Is spacetime p-adic?

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Abstract

The geometrical structure of space has always fascinated people, and Gauss was perhaps the first to try to settle this experimentally. Since then it has been **the** big open problem in fundamental physics. In this talk we examine the hypothesis of Volovich that space time is p-adic at the Planck scale. That space time geometry could be based on a p-adic or even a finite field seems to have been suggested first in the 1950's. Enrico Beltrametti and his collaborators (Cassinelli and Blasi) in the late 1960's and early 1970's were among the earliest in exploring this line of thought. Igor Volovich formulated his hypothesis in 1987.

Early ideas

- **Beltrametti-Cassinelli (1971)**

  Can the field of quantum theory be p-adic?

- **Beltrametti (1971)**

  Can the microstructure of spacetime be based on a p-adic or even a finite geometry?

- **Riemann (1854)**

  Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact, one ought to assume this as soon as it permits a simpler way of explaining phenomena ...

Volovich hypothesis

- **Volovich (1987)**

  In the late 1980's Igor Volovich made the bold hypothesis that spacetime geometry may be non archimedean. The basis of this suggestion may be found in the observation made by many people that principles of general relativity and quantum theory forbid any kind of measurement in scales of distances and times smaller than the Planck units of distance and time, so that the archimedean axiom ceases to have validity.

The Dirac Mode

"The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected... Non-euclidean geometry and noncommutative algebra, which were at one time considered to be purely fictions of the mind and pastimes for logical thinkers, have now been found to be very necessary for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.

The theoretical worker in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and *after* each success in this direction, to try to interpret the new mathematical features in terms of physical entities... "

(P. A. M. Dirac, *Quantized singularities in the electromagnetic field,* Proc. Roy. Soc. Lond., **A133** (1931), 60–72.)

Some comments on the Dirac mode

"The Dirac mode is to invent, so to speak, a new mathematical concept or framework first, and then try to find its relevance in the real world, with the expectation that (in a distorted paraphrasing of Dirac) a mathematically beautiful idea must have been adopted by God. Of course the question of what constitutes a beautiful and relevant idea is where physics begins to become an art.

I think this second mode is unique to physics among the natural sciences, being most akin to the mode practiced by the mathematicians. Particle physics, in particular, has thrived on the interplay of these two modes. Among examples of this second approach, one may cite such concepts as

Magnetic monopole

Non-Abelian gauge theory

Supersymmetry

On rare occasions, these two modes can become one and the same, as in the cases of Einstein gravity and the Dirac equation. . . "

Y. Nambu, *Broken Symmetry: Selected Papers of Y. Nambu*, 1995, World Scientific, pp. 383–389, eds. T. Eguchi and K. Nishijima.

The p-adic world

valuations

(i) $|ab|_\infty = |a|_\infty |b|_\infty$

(ii) $|a+b|_\infty \leq |a|_\infty + |b|_\infty$.

p-adic absolute value

$$|x|_p = p^{-r} \qquad \text{if } x = p^r \frac{a'}{b'} \text{ where } r, a', b' \in \mathbf{Z}, (a', b') = 1.$$

properties

(a) $|a+b|_p \leq \max(|a|_p, |b|_p)$ (ultrametric inequality)

(b) locally compact, totally disconnected

(c) non archimedean

(d) No other valuations on $\mathbf{Q}$ other than $|\cdot|_p, |\cdot|_\infty$ (Ostrowski's theorem)

Theorem. *There are no other locally compact non discrete fields densely containing $\mathbf{Q}$ other than $\mathbf{R}$ and the $\mathbf{Q}_p$. If we omit the denseness condition we must include $\mathbf{C}$ and the finite extensions of $\mathbf{Q}_p$; the latter we shall call p-adic fields.*

The Lefschetz principle

- There are the two principles which govern all the work over local fields and adele rings. They were first formulated by Harish-Chandra.

  - Whatever is true for $\mathbf{R}$ should be true for $\mathbf{Q}_p$

  - All primes must be treated on the same footing.

The p-adic Banach spaces

One has a good notion of a *Banach space* over a non archimedean local field. The norm in such a space is *ultrametric*, namely, satisfies

$$||u + v|| \leq \max(||u||, ||v||).$$

p-adic Banach spaces have come to the fore in p-adic representation theory of p-adic linear groups.

Analysis on a local field k (characteristic 0)

- **Haar measure**

  $d(ax) = |a|_k dx \quad (v = p, v = \infty)$

- **Fourier transforms of test functions**

  $\psi : k \longrightarrow T$ a non trivial additive character of k. For f any test function on k,

$$\widehat{f}(y) = \int_k f(x)\psi(xy)dx, \quad f(x) = \int_k \widehat{f}(y)\psi(-xy)dy$$

- For $k = \mathbf{R}, \mathbf{C}$ we take the test functions from the Schwartz space. For k non archimedean we take them to be elements of the *Schwartz-Bruhat space*, namely, functions which are locally constant and compactly supported. They are thus functions on B_a/B_b where $0 < a < b < \infty$, B_r is the p-adic ball of radius r. The integrals are thus sums, and are generalizations of Gauss sums. Thus the p-adic Fourier transform is arithmetic. The theory is independent of the choice of ψ and one usually makes certain normalized choices.

