SUPERGROUPS, QUANTUM SUPERGROUPS
AND THEIR HOMOGENEOUS SPACES

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We give a more algebraic definition of algebraic supergroup via its Hopf algebra. The Hopf algebra of the supergroup SL(m|n) is given together with its quantization. The example of supercoadjoint orbits is examined at the end.

1. Preliminaries

We want to recall some more or less known general facts about supergeometry.

Let $k$ be an algebraically closed field of characteristic 0 and $A$ a commutative $k$-superalgebra, $A = A_0 \oplus A_1$, $A_0$, $A_1$ respectively the even and odd parts.

In the classical setting we have an equivalence between the category $V$ of affine algebraic varieties over $k$ and the category $\mathcal{A}$ of commutative $k$-algebras finitely generated and reduced. Given $R_c$ in $\mathcal{A}$, one can recover the affine variety $V_{R_c}$ whose coordinate ring is $R_c$ using the functor of points. The $A_c$-points of the variety $V_{R_c}$ are given by:

$V_{R_c}(A_c) = \text{Hom}_{k\text{-alg}}(R_c, A_c)$ with $A_c \in \mathcal{A}$.

Hence an affine algebraic variety can also be defined as a contravariant representable functor between the above-mentioned categories (for more details see Refs. 2 and 7).

This motivates the following definition.

**Definition 1.1.** We define **affine algebraic supervariety** over $k$ a representable contravariant functor $V$ from the category of commutative $k$-superalgebras finitely generated such that modulo the ideal generated by their odd part, they are reduced to the category of sets. Let us call $k[V]$ the commutative $k$-superalgebra representing the functor $V$,

$V(A) = \text{Hom}_{k\text{-superalg}}(k[V], A)$.

We will call $V(A)$ the $A$-points of the variety $V$. If the algebra $A$ is in addition a Hopf algebra we will say that $V$ is an **affine algebraic supergroup**.

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We define the supergroup $\text{SL}(m|n)$ as the Ane Supergroup $\text{SL}(m|n)$. For more details on this equivalence see Ref. 4.

This is an Ane supervariety represented by the commutative superalgebra: $k[M(m|n)] = k[x_{ij}, \xi_{kl}]$ where $x_{ij}$'s and $\xi_{kl}$'s are respectively even and odd variables and the range of the indices $i, j, k, l$ is the one specified above.

One can show that the given definition of affine supervariety is substantially equivalent to the one found in the literature (see Refs. 1, 3, 10–12 among many others). For more details on this equivalence see Ref. 4.

### 2. The Affine Supergroup $\text{SL}(m|n)$

We define the supergroup $\text{SL}(m|n)(A)$ as the group of automorphisms of $A^{m|n}$ with berezinian equal to 1 (see Ref. 3). These are the $A$-points of the supergroup functor $\text{SL}(m|n)$. To show this we need to give a commutative Hopf superalgebra $k[\text{SL}(m|n)]$ that represents it.

$$k[\text{SL}(m|n)] \overset{\text{def}}{=} k[x_{ij}, \xi_{kl}, d_{1:\ldots:m}^{m+1\ldots}m+n^{-1}, d_{m+1\ldots}^{m+1\ldots}m+n^{-1}]/(\text{Ber} - 1)$$

with $\xi_{kl}$ odd variables, $x_{ij}$, $d_{1:\ldots:m}^{m+1\ldots}m+n^{-1}$, $d_{m+1\ldots}^{m+1\ldots}m+n^{-1}$ even variables,

$$d_{1:\ldots:m}^{m+1\ldots}m+n^{-1} d_{1:\ldots:m}^{m+1\ldots}m+n^{-1} = d_{m+1\ldots}^{m+1\ldots}m+n^{-1} d_{m+1\ldots}^{m+1\ldots}m+n^{-1}$$

