Lecture 4
Super Lie groups

In this lecture we want to take a closer look to supermanifolds with a group structure: *Lie supergroups* or *super Lie groups*. As in the ordinary setting, a super Lie group is defined as a supermanifold together with the multiplication and inverse morphisms, that satisfy the usual properties expressed in terms of certain commutative diagrams. To any Lie supergroup, we can naturally associate a Lie superalgebra, consisting of the left invariant vector fields, As in the ordinary setting, the Lie superalgebra is identified with the tangent superspace to the supergroup at the identity.

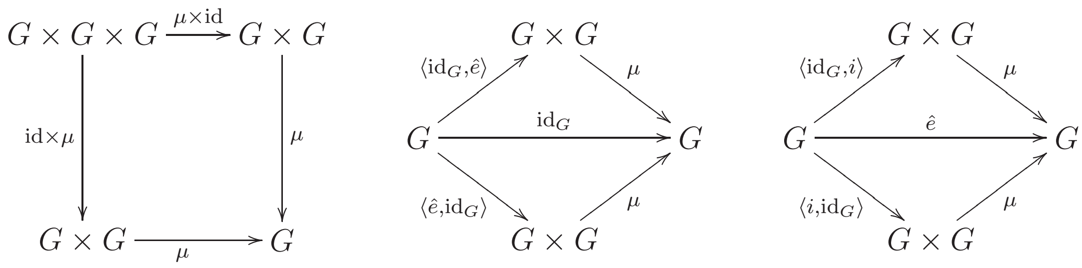
1 Super Lie Groups

A Lie group is a group object in the category of manifolds. Likewise a super Lie group is a group object in the category of supermanifolds.

Definition 1.1. A real *super Lie group* G is a real smooth super manifold G together with three morphisms:

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ i : G &\longrightarrow G \\ e : \mathbf{R}^{0|0} &\longrightarrow G \end{aligned}$$

called *multiplication*, *inverse*, and *unit* respectively satisfying the following commutative diagrams:



where $\hat{e}$ denotes the composition of the identity $e : \mathbf{R}^{0|0} \longrightarrow G$ with the unique map $G \longrightarrow \mathbf{R}^{0|0}$. $\langle \psi, \phi \rangle$ denotes the map $(\psi \times \phi) \circ \mathrm{d}_G$, $\mathrm{d}_G : G \longrightarrow G \times G$ being the canonical diagonal map.

We may of course interpret all these maps and diagrams in the language of T-points, which gives us (for any supermanifold T) morphisms $\mu_T : G(T) \times G(T) \longrightarrow G(T)$, etc. that obey the same commutative diagrams. In other words, if G is a SLG then the functor $T \to G(T)$ takes values in the category of set theoretical groups. Conversely, Yoneda's Lemma says that if the functor $T \to G(T)$ takes values in the category of set theoretical groups, then G is actually a super Lie group. This leads us to an alternative definition of a super Lie group.

Definition 1.2. A supermanifold G is a *super Lie group* if for any supermanifold T, $G(T)$ is a group, and for any supermanifold S and morphism $T \longrightarrow S$, the corresponding map $G(S) \longrightarrow G(T)$ is a group homomorphism.

In other words, G is a super Lie group if and only if $T \mapsto G(T)$ is a functor into the category of groups.

Remark 1.3. Let us notice that to each super Lie group is associated a Lie group $\widetilde{G}$. It is defined as the underlying manifold $\widetilde{G}$ with the "reduced morphisms"

$$\begin{aligned} |\mu| : \widetilde{G} \times \widetilde{G} &\longrightarrow \widetilde{G} \\ |i| : \widetilde{G} &\longrightarrow \widetilde{G} \\ |e| : \mathbf{R}^0 &\longrightarrow \widetilde{G} \end{aligned}$$

Since the map $\phi \longmapsto |\phi|$, that associates to any supermanifold morphism $\phi : M \longrightarrow N$ the morphism $|\phi| : \tilde{M} \longrightarrow \tilde{N}$ between the associated reduced manifolds, is functorial, it is immediate that $(\widetilde{G}, |\mu|, |i|, |e|)$ is a Lie group.

Example 1.4. Let us consider the super Lie group $\mathbf{R}^{1|1}$ through the symbolic language of T-points. The product morphism $\mu : \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \longrightarrow \mathbf{R}^{1|1}$ is given by

$$(t, \theta) \cdot (t', \theta') = (t + t' + \theta\theta', \theta + \theta') \tag{1}$$

where the coordinates (t, θ) and (t', θ') represent two distinct T-points for some supermanifold T. It is then clear by the formula (1) that the group axioms are satisfied. We leave to the reader as an exercise to find the inverse and antipode.

