Euler and the modern mathematician

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Abstract

It is remarkable that Euler, who preceded Gauss and Riemann, is still very much with us. The ramifications of his work are still not exhausted, three hundred years after his birth. In number theory, in algebraic geometry, in topology, in the calculus of variations, and in analysis, both conceptual and numerical, not to mention mechanics of particles and solid bodies, astronomy, hydrodynamics, and other applied areas, the ideas that he generated are still motivating mathematicians. In this talk, which will require no specialized background, I shall focus on his pioneering work in the theory of summability of divergent series, which was the precursor to the entire modern theory of divergent series and integrals.
Leonhardi Euleri Opera Omnia

• FOUR SERIES WITH SERIES I DEVOTED TO PURE MATHEMATICS

• SEVERAL TREATISES: ALGEBRA, ANALYSIS, LETTERS TO A GERMAN PRINCESS

• TOTAL NUMBER OF PAGES: 31,529+

• CORRESPONDENCE: 2,498 PAGES+

• UNPUBLISHED MANUSCRIPTS, DIARIES, ETC
Closer look

• I. Series prima: Opera Mathematica (29 vols), 14042 pages
• II. Series secunda: Opera mechanica et astronomica (30 vols), 10658 pages
• III. Series tertia: Opera Physica, Miscellena (12 vols), 4331 pages
• IV A. Series quarta A: Commercium Epistolicum (7 vols, 1 in preparation), 2498 pages
• IV B. Series quarta B: Manuscripta (unpublished manuscripts, notes, diaries, etc)
Euler on summing divergent series

• De seriebus divergentibus

• Communicated in 1755 and published in 1760.

• ... (for) the divergent series par excellence

\[
1 - 1 + 2 - 6 + 24 - 120 + 720 - 5040 + \ldots
\]

... after various attempts, the author by a wholly singular method using continued fractions found that the sum of this series is about 0.596347362123, and in this decimal fraction the error does not affect even the last digit. Then he proceeds to other similar series of wider application and he explains how to assign them a sum in the same way, where the word “sum” has that meaning which he has here established and by which all controversies are cut off.

from the translation by E. J. Barbeau and P. J. Leah
Additional comments on divergent series

... I believe that every series should be assigned a certain value. However, to account for all the difficulties that have been pointed out in this connection, this value should not be denoted by the name sum, because usually this word is connected with the notion that a sum has been obtained by a real summation: this idea however is not applicable to “seriebus divergentibus ... 

... this is language which might almost have been used by Cesàro or Borel (Hardy)
Summation of divergent series

The idea that the notion of a sum can be attached to divergent series thus goes back to Euler, who also invented many modern methods of summation. One of the simplest of the methods that Euler used is to introduce the variable $x$ and define

$$\sum_n a_n \overset{\text{def}}{=} f(1) \quad \text{where } f(x) = \sum_n a_n x^n.$$

Nowadays this is called the *Abel summation*. Thus

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots = \left(\frac{1}{1 + x}\right)_{x=1} = \frac{1}{2}. $$

$$1 - 2 + 3 - 4 + 5 - \ldots = \left(\frac{1}{(1 + x)^2}\right)_{x=1} = \frac{1}{4}. $$

The first series goes back to Leibniz who argued that the sum must be $1/2$ since the partial sums are 0 and 1 roughly half the time. Leibniz’s argument generated a big controversy in which even religious figures joined.
Summation of divergent series of positive terms

- **Failure of Euler-Abel**

  If all the $a_n$ are $\geq 0$ and $\sum_n a_n$ diverges, then the Euler–Abel method of summation will fail because $f(x)$ will not have a finite limit as $x \to 1 - 0$.

  $$a_n \geq 0, \quad \lim_{x \to 1 - 0} f(x) = s \implies \sum_n a_n = s.$$

- **Zeta function regularization**

  $$g(s) = \sum_n \frac{a_n}{n^s}$$

  $$\sum_n a_n \overset{\text{def}}{=} g(0).$$

  Usually the function $g$ will only exist on some half-plane $\text{Re}(s) > \alpha >> 0$, and one has to analytically continue $g$ to a domain including $s = 0$.

