Why do we do representation theory?

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Abstract

Years ago representation theory was a very specialized field, and very few non-specialists had much interest in it. This situation has changed profoundly in recent times. Due to the efforts of people like Gel’fand, Harish-Chandra, Langlands, Witten, and others, it has come to occupy a central place in contemporary mathematics and theoretical physics.

This talk takes a brief look at the myriad ways that representations of groups enters mathematics and physics. It turns out that the evolution of this subject is tied up with the evolution of the concept of space itself and the classification of the groups of symmetries of space.
Classical projective geometry

- The projective group $G = \text{PGL}(n + 1)$ operates on complex projective space $\mathbb{CP}^n$ and hence on algebraic varieties imbedded in $\mathbb{CP}^n$. Classical geometry was concerned with the invariants of these varieties under the projective action.
Main problems over $\mathbb{C}$

- To study the ring of invariants of the action of $G$ on the graded ring of polynomials on $\mathbb{C}^{n+1}$. The basic question is:

- Is the invariant ring finitely generated?

- The spectrum of the ring of invariants is a first approximation to a moduli space for the action.
Solutions over C

• (Hilbert-Weyl) The ring of invariants is finitely generated if $G$ is a complex semi simple group.

This is based on

• (Weyl) All representations of a complex semi simple group are completely reducible.

• (Nagata) Finite generation of invariants is not always true if $G$ is not semi simple.

• There are much simpler examples due to Steinberg.
Main problems over characteristic $p > 0$

- (Mumford) To what extent these results are true in characteristic $p > 0$?

  Unfortunately the complete reducibility theorem of Weyl is true only for tori.

- Look at action of $\text{SL}(p)$ over the space of $p \times p$ matrices by similarity. The invariant vector $I$ does not admit a complementary invariant subspace. In fact the only other proper invariant subspace is the space of matrices of trace 0 and it contains $I$!!
Solutions over fields of characteristic $p > 0$

• (Mumford) The action of a linear algebraic group $G$ on a vector space $V$ is \textit{geometrically reductive} if given any vector $0 \neq v \in V$ which is fixed by $G$, we can find a homogeneous polynomial $f$ which is invariant such that $f(v) \neq 0$.

• If $f$ above is linear, its null space is an invariant complement to $v$. In the example above we can choose $f$ to be the \textit{determinant}.

• (Mumford) Is any action of a semi simple group over characteristic $p > 0$ geometrically reductive?

• (Haboush) Yes.

Haboush’s theorem leads to

• (Nagata) The ring of invariants is finitely generated for semi simple groups in characteristic $p > 0$ also.
Group actions and representations

- The semi simple groups are precisely the groups that can act effectively and transitively on smooth projective varieties. Thus their actions on vector bundles on these varieties is essentially the theory of representations of such groups. The classification of simple groups and their representations, both over $\mathbb{C}$ (Killing, Elie Cartan) and over characteristic $p > 0$ (Borel, Chevalley), have profoundly influenced algebraic geometry.

- Complex groups are in natural bijection with compact groups (Weyl) and this has led to many deep links between homogeneous differential and algebraic geometry (Borel-Weil-Bott).
Unitary representations

• (Wigner, Dirac) In quantum physics the symmetry of a system is expressed by unitary representations of the symmetry group.

• (Gel’fand-Raikov) Every locally compact group has sufficiently many unitary irreducible representations to separate points.
Explicit formulae

• (Weyl) If $\Theta_\omega$ are the characters of the irreducible (unitary) representations $\omega$ of a compact Lie group and $\delta$ is the delta function at the identity element of the group, then

$$\delta = \sum_\omega \dim(\omega) \Theta_\omega.$$ 

• There is an explicit formula for $\Theta_\omega$ and $\dim(\omega)$.

• (Plancherel formula) For $G = SU(2)$ the characters are given on the diagonal torus $\simeq T = \{ t \in \mathbb{C} \mid |t| = 1 \}$ by

$$\Theta_{\omega_n}(t) = \frac{\chi_n(t) - \chi_n(t^{-1})}{t - t^{-1}}$$

$$\chi_n(t) = t^n$$

$$\dim(\omega_n) = n$$
Infinite dimensional representations

• Is it possible to develop a theory of infinite dimensional representations of a semi simple Lie group that generalizes the Weyl formulae for character, dimension, and the delta function?