$$d_{1:\ldots:m}^{1\ldots}m = \det(X_{11}), \quad d_{m+1\ldots}^{m+1\ldots}m+n = \det(X_{22})$$

and

$$\text{Ber} = \det(X_{11} - \Xi_{12}S_{22}(X_{22})\Xi_{21})\det(X_{22})^{-1}$$

$$= \det(X_{11})^{-1}\det(X_{22} - \Xi_{21}S_{11}(X_{11})\Xi_{12}),$$

$$X_{11} = (x_{ij})_{1\leq i, j \leq m}, \quad X_{22} = (x_{ij})_{m+1\leq i, j \leq m+n},$$

$$\Xi_{12} = (\xi_{kl})_{m+1\leq k \leq m+n, 1 \leq i \leq m}, \quad \Xi_{21} = (\xi_{kl})_{1 \leq k \leq m, m+1\leq i \leq m+n}.$$
Proposition 2.1. $k[\text{SL}(m|n)]$ is a bialgebra with counit $\varepsilon$ and comultiplication $\Delta$:

\[
\varepsilon(a_{ij}) = \delta_{ij}, \quad \varepsilon(d_{1:m}^{i:j-1}) = 1, \quad \varepsilon(d_{m+1:m+n}^{i:j-1}) = 1,
\]

\[
\Delta(a_{ij}) = \sum_{k} a_{ik} \otimes a_{kl}, \quad (a_{ij}) = \begin{pmatrix} X_{11} & \Xi_{12} \\ \Xi_{21} & X_{22} \end{pmatrix},
\]

\[
\Delta(d_{1:m}^{i:j-1}) = \sum_{i=1}^{2m+n-1} d_{1:m}^{i:j-1} \otimes d_{1:m}^{i:j-1} (\Delta(d_{1:m}^{i:m}) - d_{1:m}^{i:m} \otimes d_{1:m}^{i:m})^{-1},
\]

\[
\Delta(d_{m+1:m+n}^{i:j-1}) = \sum_{i=1}^{2m+n-1} d_{m+1:m+n}^{i:j-1} \otimes d_{m+1:m+n}^{i:j-1}
\times (\Delta(d_{m+1:m+n}^{i:m}) - d_{m+1:m+n}^{i:m} \otimes d_{m+1:m+n}^{i:m})^{-1}.
\]

Remark 2.2. The proof that $\Delta$ is well-defined makes essential use of the nilpotency of the $\xi_{kl}$’s. The details will appear in Ref. 4.

To show that $k[\text{SL}(m|n)]$ is a Hopf algebra we need to define the antipode $S$. First we define the following maps for $1 \leq i, j \leq m, m + 1 \leq k, l \leq m + n$:

\[
S_{11}(x_{ij}) \overset{\text{def}}{=} (-1)^{i-j} A_{ji}^{11} d_{1:m}^{i:j-1}, \quad S_{22}(x_{kl}) \overset{\text{def}}{=} (-1)^{k-l} A_{lk}^{22} d_{m+1:m+n}^{i:j-1},
\]

where $A_{ji}^{11}$ and $A_{lk}^{22}$ denote the determinants obtained by suppressing the $j$th row and $i$th column in $X_{11}$ and the $l$th row and $k$th column in $X_{22}$ respectively.

Define

\[
B \overset{\text{def}}{=} X_{11} - \Xi_{12} \Sigma_{22}(X_{22}) \Xi_{21}, \quad C \overset{\text{def}}{=} X_{22} - \Xi_{21} \Sigma_{11}(X_{11}) \Xi_{12},
\]

\[
S_{1}(b_{ij}) = (-1)^{i-j} A_{ji}^{B} \det(X_{11} - \Xi_{12} \Sigma_{22}(X_{22}) \Xi_{21})^{-1},
\]

\[
S_{2}(c_{kl}) = (-1)^{k-l} A_{lk}^{C} \det(X_{22} - \Xi_{21} \Sigma_{11}(X_{11}) \Xi_{12})^{-1}.
\]

$A_{ji}^{B}$ and $A_{lk}^{C}$ denote the determinants obtained by suppressing the $j$th row and $i$th column in $B$ and the $l$th row and $k$th column in $C$ respectively.

