proposition 1.2. *Let G be an affine supergroup scheme. Then $\mathcal{O}(G)$ is an Hopf algebra. Moreover we can identify the category of affine supergroups with the category of commutative Hopf superalgebras.*

Proof. We first observe that, if G is a superscheme, and $\mathcal{O}(G)$ is a Hopf superalgebra with comultiplication Δ, counit ϵ and antipode S, $h_G(A)$ has a natural group structure. In fact we can define the product of two morphisms in $h_G(A)$ in the following way:

$$x \cdot y = \mu_A \circ x \otimes y \circ \Delta : \mathcal{O}(G) \xrightarrow{\Delta} \mathcal{O}(G) \otimes \mathcal{O}(G) \xrightarrow{x \otimes y} A \otimes A \xrightarrow{\mu_A} A$$

where μ_A is the multiplication in the superalgebra A. One can immediately check that the multiplication is a morphism, that is:

$$(x \cdot y)(ab) = (x \cdot y)(a)(x \cdot y)(b), \qquad \forall a, b \in A$$

(though hidden, the sign rule plays a crucial role here).

The unit e and the inverse i in $h_G(A)$ are defined as follows:

$$e = \eta_A \circ \epsilon : \mathcal{O}(G) \xrightarrow{\epsilon} k \xrightarrow{\eta_A} A, \qquad i(x) = S \circ x,$$

where η_A is the unit in A. We leave to the reader the routine checks to check the definition 1.1.

Vice-versa, if G is a supergroup, we can define the comultiplication $\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ as the dual of the multiplication $\mu \in \mathrm{Hom}(G \times G, G)$ using the identification:

$$\mathrm{Hom}(G \times G, G) \cong \mathrm{Hom}(\mathcal{O}(G), \mathcal{O}(G \times G)),$$

(one can readily check that $\mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$). Similarly one defines the counit and the antipode ϵ and S as the duals of unit e and inverse i.

A careful look shows that formally the diagrams defining a supergroup functor are essentially the same as those defining a Hopf superalgebra, with arrows reversed.

The equivalence between the categories of affine supergroups and commutative Hopf superalgebras is an immediate consequence of the previous discussion. ■

Let us now examine some important examples of supergroup schemes and their associated Hopf superalgebras.

Example 1.3. 1. *Supermatrices* $\mathrm{M}_{m|n}$.

Consider functor of points of supermatrices:

$$\mathrm{M}_{m|n} : (\mathrm{salg}) \longrightarrow (\mathrm{sets}), \qquad A \longmapsto \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}$$

where a and b are $m \times m$, $n \times n$ block matrices with entries in A_0, while α and β are $m \times n$, $n \times m$ block matrices with entries in A_1. $\mathrm{M}_{m|n}$ is a representable functor, represented by the superalgebra of polynomials $k[x_{ij}, \xi_{kl}]$ for suitable indeces i, j, k, l. The functor $\mathrm{M}_{m|n}$ is group-valued, in fact any $\mathrm{M}_{m|n}(A)$ has an additive group structure, where the addition is simply defined as the addition of matrices. Hence by the previous proposition $k[x_{ij}, \xi_{kl}]$ is a Hopf superalgebra, where the comultiplication Δ, the counit ϵ and antipode S are given by:

$$\Delta(x_{ij}) = x_{ij} \otimes 1 + 1 \otimes x_{ij}, \qquad \Delta(\xi_{kl}) = \xi_{kl} \otimes 1 + 1 \otimes \xi_{kl},$$

$$\epsilon(x_{ij}) = \delta_{ij}, \qquad \epsilon(\xi_{ij}) = 0, \qquad S(x_{ij}) = -x_{ij}, \qquad S(\xi_{ij}) = -\xi_{ij}.$$

We leave to the reader the verification that $k[x_{ij}, \xi_{kl}]$ together with Δ, ϵ and S is a Hopf superalgebra.

2. *The general linear supergroup* $\mathrm{GL}_{m|n}$.