Remark 1.5. Notice that the properties required in definition 1.1 translate into properties of the morphisms on the global sections: $\mu^* : \mathcal{O}(G) \longrightarrow \mathcal{O}(G\times$

G), $i^* : \mathcal{O}(G) \longrightarrow \mathcal{O}(G)$, that make $\mathcal{O}(G)$ "almost" a Hopf superalgebra. One word of caution: since $\mathcal{O}(G\times G) \cong \mathcal{O}(G)\hat{\otimes}\mathcal{O}(G)$, strictly speaking $\mathcal{O}(G)$ is not a Hopf superalgebra, but a *topological Hopf superalgebra*, meaning that, since we are taking a completion of the tensor product, we are allowed to consider infinite sums.

2 The super Lie algebra of a super Lie group

For an ordinary Lie group H, we can define a morphism ℓ_h, the left multiplication by $h \in H$, as:

$$H \xrightarrow{\ell_h} H; \qquad a \longmapsto ha \tag{2}$$

(for $a \in H$). The differential of this morphism gives

$$T_a(H) \xrightarrow{(d\ell_h)_a} T_{ha}(H) \tag{3}$$

and for a vector field X on H, we say that X is *left invariant* if

$$d\ell_h \circ X = X \circ \ell_h. \tag{4}$$

We want to interpret this in the super category by saying that a left invariant vector field on G is invariant with respect to the group law μ^* "on the left".

Let G be a super Lie group with group law $\mu : G \times G \to G$ and let us denote with $\mathbb{1}$ the identity at the level of sheaf morphisms.

Definition 2.1. A vector field $X \in \mathrm{Vec_G}$ is said to be *left-invariant* if

$$(\mathbb{1} \otimes X) \circ \mu^* = \mu^* \circ X$$

Analogously a vector field $X \in \mathrm{Vec_G}$ is said to be *right-invariant* if

$$(X \otimes \mathbb{1}) \circ \mu^* = \mu^* \circ X$$

Since the bracket of left invariant vector fields is left invariant, as one can readily check, the left invariant vector fields are a super Lie subalgebra of Vec_G, which we denote by $\mathfrak{g}$.

Definition 2.2. Let G be a super Lie group. Then

$$\mathfrak{g} = \{X \in \mathrm{Vec}_G \mid (\mathbb{1} \otimes X)\mu^* = \mu^* X\}$$

is the super Lie algebra associated with the super Lie group G, and we write $\mathfrak{g} = \mathrm{Lie}(G)$ as usual.

Next proposition says that $\mathfrak{g} = \mathrm{Lie}(G)$ is a finite dimensional supervector space canonically identified with the super tangent space at the identity of the super Lie group G.

Proposition 2.3. *Let G be a super Lie group*

i) If X_e denotes a vector in T_eG, then

$$X := (\mathbb{1} \otimes X_e)\mu^*$$

is a left invariant vector field. Similarly $X^R := (X_e \otimes 1)\mu^*$ is a right invariant vector field.

ii) The map

$$\begin{aligned} T_eG &\longrightarrow \mathfrak{g} \\ X_e &\longrightarrow X := (\mathbb{1} \otimes X_e)\mu^* \end{aligned} \tag{5}$$

is an isomorphism of super vector spaces. Similarly for right vector fields.

Proof. To prove (i) for the left invariant vector fields, we need to show that:

$$[\mathbb{1} \otimes ((\mathbb{1} \otimes X_e)\mu^*)] \circ \mu^* = \mu^* \circ [(\mathbb{1} \otimes X_e)\mu^*].$$

This is a simple check that uses the coassociativity of μ^*, that is $(\mathbb{1} \otimes \mu^*)\mu^* = (\mu^* \otimes \mathbb{1})\mu^*$. In fact

$$\begin{aligned} [\mathbb{1} \otimes ((\mathbb{1} \otimes X_e)\mu^*)] \circ \mu^* &= (\mathbb{1} \otimes X_e)(\mathbb{1} \otimes \mu^*)] \circ \mu^* = \\ &= (\mathbb{1} \otimes X_e)(\mu^* \otimes \mathbb{1})] \circ \mu^* = \mu^* \circ [(\mathbb{1} \otimes X_e)\mu^*]. \end{aligned}$$