- **This is used to define infinite determinants in analysis and physics**
The summation formula $1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}$

The zeta

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \quad (\Re(s) > 1)$$

continues as a complex analytic function to the entire complex $s$-plane except for a pole at $s = 1$ where it behaves like

$$\frac{1}{s - 1}.$$

**Riemann (1826-1866)**

$$\zeta(1 - s) = 2(2\pi)^{-s} \cos(\pi s/2)\Gamma(s)\zeta(s)$$

$$1 + 2 + 3 + 4 + \ldots = \zeta(-1)$$

From the functional equation we get

$$\zeta(-1) = -\frac{1}{12}.$$

This formula is mentioned by **Ramanujan (1887–1920)** in the first of his famous letters to Hardy.
Remarks

In making the evaluation at $s = -1$ one must write

$$s = -1 + t, \quad \Gamma(s) = \Gamma(s + 2)/s(s + 1)$$

and then let $t \to 0$. The functional equation links $\zeta(-1)$ to

$$\zeta(2) = \frac{\pi^2}{6}$$

which goes back to Euler.
The factorial series

- Euler called

\[ \sum_{n=0}^{\infty} (-1)^n n! x^n = 1 - 1!x + 2!x^2 - 3!x^3 + 4!x^4 \ldots \]

the divergent series par excellence. It does not converge anywhere and so the usual methods of summability fail. The sum as a formal power series satisfies Euler’s differential equation

\[ x^2 \frac{dg}{dx} + g = x, \quad g(x) \sim e^{1/x} \int_0^x \frac{1}{t} e^{-1/t} \, dt \]

The integral is asymptotic to the factorial series and its value was computed by Euler using numerical integration:

\[ \sum_{n=0}^{\infty} (-1)^n n! \sim g(1) = 0.5963 \ldots \]

- The theory of solutions to such irregular singular differential equations would come into their own in the nineteenth century with the work of many, most notably of Poincaré who made precise the notion of asymptotic series and analytic solutions that are asymptotic to the formal solutions.
Continued fraction for the factorial series

- Euler also obtained a continued fraction for $g$:

$$g(x) = \frac{1}{1+\frac{x}{1+\frac{x}{1+\frac{2x}{1+\frac{2x}{1+\frac{3x}{1+\frac{3x}{1+\frac{\text{etc}}{\text{etc}}}}}}}}} \sim \sum_{n \geq 0} (-1)^n n! x^n.$$  

$$\sum_{n=0}^{\infty} (-1)^n n! = 0.596347362123\ldots$$

- He summed not only the factorial series but a whole class of similar series by using these methods.

- Borel discovered the Borel summation that could bring these and other such series summed by Euler into the general theory. These methods have been applied to quantum field theory and dynamical systems!

- The value for the factorial series is also given by Ramanujan in that first letter of his to Hardy, but only to the first 3 decimal places!

- Euler’s value : 0.596347362123.....

- Mathematika : 0.596347362319.....
Euler’s formula

From Euler’s work it is clear that he is assuming without proof that the continued fraction and integral are both equal and also that the formal series has the given continued fraction expansion. To state formally, one must show that

\[ \sum_{n \geq 0} (-1)^n n! x^n = \frac{1}{1 + \frac{x}{1 + \frac{x}{2 + \frac{2x}{1 + \frac{3x}{1 + \frac{3x}{1 + \frac{\quad \ddots}{\quad \ddots}}}}}} \]

when one works with formal power series, and further that when we work pointwise over the positive real axis where \( x > 0 \), we also have

\[ \frac{1}{1 + \frac{x}{1 + \frac{x}{2 + \frac{2x}{1 + \frac{3x}{1 + \frac{3x}{1 + \frac{\quad \ddots}{\quad \ddots}}}}}} = e^{1/x} \int_0^x \frac{1}{t} e^{-1/t} dt \]

where the left side must converge for all \( x > 0 \).