Let $A \in$ (salg). Let us define $\mathrm{GL}_{m|n}(A)$ as $\mathrm{GL}(A^{m|n})$ the set of automorphisms of the A-supermodule $A^{m|n}$. Choosing coordinates we can write

$$\mathrm{GL}_{m|n}(A) = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \right\} \subset \mathrm{M}_{m|n}.$$

As we have previously noticed, $\mathrm{GL}_{m|n}(A)$ are the invertible transformations of $k^{m|n}(A)$ preserving parity.

This is the functor of points of an affine supergroup $\mathrm{GL}_{m|n}$ represented by the Hopf superalgebra

$$k[\mathrm{GL}_{m|n}] = k[x_{ij}, \xi_{kl}][U, V]/(Ud_1 - 1, Vd_2 - 1)$$

where x_{ij}'s, U, V and ξ_{kl}'s are respectively even and odd variables with $1 \leq i, j \leq m$ or $m+1 \leq i, j \leq m+n$, $1 \leq k \leq m$, $m+1 \leq l \leq m+n$ or $m+1 \leq k \leq m+n$, $1 \leq l \leq m$ and

$$d_1 = \textstyle\sum_{s \in S_m} (-1)^{l(s)} x_{1,s(1)} \dots x_{m,s(m)},$$

$$d_2 = \textstyle\sum_{t \in S_n} (-1)^{l(t)} x_{m+1,m+t(1)} \dots x_{m+n,m+t(n)}.$$

It is customary to write d_1^{-1} and d_2^{-1} in place of U and V, so we shall write:

$$k[\mathrm{GL}_{m|n}] = k[x_{ij}, \xi_{kl}][d_1^{-1}, d_2^{-1}]$$

Notice that the Berezinian function is well defined in $k[\mathrm{GL}_{m|n}]$, in fact:

$$\mathrm{Ber} = d_2^{-1} \det(a - \beta b^{-1} \alpha).$$

The bialgebra structure of $k[\mathrm{GL}_{m|n}]$ is explicitly described in [5].

3. *The special linear group* $\mathrm{SL}_{m|n}$.

For a superalgebra A, let us define $\mathrm{SL}_{m|n}(A)$ to be the subset of $\mathrm{GL}_{m|n}(A)$ consisting of matrices with Berezinian equal to 1. This is the functor of points of an affine supergroup and it is represented by the Hopf superalgebra:

$$k[\mathrm{SL}_{m|n}] = k[x_{ij}, \xi_{kl}][d_1^{-1}, d_2^{-1}]/(\mathrm{Ber} - 1),$$

where the comultiplication, counit and antipode are inherited naturally from the ones in $\mathrm{GL}_{m|n}$.

2 Lie Superalgebras

Consider the functor of points of the superscheme $\mathbf{A}^{1|0}$, the affine line, $\mathcal{O}_k$: (salg) $\longrightarrow$ (sets), $\mathcal{O}_k(A) = \mathrm{Hom}(k[x], A) \cong A_0$ For notational purposes we use the symbol $\mathcal{O}_k$ to denote it, instead of $h_{\mathbf{A}^{1|0}}$.

Definition 2.1. Let $\mathfrak{g}$ be a super vector space. We say that the functor

$$L_{\mathfrak{g}} : (\mathrm{salg}) \longrightarrow (\mathrm{sets}), \qquad L_{\mathfrak{g}}(A) = (A \otimes \mathfrak{g})_0$$

is Lie algebra valued if there is a $\mathcal{O}_k$-linear natural transformation

$$[\ ,\ ] : L_{\mathfrak{g}} \times L_{\mathfrak{g}} \longrightarrow L_{\mathfrak{g}}$$

that satisfies commutative diagrams corresponding to the ordinary antisymmetric property and the Jacobi identity. In other words, for each superalgebra A, the bracket $[\ ,\ ]_A$ defines a Lie algebra structure on the A_0-module $L_{\mathfrak{g}}(A)$. We will drop the suffix A from the bracket and the natural transformations to ease the notation.