As for (ii) we notice that the injectivity of the map 5 is immediate. Let us hence focus on the surjectivity. Suppose X is a left invariant vector field, i. e. $(\mathbb{1} \otimes X)\mu^* = \mu^* X$. Apply $\mathbb{1} \otimes e^*$ to this equality:

$$(\mathbb{1} \otimes e^*)(\mathbb{1} \otimes X)\mu^* = (\mathbb{1} \otimes e^*)\mu^* X$$

we get $X = (\mathbb{1} \otimes X_e)\mu^*$, since $(\mathbb{1} \otimes e^*)\mu^* = \mathbb{1}$. So we are done. ■

We can use previous proposition to endow T_eG with the structure of a super Lie algebra and to identify it with $\mathfrak{g}$. From now on we shall use such an identification freely without an explicit mention.

Example 2.4. We want to calculate the left invariant vector fields on $\mathbf{R}^{1|1}$ with the group law from example (1.4)

$$(t,\theta) \cdot_\mu (t',\theta') = (t + t' + \theta\theta', \theta + \theta'). \tag{6}$$

In terms of μ^* the group law reads:

$$\begin{aligned} \mu^*(t) &= t \otimes \mathbb{1} + \mathbb{1} \otimes t + \theta \otimes \theta \\ \mu^*(\theta) &= \theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta. \end{aligned}$$

From proposition (2.3), we know that the Lie algebra of left invariant vector fields can be extracted from $T_eG = \text{span}\{\frac{\partial}{\partial t}|_e, \frac{\partial}{\partial \theta}|_e\}$. We use the identity 5 to calculate the corresponding left invariant vector fields:

$$\left(\mathbb{1} \otimes \frac{\partial}{\partial t}\bigg|_e\right) \circ \mu^*, \qquad \left(\mathbb{1} \otimes \frac{\partial}{\partial \theta}\bigg|_e\right) \circ \mu^*. \tag{7}$$

To get coordinate representations of (7), we apply them to coordinates (t,θ):

$$\begin{aligned} (\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e) \circ \mu^*(t) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e)(t \otimes \mathbb{1} + \mathbb{1} \otimes t + \theta \otimes \theta) = 1 \\ (\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e) \circ \mu^*(\theta) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial t}\big|_e)(\theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta) = 0; \end{aligned} \tag{8}$$

$$\begin{aligned} (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e) \circ \mu^*(t) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e)(t \otimes \mathbb{1} + \mathbb{1} \otimes t + \theta \otimes \theta) = -\theta \\ (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e) \circ \mu^*(\theta) &= (\mathbb{1} \otimes \tfrac{\partial}{\partial \theta}\big|_e)(\theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta) = 1. \end{aligned} \tag{9}$$

Thus the left invariant vector fields on $(\mathbf{R}^{1|1}, \mu)$ are

$$\frac{\partial}{\partial t}, \qquad -\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}. \tag{10}$$

A quick check using the definition shows that (10) are in fact left invariant.

Proposition 2.5. *Let G and H be super Lie groups and let $\phi : G \longrightarrow H$ be a morphism of super Lie groups. The map*

$$(\mathrm{d}\phi)_e : \mathfrak{g} \longrightarrow \mathfrak{h}$$

is a super Lie algebra homomorphism.

Proof. The only thing to check is that $(\mathrm{d}\phi)_e$ preserves the super Lie bracket. We leave this to the reader as an easy exercise, recalling that the bracket has always to be computed on the left invariant vector fields. ■

Corollary 2.6. *The even part of the super Lie algebra* $\mathrm{Lie}(G)$ *canonically identifies with* $\mathrm{Lie}(\widetilde{G})$.

Proof. The thesis is immediate considering the canonical inclusion $j : \widetilde{G} \to G$ and the previous proposition. ■

We end this section showing that the reduced Lie group $\widetilde{G}$ acts on G in a natural way.

Definition 2.7. If G is a super Lie group and M is a supermanifold, we say that G *acts* on M if we have a morphism

$$a : G \times M \longrightarrow M$$

defined on the functor of points as:

$$\begin{aligned} a_T : G(T) \times M(T) &\longrightarrow M(T) \\ (g, m) &\longmapsto g \cdot m \end{aligned}$$

such that:

1. $1 \cdot m = m$,
2. $g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m$.