Choices for ψ

For $\mathbf{R}$ we take

$$\psi(x) = e^{2\pi i x}.$$

For $\mathbf{Q}_p$ we write any $x \in \mathbf{Q}_p$ as

$$x = \sum_{r >> -\infty} m_r p^r \qquad (m_r \in \{0, 1, \ldots, p-1\})$$

so that

$$x \equiv m = \sum_{r<0} m_r p^r (\text{mod } \mathbf{Z}_p) \qquad (m \in \mathbf{Z}[1/p])$$

where $\mathbf{Z}_p$ is the ring of p-adic integers. The number m is unique up to addition by an ordinary integer so that

$$\psi_p(x) = e^{-2\pi i m}$$

is well defined and is a non-trivial continuous character of $\mathbf{Q}_p$.

For a finite extension k of $\mathbf{Q}_p$ we take

$$\psi_k(x) = \psi_p(\mathrm{Tr}_{k/\mathbf{Q}_p}(x)).$$

Some ingredients for a quantum theory over p-adic spacetime

- The Poincaré group P
- Projective unitary representations (PUR) of P
- Particles (PUIR's)
- Conformal compactification of spacetime
- Conformal symmetry of massless paticles
- QFT and CQFT

Rational algebraic groups

If we take a group like $\mathbf{GL}(n)$ or more generally, an algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$ (think $\mathrm{SL}(n), \mathrm{SO}(n), \mathrm{Sp}(n)$), we get the *locally compact groups* $G_p = \mathbf{G}(\mathbf{Q}_p)$ which can be studied in great depth. These are examples of lcsc groups which are p-adic Lie groups:

(a) They are manifolds, i.e., look locally like $\mathbf{Q}_p^n$

(b) Multiplication and inversion are morphisms

The structure of the p-adic Lie groups was studied many years ago by Lazard who proved a version of Hilbert's fifth problem for these. The *complex* representation theory of these groups, especially the semi simple ones, both in Hilbert spaces, and in recent years, in Banach spaces over p-adic fields, has been intensively studied .

Key examples are (1) the orthogonal group $\mathrm{SO}(V)$ where V is a finite dimensional vector space over a ground field k of characteristic $\neq 2$, equipped with a non-degenerate quadratic form (a quadratic vector space) and (2) the Poincare group $P(V) = V \times' \mathrm{SO}(V)$ where $\times'$ referes to the semi direct product. We may replace $\mathrm{SO}(V)$ by $\mathrm{Spin}(V)$, the 2-fold cover of $\mathrm{SO}(V)$.

Topology

A Hausdorff topological space is called *totally disconnected* (t.d) if the sets which are closed *and* open (called *clopen*) form a basis for the topology. If the space is in addition locally compact we can replace the clopen sets by the sets which are *compact* and open. Closed subgroups of $\mathrm{GL}(n, k)$ are t.d if k is a non archimedean local field, and homogeneous spaces for t.d groups are t.d.

Some problems. I.

- **Multipliers of p-adic algebraic groups**

  Symmetries in quantum theory are implemented by projective unitary representations which are unitary representations up to phase factors. The phase factors are identified with elements of the cohomology group $H^2(G)$ where G is the symmetry group.

- **Problem**

  If $\mathbf{G}$ is an affine algebraic group (Poincaré, Galilean, etc) defined over a local field k and $G = \mathbf{G}(k)$, find $H^2(G)$ and determine, if possible, the universal topological central extension of G to which all projective unitary representations of G can be lifted to become ordinary unitary representations.

- This can be done for the Poincaré and Galilean groups over $\mathbf{Q}_p$.

QFT

- Build p-adic analogues of quantum field theories a la Wightman and Schwinger when spacetime is p-adic or finite.

- A partial solution has been obtained by A. Kochubei and M. R. Sait-Ametov. They construct probability measures on the space of distributions on spacetime which have genuine interactions. However they have a cut off which means their theory does not have relativistic invariance.

- The field strengths are not measurable at points and only their average values over spacetime regions can be observed. A quantum field is thus an operator-valued distribution. Wightman and Schwinger formulated the principles defining such fields.

- (Wightman) Quantum fields in Minkowski spacetime.

- (Schwinger) Quantum fields in Euclidean spacetime.

Conformal symmetry

- The Poincaré group P in Minkowski spacetime V can be imbedded in $\mathrm{SO}(U)$ where $U = V \oplus H$, H being a hyperbolic plane. For a quantum system over V with a unitary representation that describes its covariance with respect to P, it makes sense to ask if the representation can be extended to $\mathrm{SO}(U)$. If so, we say the system has *conformal symmetry.* Over $\mathbf{R}$ massless particles have conformal symmetry. Is this true over a non-archimedean local field?

CQFT

In dimensions ≥ 3 the conformal group is finite dimensional. But in dimension 2 holomorphic maps are conformal and so the conformal group is infinite dimensional. CQFT is thus very deep in dimension 2. It is tied to the representation theory of the Loop and the Virasoro algebras. Nothing is known about CQFT over local fields.