Remark 2.3. Note that the determinants that appear in the definition of $S_{1}$ and $S_{2}$ are invertible in the ring $k[\text{SL}(m|n)]$.

Proposition 2.4. $k[\text{SL}(m|n)]$ is a Hopf algebra with antipode $S$:

\[
S(X_{11}) = S_{1}(X_{11} - \Xi_{12} \Sigma_{22}(X_{22}) \Xi_{21}),
\]

\[
S(X_{22}) = S_{2}(X_{22} - \Xi_{21} \Sigma_{11}(X_{11}) \Xi_{12}),
\]

\[
S(\Xi_{12}) = -S_{11}(X_{11}) \Xi_{12} \Sigma_{22}(X_{22} - \Xi_{21} \Sigma_{11}(X_{11}) \Xi_{12}),
\]

\[
S(\Xi_{21}) = -S_{22}(X_{22}) \Xi_{21} \Sigma_{11}(X_{11} - \Xi_{12} \Sigma_{22}(X_{22}) \Xi_{21}),
\]

\[
S(d_{1:m}^{i:j-1}) = d_{m+1:m+n}^{i:j-1}, \quad S(d_{m+1:m+n}^{i:j-1}) = d_{1:m}^{i:j-1}.
\]
Theorem 2.5. \( \text{Hom}_{k-\text{superalg}}(k[\text{SL}(m|n)], A) \) is the group of automorphisms of \( A^{m|n} \) with berezinian 1, hence \( k[\text{SL}(m|n)] \) represents the functor \( \text{SL}(m|n) \) (see Ref. 4).

3. The Quantum \( \text{SL}(m|n) \)

A quantum group is a Hopf algebra which is neither commutative nor cocommutative and depends on a parameter that specialized to 1 gives a commutative (cocommutative) Hopf algebra. Following this philosophy we can define a quantum supergroup in the same way.

Definition 3.1. Let \( A \) be a commutative (super)algebra over \( k \). A formal deformation of \( A \) is a noncommutative (super)algebra \( A_q \) over \( k_q = k[q, q^{-1}] \) such that \( A_q/(q - 1) \cong A \). We will refer to such deformation as a quantum (super)group.

We want to construct a formal deformation of \( k[\text{SL}(m|n)] \), the quantum special linear group \( k_q[\text{SL}(m|n)] \). Since \( \text{SL}(m|n)(A) \) is the group of automorphisms with \( \text{Ber} = 1 \) of \( A^{m|n} \), we ask that its quantization \( k_q[\text{SL}(m|n)] \) has a coaction on a certain quantum space. We first need to quantize the supervariety \( M(m|n) \) and the superspace \( A^{m|n} \).

\[
k_q[M(m|n)] \overset{\text{def}}{=} k_q[x_{ij}, \xi_{kl}]/I_M,
\]

where

\[
(X_{11} \Xi_{12}) = \left( \begin{array}{cc}
(x_{ij}) & (\xi_{il}) \\
(\xi_{lj}) & (x_{kl})
\end{array} \right) = (a_{rs}).
\]

The ideal \( I_M \) is generated by the relations\(^\text{12}\):

\[
a_{ij}a_{il} = (-1)^{\pi(a_{ij})} \pi(a_{il}) q^{(-1)^{p(i)+1}} a_{il}a_{ij}, \quad j < l,
\]

\[
a_{ij}a_{kj} = (-1)^{\pi(a_{ij})} \pi(a_{kj}) q^{(-1)^{p(i)+1}} a_{kj}a_{ij}, \quad i < k,
\]