In other words:

$$a \circ \langle \hat{e}, \mathbb{1}_M \rangle = \mathbb{1}_M \tag{11a}$$

$$a \circ (\mu \times \mathbb{1}_M) = a \circ (\mathbb{1}_G \times a) \tag{11b}$$

where $\mathbb{1}_M : M \longrightarrow M$ denotes the identity morphism of a supermanifold M and $\hat{e} : G \longrightarrow G$ is a super morphism defined, in the functor of points notation, as $\hat{e}_T(g) = e_T$, where e_T is the identity element in the group $G(T)$.

a is called an *action* of G on M.

Example 2.8. 1. *Left multiplication.* Since G is a super Lie group, it acts on itself through group multiplication $\mu : G \times G \to G$. Fix an element $\overline{g} : \mathbf{R}^{0|0} \longrightarrow G$ in $|G|$ and define the *left translation* by $\overline{g}$:

$$l_{\overline{g}} : G \simeq \mathbf{R}^{0|0} \times G \xrightarrow{i_{\overline{g}} \times \mathrm{id}} G \times G \xrightarrow{\mu} G$$

This induces an action:

$$\underline{a} : \widetilde{G} \times G \quad \longrightarrow \quad G$$

which we call left multiplication by $\widetilde{G}$. At the level of sections we have

$$l_{\overline{g}}^* f \quad = \quad (\mathrm{ev}_{\overline{\mathrm{g}}} \otimes \mathbb{1}) \circ \mu^* \,(\mathrm{f}) \tag{12}$$
$$\underline{a}^* f \quad = \quad (j \otimes \mathbb{1}) \circ \mu^*(f). \tag{13}$$

where j denotes the embedding of $\widetilde{G}$ in G.

2. *Adjoint representation.* We can define for each $g \in |G|$ a morphism $c_g : G \longrightarrow G$, $c_g(x) = gxg^{-1}$ (recall that any topological point of G can be viewed naturally as a T-point of G for all T). We define $Ad(g) = (\mathrm{d}c_g)_e$. One can check that $dAd(X)(Y) = [X, Y]$ for $X \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}$, with $\mathfrak{g} = \mathrm{Lie}(G)$.

3 Super Harish-Chandra pairs

Super Harish-Chandra pairs (SHCP for short) give an equivalent way to approach the theory of Lie supergroups. A SHCP essentially consists of a pair $(G_0, \mathfrak{g})$, where G_0 is an ordinary Lie group and $\mathfrak{g}$ a Lie superalgebra, such that $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, together with some natural compatibility conditions. The name comes from the analogy with the theory of Harish-Chandra pairs, that is the pairs consisting of a compact Lie group K and a Lie algebra $\mathfrak{g}$, with a Cartan involution corresponding to the compact form $\mathfrak{k} = Lie(K)$. Harish-Chandra pairs are very important in the theory of representation of Lie groups and we shall see in the next chapter that SHCP provide an effective method to study the representations of Lie supergroups.

Definition 3.1. Suppose $(G_0, \mathfrak{g}, \sigma)$ are respectively a Lie group, a super Lie algebra and a representation of G_0 on $\mathfrak{g}$ such that

1. $\mathfrak{g}_0 \simeq \mathrm{Lie}(G_0)$,
2. σ acts on $\mathfrak{g}_0$ as the adjoint representation of G_0 on $\mathrm{Lie}(G_0)$.

$(G_0, \mathfrak{g}, \sigma)$ is called a *a super Harish-Chandra pair (SHCP).*

In order to ease the notations, we will often refer to a SHCP simply by $(G_0, \mathfrak{g})$. Moreover we will never mention explicitly the isomorphism in items 1. and 2. .

A morphism of SHCPs is simply a pair of morphisms $\psi = \left(\psi_0, \rho^\psi\right)$ preserving the SHCP structure.

Definition 3.2. Let $(G_0, \mathfrak{g}, \sigma)$ and $(H_0, \mathfrak{h}, \tau)$ be SHCP. A *morphism* between them is a pair (ψ_0, ρ^ψ) such that

1. $\psi_0 : G_0 \to H_0$ is a Lie groups homomorphism
2. $\rho^\psi : \mathfrak{g} \to \mathfrak{h}$ is a super Lie algebra homomorphism
3. ψ_0 and ρ^ψ are compatible in the sense that:

$$\rho^\psi_{|\mathfrak{g}_0} \simeq \mathrm{d}\psi_0 \qquad \tau(\psi_0(g)) \circ \rho^\psi = \rho^\psi \circ \sigma(g).$$