Example 2.2. *The supermatrices as a Lie algebra valued functor.* Consider again the functor of points of supermatrices:

$$A \longmapsto \mathrm{M}_{m|n}(A) = \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right\}$$

where P, Q, R, S are respectively $m \times m$, $m \times n$, $n \times m$, $n \times n$ matrices with entries: P and S in A_0, R and Q in A_1.

This is a Lie algebra valued functor once we define as Lie bracket on each $\mathrm{M}_{m|n}(A)$:

$$[X, Y] = XY - YX, \qquad \forall X, Y \in \mathrm{M}_{m|n}(A).$$

Since $\mathrm{M}_{m|n}$ is a representable functor, it corresponds to a super vector space $\mathrm{M}(m|n)$ (by an abuse of notation we may at times use the same letter). The super vector space $\mathrm{M}(m|n)$ is a super Lie algebra with bracket:

$$[X, Y] = XY - (-1)^{p(x)p(y)} YX, \qquad \forall X, Y \in \mathrm{M}(m|n).$$

One word of warning: the super vector space $\mathrm{M}(m|n)$ is *not* $\mathrm{M}_{m|n}(k)$. In fact $\mathrm{M}_{m|n}(k)$ consists only the even part of the super Lie algebra $\mathrm{M}(m|n)$ and consists of the diagonal block matrices with entries in k, while $\mathrm{M}(m|n)$, as a vector space, consists only of $m + n \times m + n$ matrices with entries in k and has superdimension $m^2 + n^2 | 2mn$.

In ordinary geometry we can associate to any group scheme G a Lie algebra, commonly denoted by $\mathrm{Lie}(G)$, which is identified with the tangent space to the group scheme G at the identity. This is an extremely important construction, since it allows us to linearize problems, by transfering our questions from the group to its Lie algebra.

Let G be a supergroup functor.

Let A be a commutative superalgebra and let $A(\epsilon) =_{\mathrm{def}} A[\epsilon]/(\epsilon^2)$ be the algebra of dual numbers, where ϵ is an *even* indeterminate. We have that $A(\epsilon) = A \oplus \epsilon A$ and there are two natural morphisms:

$$i : A \to A(\epsilon), \quad i(1) = 1$$

$$p : A(\epsilon) \to A, \quad p(1) = 1, \quad p(\epsilon) = 0, \quad p \cdot i = \mathrm{id}_A.$$

Definition 2.3. Consider the homomorphism $G(p) : G(A(\epsilon)) \longrightarrow G(A)$. For each G there is a supergroup functor,

$$\mathrm{Lie}(G) : (\mathrm{salg}) \longrightarrow (\mathrm{sets}), \qquad \mathrm{Lie}(G)(A) =_{\mathrm{def}} \ker(G(p)).$$

If G is a supergroup scheme, we denote $\mathrm{Lie}(h_G)$ by $\mathrm{Lie}(G)$.

If $f : G \longrightarrow H$ is a natural transformation of supergroup functors we have the following commutative diagram (where the vertical arrows form exact sequences):

$$\begin{tikzcd}
1 \arrow[r] & 1 \\
G(A) \arrow[u] \arrow[r, "f_A"] & H(A) \arrow[u] \\
G(A(\epsilon)) \arrow[u, "G(p)"] \arrow[r, "f_{A(\epsilon)}"] & H(A(\epsilon)) \arrow[u, "H(p)"'] \\
\mathrm{Lie}(G)(A) \arrow[u] \arrow[r, "f_{A(\epsilon)}|_{\mathrm{Lie}(G)(A)}"] & \mathrm{Lie}(H)(A) \arrow[u] \\
1 \arrow[u] \arrow[r] & 1 \arrow[u]
\end{tikzcd}$$

We hence define: $\mathrm{Lie}(f))(A) = f_{A(\epsilon)}|_{\mathrm{Lie}(G)(A)}$. We immediately have the following proposition.