\[
a_{ij}a_{kl} = (-1)^{\pi(a_{ij})} \pi(a_{kl}) a_{kl}a_{ij}, \quad i < k, j > l \text{ or } i > k, j < l,
\]

\[
a_{ij}a_{kl} - (-1)^{\pi(a_{ij})} \pi(a_{kl}) a_{kl}a_{ij} = (-1)^{p(i)p(k)+p(j)p(l)+p(k)p(l)}
\]

\[
\times (q^1 - (-1)^{\pi(a_{ij})} \pi(a_{kl}) q)a_{jk}a_{il}, \quad i < k, j < l,
\]

where \( p(i) = 0 \) if \( 1 \leq i \leq m \), \( p(i) = 1 \) otherwise and \( \pi(a_{ij}) \) denotes the parity of \( a_{ij} \).

Definition 3.2. Define the quantum superspace \( k_q^{m|n} \) as the ring generated over \( k \) by \( x_1 \cdots x_m \) and \( \xi_1 \cdots \xi_n \) subject to the relations\(^\text{12}\):

\[
x_i x_j - q^{-1} x_j x_i, \quad x_i \xi_k - q^{-1} \xi_k x_i, \quad 1 \leq i < j \leq m, \quad m + 1 \leq k \leq n + m,
\]

\[
\xi_k \xi_l + q^{-1} \xi_l \xi_k, \quad m + 1 \leq k < l \leq m + n, \quad \xi_k^2, \quad m + 1 \leq k \leq m + n.
\]

Define also dual quantum superspace \( (k_q^{m|n})^* \) as the ring generated over \( k \) by \( y_1 \cdots y_m \) and \( \eta_1 \cdots \eta_m \) subject to the relations\(^\text{12}\):

\[
y_i y_j - q y_j y_i, \quad y_i \eta_k - q \eta_k y_i, \quad 1 \leq i < j \leq m, \quad m + 1 \leq k \leq n + m,
\]

\[
\eta_i \eta_j + q \eta_j \eta_i, \quad m + 1 \leq k < l \leq m + n, \quad \eta_k^2, \quad m + 1 \leq k \leq m + n.
\]
**Observation 3.3.** The superalgebra $k_q[M(m|n)]$ admits a bialgebra structure with comultiplication and counit given by $\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj}$ and $\varepsilon(a_{ij}) = \delta_{ij}$ and a coaction on the quantum spaces $k_q^{m|n}$ and $(k_q^{m|n})^*$. (See Ref. 12 for more details.)

**Definition 3.4.** Quantum special linear supergroup.

$$k_q[\text{SL}(m|n)] \equiv \frac{k_q(\pi_{ij}: D_{11}^{m\cdots m-1} \otimes D_{m+1\cdots m+n}^{m+1\cdots m+n-1})}{(\text{Ber}_q - 1)},$$

$$D_{11}^{m\cdots m} D_{11}^{1\cdots m-1} = D_{11}^{1\cdots m-1} D_{11}^{1\cdots m},$$

$$D_{m+1\cdots m+n}^{m+1\cdots m+n} D_{m+1\cdots m+n}^{m+1\cdots m+n-1} = D_{m+1\cdots m+n}^{m+1\cdots m+n-1} D_{m+1\cdots m+n},$$

where $\text{Ber}_q \equiv \det_q(X_{11})^{-1} \det_q(X_{22} - \Xi_{21} S_{11}^q (X_{11}) \Xi_{12})$ is the quantum berezinian, (see Ref. 13) and $D_{11}^{m\cdots m}$, $D_{m+1\cdots m+n}$ denote respectively the quantum determinants obtained by taking the first $m$ rows and columns and the last $n$ rows and columns.