Example 3.3. If G is a super Lie group, the pair $(\widetilde{G}, \mathfrak{g})$ given by the reduced Lie group of G and the super Lie algebra $\mathfrak{g}$ is a super Harish-Chandra pair with respect to the adjoint action of $\widetilde{G}$ on $\mathfrak{g}$ as defined in def. **??**. Moreover, given a morphism $\phi : G \longrightarrow H$ of super Lie groups, ϕ determines the morphism of the corresponding super Harish-Chandra pairs:

$$\left(|\phi|, (\mathrm{d}\phi)_e\right).$$

We can summarize our previous considerations by saying that we have defined a functor

$$\begin{aligned} \mathcal{H} : \ \mathbf{SGrp} &\longrightarrow (\mathrm{shcps}) \\ G &\longrightarrow (\widetilde{G}, \mathfrak{g}, \mathrm{Ad}) \end{aligned}$$

from the category of super Lie groups to the category of super Harish Chandra pairs. The most important results of this section is the following:

Theorem 3.4. *The category of super Lie groups is equivalent to the category of super Harish Chandra pairs.*

Roughly speaking this theorem says that each problem in the category of super Lie groups can be reformulated as an equivalent problem in the language of SHCP. We shall not prove this theorem here, but let us just outline the path to follow. We need show that:

i) given a SHCP $(G_0, \mathfrak{g})$ there exists a super Lie group G whose associated SHCP is isomorphic to $(G_0, \mathfrak{g})$

ii) given a morphism of SHCP $(\psi_0, \rho_\psi) : (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$ there exists a unique morphism ψ of the corresponding super Lie groups from which (ψ_0, ρ_ψ) arises.

iii) due to points i) ad ii) we have a functor

$$\mathcal{K} : \text{(shcps)} \longrightarrow \textbf{SGrp}$$

In order to prove the theorem we have to show that $\mathcal{K} \circ \mathcal{H} \simeq 1_{\textbf{SGrp}}$ and $\mathcal{H} \circ \mathcal{K} \simeq 1_{\text{(shcps)}}$. This means that, for each $G \in \textbf{SGrp}$ and $(G_0, \mathfrak{g}) \in$ (shcps), $\mathcal{K} \circ \mathcal{H}(G) \simeq G$ and $(\mathcal{H} \circ \mathcal{K})((G_0, \mathfrak{g})) \simeq (G_0, \mathfrak{g})$, and moreover the diagrams

$$\begin{array}{ccc} (\mathcal{K}\circ\mathcal{H})(G) & \xrightarrow{\sim} & G \\ \downarrow{\scriptstyle (\mathcal{K}\circ\mathcal{H})(\phi)} & & \downarrow{\scriptstyle \phi} \\ (\mathcal{K}\circ\mathcal{H})(H) & \xrightarrow{\sim} & H \end{array} \qquad \begin{array}{ccc} (\mathcal{H}\circ\mathcal{K})((G_0,\mathfrak{g})) & \xrightarrow{\sim} & (G_0,\mathfrak{g}) \\ \downarrow{\scriptstyle (\mathcal{H}\circ\mathcal{K})(\chi)} & & \downarrow{\scriptstyle \chi} \\ (\mathcal{H}\circ\mathcal{K})((H_0,\mathfrak{h})) & \xrightarrow{\sim} & (H_0,\mathfrak{h}) \end{array} \tag{14}$$

commute for each $\phi \colon G \to H$ and $\chi \colon (G_0, \mathfrak{g}) \to (H_0, \mathfrak{h})$.

We start with the reconstruction of a super Lie group from a SHCP. Suppose hence a SHCP $(G_0, \mathfrak{g}, \sigma)$ is given and notice that

1. $\mathfrak{U}(\mathfrak{g})$ is naturally a left $\mathfrak{U}(\mathfrak{g}_0)$-module;

2. for each open set $U \subseteq G_0$, $\mathcal{C}^\infty_{G_0}(U)$ is a left $\mathfrak{U}(\mathfrak{g}_0)$ module. In fact (see for example, [?]) each $X \in \mathfrak{U}(\mathfrak{g}_0)$ acts from the left on smooth functions on G_0 as the left invariant differential operator[1]

$$(\widetilde{D}^L_X f)(g) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} f(g e^{tX}) \Big|_{t=0}$$

[1] Notice that, as already remarked, here and in the following we don't mention explicitly the isomorphism $\mathrm{Lie}(G_0) \simeq \mathfrak{g}_0$ appearing in the definition of a SHCP.