Proposition 2.4. Lie *is a functor from the category of supergroup functors to the category of sets. Moreover, if G is an algebraic supergroup scheme,* $\mathrm{Lie}(G)$ *is a Lie algebra valued functor, thus associating to any algebraic supergroup scheme a Lie superalgebra.*

Example 2.5. 1. *The super general linear algebra.*

We want to determine the functor $\mathrm{Lie}(\mathrm{GL}_{m|n})$. Consider the morphism:

$$\begin{array}{cccc} \mathrm{GL}_{m|n}(p): & \mathrm{GL}_{m|n}(A(\epsilon)) & \longrightarrow & \mathrm{GL}_{m|n}(A) \\ & & & \\ & \begin{pmatrix} p+\epsilon p' & q+\epsilon q' \\ r+\epsilon r' & s+\epsilon s' \end{pmatrix} & \mapsto & \begin{pmatrix} p & q \\ r & s \end{pmatrix} \end{array}$$

with p, p', s, s' having entries in A_0 and q, q', r, r' having entries in A_1; the blocks p and s are invertible matrices. One can see immediately that

$$\mathrm{Lie}(\mathrm{GL}_{m|n})(A) = \ker(\mathrm{GL}_{m|n}(p)) = \left\{ \begin{pmatrix} I_m + \epsilon p' & \epsilon q' \\ \epsilon r' & I_n + \epsilon s' \end{pmatrix} \right\}$$

where I_n is a $n \times n$ identity matrix. The functor $\mathrm{Lie}(\mathrm{GL}_{m|n})$ is clearly group valued and can be identified with the (additive) group functor $\mathrm{M}_{m|n}$ defined as (see example 2.2):

$$\mathrm{M}_{m|n}(A) = \mathrm{Hom}_{(\mathrm{smod})}(\mathrm{M}(m|n)^*, A) = \mathrm{Hom}_{(\mathrm{salg})}(\mathrm{Sym}(\mathrm{M}(m|n)^*), A)$$

where $\mathrm{M}(m|n)$ is the supervector space

$$\mathrm{M}(m|n) = \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right\} \cong k^{m^2+n^2|2mn}$$

where P, Q, R, S are respectively $m \times m$, $m \times n$, $n \times m$, $n \times n$ matrices with entries in k. An element $X \in \mathrm{M}(m|n)$ is even if $Q = R = 0$, odd if $P = S = 0$.

As we already noticed in the example 2.2, $\mathrm{M}(m|n)$ is a Lie superalgebra with superbracket:

$$[X, Y] = XY - (-1)^{p(X)p(Y)} YX$$

So $\mathrm{Lie}(\mathrm{GL}_{m|n})$ is a Lie superalgebra. In the next section we will see that in general we can give a Lie superalgebra structure to $\mathrm{Lie}(G)$ for any group scheme G.

2. *The special linear superalgebra.*
A similar computation shows that

$$\mathrm{Lie}(\mathrm{SL}_{m|n})(A) = \left\{ W = \begin{pmatrix} I_m + \epsilon p' & \epsilon q' \\ \epsilon r' & I_n + \epsilon s' \end{pmatrix} \quad | \quad \mathrm{Ber}(W) = 1 \right\}.$$

The condition on the Berezinian is equivalent to:

$$\det(1 - \epsilon s') \det(1 + \epsilon p') = 1$$

which gives:

$$\mathrm{tr}(p') - \mathrm{tr}(s') = 0.$$

Hence

$$\mathrm{Lie}(\mathrm{SL}_{m|n})(A) = \{X \in \mathrm{M}_{m|n}(A) \quad | \quad \mathrm{str}(X) = 0\}.$$

where str is the supertrace, i.e. $\mathrm{str}\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \mathrm{tr}(a) - \mathrm{tr}(d)$.

3 Linear Representations

We now want to discuss linear representations, in particular we will show that, as in the classical case, every affine algebraic supergroup G can be embedded into some $\mathrm{GL}_{m|n}$.

Definition 3.1. Let $X = (|X|, \mathcal{O}_X)$ and $Y = (|Y|, \mathcal{O}_Y)$ be two affine superschemes and let $f : X \longrightarrow Y$ be a superscheme morphism. We say that f is a *closed embedding* if it induces a surjective morphism $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$, in other words $\mathcal{O}(X) \cong \mathcal{O}(Y)/I$.