**Proposition 3.5.** Coalgebra structure for $k_q[\text{SL}(m|n)]$, $k_q[\text{SL}(m|n)]$ is a coalgebra with counit $\varepsilon$ and comultiplication $\Delta$:

$$\varepsilon(a_{ij}) = \delta_{ij}, \quad \varepsilon(D_{11}^{1\cdots m-1}) = 1, \quad \varepsilon(D_{m+1\cdots m+n}^{m+1\cdots m+n-1}) = 1,$$

$$\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj},$$

$$\Delta(D_{11}^{1\cdots m-1}) = \sum_{i=1}^{2m+1} D_{11}^{1\cdots m-i} \otimes D_{11}^{1\cdots m-i} (\Delta(D_{11}^{1\cdots m}) - D_{11}^{1\cdots m} \otimes D_{11}^{1\cdots m})^{i-1},$$

$$\Delta(D_{m+1\cdots m+n}^{m+1\cdots m+n-1}) = \sum_{i=1}^{2m+1} D_{m+1\cdots m+n}^{m+1\cdots m+n-i} \otimes D_{m+1\cdots m+n}^{m+1\cdots m+n-i}$$

$$\times (\Delta(D_{m+1\cdots m+n}) - D_{m+1\cdots m+n}^{m+1\cdots m+n} \otimes D_{m+1\cdots m+n}^{m+1\cdots m+n})^{i-1}.$$

We now want to examine the Hopf algebra structure for $k_q[\text{SL}(m,n)]$.

**Proposition 3.6.** $k_q[\text{SL}(m|n)]$ is a Hopf algebra with antipode $S^q$:

$$S^q(X_{11}) = S_{11}^q (X_{11} - \Xi_{12} S_{22}^q (X_{22}) \Xi_{21}),$$

$$S^q(\Xi_{12}) = -S_{12}^q (X_{11}) \Xi_{12} S_{22}^q (X_{22} - \Xi_{21} S_{11}^q (X_{11}) \Xi_{12}),$$

$$S^q(\Xi_{21}) = -S_{21}^q (X_{22}) \Xi_{21} S_{11}^q (X_{11} - \Xi_{12} S_{22}^q (X_{22}) \Xi_{21}),$$

$$S^q(X_{22}) = S_{22}^q (X_{22} - \Xi_{21} S_{11}^q (X_{11}) \Xi_{12}),$$

$$S^q(D_{11}^{1\cdots m-1}) = D_{m+1\cdots m+n}^{m+1\cdots m+n}, \quad S^q(D_{m+1\cdots m+n}^{m+1\cdots m+n-1}) = D_{11}^{1\cdots m},$$

where $S_{11}^q$ and $S_{22}^q$ denote respectively the antipode for the quantum matrices $X_{11}$ and $X_{22}$, while $S_{12}^q$ and $S_{21}^q$ are the antipodes for $X_{11} - \Xi_{12} S_{22}^q (X_{22}) \Xi_{21}$ and $X_{22} - \Xi_{21} S_{11}^q (X_{11}) \Xi_{12}$. (See Ref. 13 for more details.)
4. Homogeneous Spaces for the Supergroup $\text{SL}(m|n)$

Let us consider the adjoint action of the group $\text{SL}(m|n)(A)$ on $\text{sl}(m|n)(A)$ the supertraceless matrices in $M(m|n)(A)$. Fix $X_0 \in \text{sl}(m|n)(A)$ and define:

$$O_{X_0}(A) \overset{\text{def}}{=} \{ gX_0g^{-1} | g \in \text{SL}(m|n)(A) \}.$$ 

**Proposition 4.1.** $O_{X_0}$ is a supervariety in the sense of Definition 1.1 and it is given by the superring:

$$\mathbb{C}[M(m|n)]/(\text{str}(M) - c_1, \ldots, \text{str}(M^{m+n}) - c_{m+n}), \quad c_1, \ldots, c_{m+n} \in \mathbb{C},$$

where $\text{str}$ denotes the supertrace and $M$ is the matrix of indeterminates that generate $\mathbb{C}[M(m|n)]$.

The proof of this result together with some generalizations and remarks can be found in Ref. 5.

References