Hence, for each open subset $U \subseteq G_0$, we can define the assignment:

$$U \longmapsto \mathcal{O}_G(U) := \underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\big(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty_{G_0}(U)\big)$$

where the r. h. s. is the subset of $\underline{\mathrm{Hom}}\big(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty_{G_0}(U)\big)$ consisting of $\mathfrak{U}(\mathfrak{g}_0)$-linear morphisms. (Notice that, for the moment, G is just a letter and we have not defined any supergroup structure on G_0).

Remark 3.5. If $\mathfrak{g} = \mathfrak{g}_0$ we have:

$$\underline{\mathrm{Hom}}_{\mathfrak{U}(\mathfrak{g}_0)}\big(\mathfrak{U}(\mathfrak{g}), \mathcal{C}^\infty_{G_0}(U)\big) \cong \mathcal{C}^\infty_{G_0}(U)$$

In fact a $\mathfrak{U}(\mathfrak{g}_0)$-linear map is uniquely determined by the image of $1 \in \mathfrak{U}(\mathfrak{g})$ and consequently we can uniquely associate to any morphism an element of $\mathcal{C}^\infty_{G_0}(U)$.

All the details for this construction can be found in [3].

4 Homogeneous spaces for super Lie groups

We are now interested in the construction of homogeneous spaces for super Lie groups.

Let G be a Lie supergroup and H a closed Lie subsupergroup. We want to define a supermanifold structure on the topological space $|G|/|H|$. This structure will turn out to be unique once we impose some natural conditions on the action of G on its quotient. In order to do this we first define a supersheaf $\mathcal{O}_X$ on $|G|/|H|$, in other words we define a superspace $X = (|G|/|H|, \mathcal{O}_X)$. We then prove the local splitting property for X, that is we show that X is locally isomorphic to domain in $k^{p|q}$ for some p and q. Here $k = \mathbf{R}$ or $\mathbf{C}$ according as G is a real or complex analytic Lie supergroup. We start by defining the supersheaf $\mathcal{O}_X$ on $|G|/|H|$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. For each $Z \in \mathfrak{g}$, let D_Z be the left invariant vector field on G defined by Z. For $x_0 \in |G|$ let ℓ_{x_0} and r_{x_0} be the left and right translations of G by x_0. We denote by $i_{x_0} = \ell_{x_0} \circ r_{x_0}^{-1}$ the inner automorphism defined by x_0. It fixes the identity and induces the transformation Ad_{x_0} on $\mathfrak{g}$.

Definition 4.1. For any sub-superalgebra $\mathfrak{k}$ of $\mathfrak{g}$ we define the subsheaf $\mathcal{O}_{\mathfrak{k}}$ of $\mathcal{O}_G$ as:

$$\mathcal{O}_{\mathfrak{k}}(U) = \{f \in \mathcal{O}_G(U) \mid D_Z f = 0 \text{ on } U \text{ for all } Z \in \mathfrak{k}\}, \qquad \forall U \quad \text{open} \quad \subset |G|$$

On the other hand, for any open subset $\hat{W} \subset |G|$, invariant under right translations by elements of $|H|$, we put

$$\mathcal{O}_{inv}(\hat{W}) = \{f \in \mathcal{O}_G(\hat{W}) \mid f \text{ is invariant under } r_{x_0} \text{ for all } x_0 \in |H|\}.$$

If $|H|$ is connected we have

$$\mathcal{O}_{inv}(\hat{W}) = \mathcal{O}_{\mathfrak{h}_0}(\hat{W}).$$

For any open set $|W| \subset |X| = |G|/|H|$ with $|\hat{W}| = \pi_0^{-1}(|W|)$ we put

$$\mathcal{O}_X(|W|) = \mathcal{O}_{inv}(\hat{W}) \cap \mathcal{O}_{\mathfrak{h}}(\hat{W}).$$

Clearly $\mathcal{O}_X(|W|) = \mathcal{O}_{\mathfrak{h}}(\hat{W})$ if $|H|$ is connected. The subsheaf $\mathcal{O}_X$ is a supersheaf on $|X|$. We have thus defined a ringed superspace $X = (|X|, \mathcal{O}_X)$. Our aim is to prove that X is a supermanifold with $\mathcal{O}_X$ as its structure sheaf.

It is clear that the left action of the group $|G|$ on $|X|$ leaves $\mathcal{O}_X$ invariant and so it is enough to prove that there is an open neighborhood $|W|$ of $|\pi|(1) \equiv \bar{1}$ with the property that $(|W|, \mathcal{O}_X|_{|W|})$ is a super domain, i. e., isomorphic to an open submanifold of $k^{p|q}$.