We now want to introduce the notion of linear representation of a supergroup. Let $h_G : (\mathrm{salg}) \longrightarrow (\mathrm{sets})$, $h_G(A) = \mathrm{Hom}(k[G], A)$ be the functor of points of our affine algebraic supergroup G, with Hopf superalgebra $k[G]$.

Definition 3.2. Let V be a super vector space. We define *linear representation* of G in V a natural transformation ρ preserving the product and the identity element:

$$\rho : h_G \longrightarrow \mathrm{End}(V),$$

where $\mathrm{End}(V)$ is the functor

$$\mathrm{End}(V) : (\mathrm{salg}) \longrightarrow (\mathrm{sets}), \qquad \mathrm{End}(V)(A) = \mathrm{End}(A \otimes V).$$

Here $\mathrm{End}(A \otimes V)$ denotes the endomorphisms of the A-module $A \otimes V$ preserving the parity. We will also say that G *acts* on V.

Definition 3.3. Let V be a supervector space. We say that V is a *left G-comodule* if there exists a linear map:

$$\Delta_V : V \longrightarrow k[G] \otimes V$$

called a *comodule map* with the properties:
1) $(\mathrm{id}_G \otimes \Delta_V)\Delta_V = (\Delta \otimes \mathrm{id}_V)\Delta_V$
2) $(\epsilon \otimes \mathrm{id}_V)\Delta_V = \mathrm{id}_V$,
where $\mathrm{id}_G : k[G] \longrightarrow k[G]$ is the identity map.

One can also define a right G-comodule in the obvious way.

Observation 3.4. The two notions of G acting on V and V being a (left) G-comodule are essentially equivalent. In fact, given a representation $\rho : G \longrightarrow \mathrm{End}(V)$, it defines immediately a comodule map:

$$\Delta_V(v) = \rho_{k[G]}(\mathrm{id}_G)v, \qquad \mathrm{id}_G \in h_G(k[G]) = \mathrm{Hom}_{(\mathrm{salg})}(k[G], k[G])$$

where we are using the natural identification (for $A = k[G]$)

$$\mathrm{End}(V)(A) \cong \mathrm{Hom}_{(\mathrm{smod})}(V, A \otimes V).$$

Vice-versa if we have a comodule map Δ_V we can define a representation in the following way:

$$\begin{array}{rcc} \rho_A : h_G(A) & \longrightarrow & \mathrm{End}(V)(A) \cong \mathrm{Hom}_{(\mathrm{smod})}(V, A \otimes V) \\ g & \mapsto & v \mapsto (g \otimes \mathrm{id})(\Delta_V(v)) \end{array}$$

where $g \in h_G(A) = \mathrm{Hom}_{(\mathrm{salg})}(k[G], A)$.

Let us see this correspondence in a special, but important case.

Example 3.5. Let us consider the natural action of $\mathrm{GL}_{m|n}$ on $k^{m|n}$:

$$\begin{array}{rcc} \rho_A : \mathrm{GL}_{m|n}(A) & \longrightarrow & \mathrm{End}(k^{m|n})(A) \\ g = (g_{ij}) & \mapsto & e_j \mapsto \sum g_{ij} \otimes e_i \end{array}$$

where $\{e_j\}$ is the canonical homogeneous basis for the framed supervector space $k^{m|n}$. We identify the morphism $g \in \mathrm{GL}_{m|n}(A) = \mathrm{Hom}_{(\mathrm{salg})}(k[G], A)$ with the matrix with entries $g_{ij} = g(x_{ij})$, where x_{ij}'s are the generators of $\mathcal{O}(\mathrm{GL}_{m|n}) = k[\mathrm{GL}_{m|n}]$.