To prove these one exhibits the Euler series as a limit of deformations of the hypergeometric series and use the classical work of Gauss and Perron on continued fractions for ratios of hypergeometric series.
Hypergeometric series

For $|z| < 1, c \notin \{0, -1, -2, \ldots\}$ we have

$$F(a, b, c : z) = 1 + \frac{a.b}{c.1} z + \frac{a(a + 1)b(b + 1)}{c(c + 1)1.2} z^2 + \ldots$$
Continued fractions for HG series

For brevity let

\[ [1, a_1, a_2, a_3, \ldots, a_n, \ldots] := \frac{1}{1 + \frac{a_1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \frac{a_4}{\ddots}}}}} \text{ etc} \]

From the Gauss contiguity relations we get

\[ F(a, 1, c + 1; -cz) = [1, a_1z, a_2z, \ldots] \]

where

\[ a_{2n} = \frac{nc(c - a + n)}{(c + 2n - 1)(c + 2n)}, \quad a_{2n+1} = \frac{(a + n)c(c + n)}{(c + 2n)(c + 2n + 1)} \]
Integral representation for HG series

- The HG series $F$ is convergent for $|z| < 1$ and continues analytically to $\mathbb{C}_{[1,\infty]} = \mathbb{C} \setminus [1, \infty]$.

- If $c > b > 0$ this analytic continuation is given by Euler’s integral

\[
F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt
\]

- Here $(1-zt)^{-a} = e^{-a\log(1-zt)}$, where log is 0 at 1 and is analytic on the cut plane $\mathbb{C}_{[-\infty,0]}$, is thus defined on $\mathbb{C}_{[1,\infty]}$ and gives the binomial expansion of $(1 - zt)^{-a}$ or $|z| < 1$. Integrating this expansion gives the series for $F$.

- The HG function has attracted the greatest of mathematicians: Euler, Gauss, Kummer, Riemann, Deligne, Mostow........
C.F.s and integrals for ratios of HG series

From the Gauss-Perron formulae we get

\[ [1, a_1 x, a_2 x, \ldots] = \int_0^c (1 - (u/c))^{c-1}(1 + xu)^{-a} du \]

with

\[ a_{2n} = \frac{n(c - a + n)c}{(c + 2n - 1)(c + 2n)}, \quad a_{2n+1} = \frac{(a + n)(c + n)c}{(c + 2n)(c + 2n + 1)}. \]
**Limits when \( c \to \infty \): Euler’s formulae**

We now treat \( c \) as a real parameter and let it go to \(+\infty\).

We have

\[
F(a, 1, c + 1; -cz) = 1 + \sum_{n \geq 1} (-1)^n \frac{a(a + 1) \ldots (a + n - 1)}{c(c + 1) \ldots (c + n - 1)} c^n z^n
\]

When \( c \to \infty \) we get

\[
1 - az + a(a + 1)z^2 - \ldots + (-1)^n a(a + 1) \ldots (a + n - 1) z^n + \ldots.
\]

Thus (with \( a = 1 \)), as \( a_{2n} \to n \), \( a_{2n+1} \to n+1 \) we have

\[
[1, z, z, 2z, 2z, \ldots] = 1 - z + \ldots + (-1)^n n! z^n + \ldots.
\]

At the analytical level (after making the change of variable \( s = ct \))

\[
[1, x, x, 2x, 2x, \ldots] = \lim_{c \to \infty} \int_0^c \left( 1 - \frac{u}{c} \right)^{c-1} (1 + ux)^{-1} du
\]

Thus

\[
[1, x, z, 2x, 2x, \ldots] = \int_0^\infty \frac{e^{-u}}{(1 + xu)} du = \frac{e^{1/x}}{x} \int_0^x \frac{e^{1/t}}{t} \quad (x > 0).
\]
Originally continued fractions arose as a method of representing individual real numbers and approximating them by rational numbers. In Euler’s work we see for the first time that they can be used to represent functions. It was T. J. Stieltjes who developed a far-reaching theory of functions of a complex variable approximated by continued fractions. The question of the relationship of the theory of Stieltjes to solutions of irregular singular differential equations, similar to the one treated above due to Euler, is still wide open.
I give three examples where analytic continuation is used as the method of summation.