We will do this using the local Frobenius Theorem (see ch. **??**). Also, we identify as usual $\mathfrak{g}$ with the space of all left invariant vector fields on G, thereby identifying the tangent space of G at every point canonically with $\mathfrak{g}$ itself.

On G we have a distribution spanned by the vector fields in $\mathfrak{h}$. We denote it by $\mathcal{D}_{\mathfrak{h}}$.

On each $|H|$-coset $x_0|H|$ we have a supermanifold structure which is a closed submanifold of G. It is an integral manifold of $\mathcal{D}_{\mathfrak{h}}$, i. e. the tangent space at any point is the subspace $\mathfrak{h}$ at that point. By the local Frobenius theorem there is an open neighborhood U of 1 and coordinates x_i, $1 \leq i \leq n$ and θ_α, $1 \leq \alpha \leq m$ on U such that on U, $\mathcal{D}_{\mathfrak{h}}$ is spanned by $\partial/\partial x_i, \partial/\partial\theta_\alpha (1 \leq i \leq r, 1 \leq \alpha \leq s)$. Moreover, from the theory on $|G|$ we may assume that the slices $L(\mathbf{c}) := \{(x_1, \dots, x_n) \mid \ x_j = c_j, \quad r+1 \leq j \leq n\}$ are open subsets of distinct $|H|$-cosets for distinct $\mathbf{c} = (c_{r+1}, \dots, c_n)$. These slices are therefore supermanifolds with coordinates x_i, θ_α, $1 \leq i \leq r$, $1 \leq \alpha \leq s$. We have a submanifold W' of U defined by $x_i = 0$ with $1 \leq i \leq r$ and $\theta_\alpha = 0$ with $1 \leq \alpha \leq s$. The map $|\pi| : |G| \longrightarrow |X|$ may be assumed to be a diffeomorphism of $|W'|$ with its image $|W|$ in $|X|$ and so we may view $|W|$ as a superdomain, say W. The map $|\pi|$ is then a diffeomorphism of W' with W. What we want to show is that $W \cong (|W|, \mathcal{O}_X|_{|W|})$.

Lemma 4.2. *The map*

$$\begin{array}{ccc} W' \times H & \stackrel{\gamma}{\longrightarrow} & G \\ w, h & \longrightarrow & wh \end{array}$$

is a super diffeomorphism of $W' \times H$ onto the open sub-supermanifold of G with reduced manifold the open subset $|W'||H|$ of $|G|$.

Proof. The map γ in question is the informal description of the map $\mu \circ (i_{W'} \times i_H)$ where i_M refers to the canonical inclusion $M \hookrightarrow G$ of a sub-supermanifold of G into G, and $\mu : G \times G \longrightarrow G$ is the multiplication morphism of the Lie supergroup G. We shall use such informal descriptions without comment from now on.

It is classical that the reduced map $|\gamma|$ is a diffeomorphism of $|W'| \times |H|$ onto the open set $U = |W|'|H|$. This uses the fact that the cosets $w|H|$ are distinct for distinct $w \in |W'|$. It is thus enough to show that $\mathrm{d}\gamma$ is surjective at all points of $|W'| \times |H|$. For any $h \in |H|$, right translation by h (on the second factor in $W' \times H$ and simply r_h on G) is a super diffeomorphism commuting with γ and so it is enough to prove this at $(w, 1)$. If $X \in \mathfrak{g}$ is tangent to W' at w and $Y \in \mathfrak{h}$, then

$$\mathrm{d}\gamma(X, Y) = \mathrm{d}\gamma(X, 0) + \mathrm{d}\gamma(0, Y) = \mathrm{d}\mu(X, 0) + \mathrm{d}\mu(0, Y) = X + Y.$$

Hence the range of $d\gamma$ is all of $\mathfrak{g}$ since, from the coordinate chart at 1 we see that the tangent spaces to W' and $w|H|$ at w are transversal and span the tangent space to G at w which is $\mathfrak{g}$. This proves the lemma. ■

Lemma 4.3. *We have*

$$\gamma^* \mathcal{O}_X \Big|_{|W|} = \mathcal{O}_{W'} \otimes 1,$$

where $\gamma^* : \mathcal{O}_G \longrightarrow \gamma_* \mathcal{O}_{W' \times H}$.

Proof. To ease the notation we drop the open set in writing a sheaf superalgebra, that is we will write $\mathcal{O}_X$ instead of $\mathcal{O}_X(U)$.