This corresponds to the comodule map

$$\begin{array}{rcl} \Delta_{k^{m|n}} : k^{m|n} & \longrightarrow & k[\mathrm{GL}_{m|n}] \otimes k^{m|n} \\ e_j & \mapsto & \sum x_{ij} \otimes e_i. \end{array}$$

Vice-versa, the comodule map $e_j \mapsto \sum x_{ij} \otimes e_i$ corresponds to the representation:

$$\begin{array}{rlcr} \rho_A & : \mathrm{GL}_{m|n}(A) & \longrightarrow & \mathrm{End}(k^{m|n})(A) \\ & g = (g_{ij}) & \mapsto & e_j \mapsto (g \otimes \mathrm{id})(\sum x_{ij} \otimes e_i) = \ \sum g_{ij} \otimes e_i. \end{array}$$

Definition 3.6. Let G act on the superspace V, via a representation ρ corresponding to the comodule map Δ_V. We say that the subspace $W \subset V$ is *G-stable* if $\Delta_V(W) \subset k[G] \otimes W$. Equivalently W is G-stable if $\rho_A(g)(A \otimes W) \subset A \otimes W$.

Definition 3.7. The *right regular representation* of the affine algebraic group G is the representation of G in the (infinite dimensional) super vector space $k[G]$ corresponding to the comodule map:

$$\Delta : k[G] \longrightarrow k[G] \otimes k[G].$$

Proposition 3.8. *Let ρ be a linear representation of an affine algebraic supergroup G. Then each finite dimensional subspace W of V is contained in a finite dimensional G-stable subspace of V.*

Proof. It is the same as in the commutative case. Let us sketch it. It is enough to prove for W generated by one element $x \in V$. Let $\Delta_V : V \longrightarrow V \otimes k[G]$ be the comodule structure associated to the representation ρ. Let

$$\Delta_V(x) = \sum_i x_i \otimes a_i$$

where $\{a_i\}$ is a basis for $k[G]$.

We claim that $\mathrm{span}_k\{x_i\}$ is a G-stable subspace.

By definition of comodule we have:

$$(\Delta_V \otimes \mathrm{id}_G)(\Delta_V(x)) = (\mathrm{id}_V \otimes \Delta)(\Delta_V(x)),$$

that is

$$\sum_j \Delta_V(x_j) \otimes a_j = \sum_j x_j \otimes \Delta(a_j) = \sum_{i,j} x_j \otimes b_{ij} \otimes a_i.$$

Hence

$$\Delta_V(x_i) = \sum_j x_j \otimes b_{ij}.$$

The finite dimensional stable subspace is given by the span of the x_i's.

Moreover we have that $x \in \mathrm{span}_k\{x_i\}$. In fact $x = \mathrm{id}(x) = (\mathrm{id} \otimes \epsilon)\Delta_V(x) = \sum x_i \epsilon(a_i)$. ■

Theorem 3.9. *Let G be an affine supergroup variety. Then there exists a closed embedding:*

$$G \subset \mathrm{GL}_{m|n}$$

for suitable m and n.

Proof. By 3.1 we need to find a surjective superalgebra morphism $k[\mathrm{GL}_{m|n}] \longrightarrow k[G]$ for suitable m and n. Let $k[G] = k[f_1 \dots f_n]$, where f_i are homogeneous and chosen so that $W = \mathrm{span}\{f_1 \dots f_n\}$ is G-stable, according to the right regular representation. This choice is possible because of Proposition 3.8. We have:

$$\Delta_{k[G]}(f_i) = \sum_j f_j \otimes a_{ij}.$$

Define the morphism:

$$\begin{array}{ccc} k[\mathrm{GL}_{m|n}] & \longrightarrow & k[G] \\ x_{ij} & \mapsto & a_{ij} \end{array}$$

where x_{ij} are the generators for $k[\mathrm{GL}_{m|n}]$. This is the required surjective superalgebra morphism. In fact, since $k[G]$ is both a right and left G-comodule we have:

$$f_i = (\epsilon \otimes \mathrm{id})\Delta(f_i) = (\epsilon \otimes \mathrm{id})(\sum_j f_j \otimes a_{ij}) = \sum_j \epsilon(f_j) a_{ij}$$

which proves the surjectivity. ■

We hence have proved the following.

Corollary 3.10. *G is an affine supergroup scheme if and only if it is a closed subgroup of* $\mathrm{GL}_{m|n}$*, for suitable m and n.*

References

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