- The Gamma function of Euler:
  \[ \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \quad (\Re(s) > 0). \]

- Continues analytically to the entire \( s \)-plane with simple poles at the negative integers.
Divergent integrals. II.

- The Fresnel integrals in wave propagation of light:

\[
\int_{-\infty}^{\infty} e^{iax^2} \, dx = \sqrt{\pi} |a|^{-1/2} e^{-\frac{i\pi}{4} \text{sgn}(a)} \quad (a \in \mathbb{R} \setminus \{0\})
\]

- The convergent integral is

\[
\int_{-\infty}^{\infty} e^{-\tau x^2} \, dx = \sqrt{\pi} |\tau|^{-1/2} e^{-\frac{i}{2} \text{arg}(\tau)}
\]

on the domain

\[
(\Re(\tau) > 0, |\text{arg}(\tau)| < \pi/2).
\]

We then let \( \tau \to -ia \).
Divergent integrals. III.

Feynman and Feynman-Kac path integral

for the quantum propagator

• The Feynman path integral in the simple case of a particle in quantum theory moving on the real line for the kernel of the integral operator defining the dynamics is

\[ K_t(a, b) = \int_{P(a,b)} e^{iS(x)/\hbar} \mathcal{D}x \]

where the integration is over all paths connecting \((0, a)\) with \((t, b)\).

• The Feynman measure \(\mathcal{D}x\) on the space \(P(a, b)\) is non-existent.

• The Feynman-Kac path integral is with respect to a genuine probability measure \(W\) first constructed by Wiener on \(P(a, b)\). But we have to make time imaginary. The results for real time are obtained by continuation:

\[ K_{it}(a, b) = \int_{P(a,b)} e^{-\int_0^t V(x(s))ds} dW \]

where \(V\) is the potential.
Smeared summation

• When does the trigonometric series

\[ f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} \]

make sense?

• (L. Schwartz) If the \( a_n = O(n^r) \) for some \( r \geq 0 \).

• The summation procedure used is called smeared summation. The intuition behind it is that measurements (of electric or magnetic fields for example) require always a test body of finite size and so are only averages over small regions. Thus we define, for smooth periodic \( \varphi \),

\[
\int f(x) \varphi(x) dx = \sum_{n\in\mathbb{Z}} a_n \int e^{inx} \varphi(x) dx.
\]

• The LHS is interpreted as a linear functional on the space of \( \varphi \)'s, i.e., a distribution. The integrals on the right are rapidly decreasing in \( n \) and so there is convergence for polynomially growing \( a_n \).
Plancherel formula

\[ \delta(x) = \sum_{n \in \mathbb{Z}} e^{inx} \quad (\text{Euler}) \]

\[ \delta(x) = \int_{-\infty}^{\infty} e^{i\xi x} d\xi \quad (\text{Fourier-Schwartz}) \]

Let \( G \) be a compact Lie group, \( \omega \) denote its irreducible representations up to equivalence and \( \Theta_\omega \) the character of \( \omega \). Then

\[ \delta_G = \sum_{\omega} \dim(\omega) \Theta_\omega \quad (\text{Hermann Weyl}) \]
Harish-Chandra character

- If $L$ is an infinite dimensional unitary representation of a Lie group $G$ in a Hilbert space $\mathcal{H}$, the attempt to define its character by

$$\text{Tr}(L(x)) = \sum_n (L(x)e_n, e_n) \quad (\{e_n\} \text{ an ON basis of } \mathcal{H})$$

fails. Harish-Chandra used smeared summation:

$$\int_G \text{Tr}(L(x)) \varphi(x) dx = \sum \int_G (L(x)e_n, e_n) \varphi(x) dx.$$

- The left side is a distribution, thus defining the Harish-Chandra character or the distribution character of $L$. It exists if $G$ is semi simple and $L$ is irreducible.

- Harish-Chandra discovered a Plancherel formula for all semi simple Lie groups.