We want to show that for any g in $\mathcal{O}_X\big|_U$, $\gamma^* g$ is of the form $f \otimes 1$ and that the map $g \longmapsto f$ is bijective with $\mathcal{O}_{W'}$. Now γ^* intertwines $D_Z (Z \in \mathfrak{h})$ with $1 \otimes D_Z$ and so $(1 \otimes D_Z)\gamma^* g = 0$. Since the D_Z span all the super vector fields on $|H|$ it follows using charts that for any $p \in |H|$ we have $\gamma^* g = f_p \otimes 1$ locally around p for some $f_p \in \mathcal{O}_{W'}$. Clearly f_p is locally constant in p. Hence f_p is independent of p if $|H|$ is connected. If we do assume that $|H|$ is connected, the right invariance under $|H|$ shows that f_p is independent of p. In the other direction it is obvious that if we start with $f \otimes 1$ it is the image of an element of $\mathcal{O}_X\big|_U$. ■

Theorem 4.4. *The superspace $(|X|, \mathcal{O}_X)$ is a supermanifold.*

Proof. At this stage by the previous lemmas we know that $(|X|, \mathcal{O}_X)$ is a super manifold at $\bar{1}$. The left invariance of the sheaf under $|G|$ shows this to be true at all points of $|X|$. ■

We now want to describe the action of G on the supermanifold $X = (|G|/|H|, \mathcal{O}_X)$ we have constructed. Notice that in the course of our discussion we have also shown that there is a well defined morphism $\pi : G \longrightarrow X$.

Proposition 4.5. *There is a unique morphism $\beta : G \times X \longrightarrow X$ such that the following diagram*

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\beta} & X \end{array}$$

commutes.

Proof. Let $\alpha := \pi \circ \mu : G \times G \longrightarrow X$. The action of $|G|$ on $|X|$ shows that such a map $|\beta|$ exists at the reduced level. So it is a question of constructing the pull-back map

$$\beta^* : \mathcal{O}_X \longrightarrow \mathcal{O}_{G \times X}$$

such that

$$(1 \times \pi)^* \circ \beta^* = \alpha^*.$$

Now π^* is an *isomorphism* of $\mathcal{O}_X$ onto the sheaf $\mathcal{O}_G$ restricted to a sheaf on X $(W \longmapsto \mathcal{O}_G(|\pi|^{-1}(W)))$, an so to prove the *existence and uniqueness* of β^* it is a question of proving that α^* and $(1 \times \pi)^*$ have the same image in $\mathcal{O}_{G\times G}$. It is easy to see that $(1\times\pi)^*$ has as its image the subsheaf of sections f killed by $1 \otimes D_X(X \in \mathfrak{h})$ and invariant under $1 \times r_h(h \in |H|)$. It is not difficult to see that this is also the image of α^*. ■

We tackle now the question of the uniqueness of X.

Proposition 4.6. *Let X' be a super manifold with $|X'| = |X|$ and let π' be a morphism $G \longrightarrow X'$. Suppose that*

(a) π' is a submersion.

(b) The fibers of π' are the super manifolds which are the cosets of H.

Then there is a natural isomorphism

$$X \simeq X'.$$

Proof. Indeed, from the local description of submersions as projections it is clear that for any open $|W| \subset |X|$, the elements of $\pi'^*(\mathcal{O}_{X'}(|W|))$ are invariant under $r_h, (h \in |H|)$ and killed by $D_X(X \in \mathfrak{h})$. Hence we have a natural map $X' \longrightarrow X$ commuting with π and π'. This is a submersion, and by dimension considerations it is clear that this map is an isomorphism. ■

We have proved the following result:

Theorem 4.7. *Let G be a Lie supergroup and H a closed Lie subsupergroup. There exist a supermanifold $X = (|G|/|H|, \mathcal{O}_X)$ and a morphism $\pi : G \longrightarrow G/H$ such that the following properties are satisfied:*

1. *The reduction of π is the natural map $|\pi| : |G| \longrightarrow |X|$.*
2. *π is a submersion.*
3. *There is an action β from the left of G on X reducing to the action of $|G|$ on $|X|$ and compatible with the action of G on itself from the left through π:*

$$\begin{array}{ccc} G\times G & \xrightarrow{\mu} & G \\ {\scriptstyle 1\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G\times X & \xrightarrow{\beta} & X \end{array}$$

Moreover, the pair (X, π), subject to the properties 1, 2, and 3 is unique up to isomorphism. The isomorphism between two choices is compatible with the actions, and it is also unique.

Proof. Immediate from previous lemmas and propositions. ■

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