• Harish-Chandra erected a monumental theory of representations which led to beautiful generalizations of the Weyl formulae.

• Every irreducible unitary representation of a semi simple Lie group has a character which is a distribution on the group. It is a locally integrable function on the group, analytic at the regular points.
Example: $G = \text{SL}(2, \mathbb{C})$

- For $G = \text{SL}(2, \mathbb{C})$ the characters of the principal series are given on the diagonal torus $A \simeq \mathbb{C}^\times$ by

$$T_\chi(t) = \frac{\chi(t) + \chi(t^{-1})}{|t - t^{-1}|^2}$$

where $\chi$ is a unitary character of $A$.

- $\chi_{m,\rho}(t) = \left(\frac{t}{|t|}\right)^m |t|^{i\rho}$ ($m \in \mathbb{Z}, \rho \in \mathbb{R}$)

- The squaring in the denominator is due to the fact that we view $G$ as a real group.

- (Plancherel formula)

$$c(G') \delta = \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} (m^2 + \rho^2) T_{\chi_{m,\rho}} d\rho.$$
Example: $G = \text{SL}(2, \mathbb{R})$ (Principal series)

- For $G = \text{SL}(2, \mathbb{R})$ there are two kinds of characters.

- The ones analogous to the $T_\chi$ of the complex group, the principal series, are given on the diagonal torus $A \cong \mathbb{R}^\times$ by

$$T_\chi(t) = \frac{\chi(t) + \chi(t^{-1})}{|t - t^{-1}|}.$$

- They vanish on the elements of the group having distinct non real eigenvalues.
Discrete series

• There are additional ones, the discrete series, so named because they occur as direct summands of the regular representation, with characters $\Theta_n(n \neq 0)$ given by

$$\Theta_n(u_{\theta}) = -\text{sgn}(n) \frac{e^{in\theta}}{e^{in\theta} - e^{-in\theta}}$$

$$\Theta_n((-1)^\varepsilon a_t) = (-1)^{(n-1)\varepsilon} \frac{e^{-|n||t|}}{|e^t - e^{-t}|}$$

• Here

$$u_{\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad \varepsilon = 0, 1$$

• Note the analogy with Weyl’s formulae.
Plancherel formula for $G = \text{SL}(2, \mathbb{R})$

- We have
  \[
  c(G)\delta = \sum_{n \neq 0} |n|\Theta_n + \frac{1}{2} \int_0^\infty r \tanh(r\pi/2)T_{0,ir}dr
  \]
  \[
  + \frac{1}{2} \int_0^\infty r \coth(r\pi/2)T_{1,ir}dr.
  \]

- $T_{\varepsilon,ir} = T_{\chi_{\varepsilon,ir}}$

- $\chi_{0,ir}((-1)^\varepsilon a_t) = e^{irt}$

- $\chi_{1,ir}((-1)^\varepsilon a_t) = (-1)^\varepsilon e^{irt}$
Number theory: Galois extensions of $\mathbb{Q}$

- (Kronecker-Weber) Abelian Galois extensions are subextensions of cyclotomic extensions.

- How to describe all Galois extensions of $\mathbb{Q}$ in terms of data constructed from the ground field $\mathbb{Q}$?

- For abelian extensions such a description exists and is called class field theory.

- How do we do non-abelian class field theory?
Groups over local fields and adele rings

• Given \( \mathbb{Q} \) we have its completions \( \mathbb{R} \) and \( \mathbb{Q}_p \) which are denoted by \( \mathbb{Q}_v \) where \( v = \infty \) denotes completion at the usual absolute value and \( v = p \) denotes completion at the \( p \)-adic absolute value. The \( \mathbb{Q}_p \) have compact open subrings \( \mathbb{Z}_p \) and we can form the restricted direct product

\[
\mathbb{A} = \prod'_v \mathbb{Q}_v.
\]

An element of the full direct product \( (x_v) \) lies in \( \mathbb{A} \) if and only if \( x_p \in \mathbb{Z}_p \) for all but finitely many \( p \). The elements of \( \mathbb{A} \) are called adeles and \( \mathbb{A} \) the adele ring.

• If \( G \) is a linear algebraic group defined over \( \mathbb{Q} \) it makes sense to speak of its points over \( \mathbb{Q}_v \) and \( \mathbb{A} \).
The fundamental homogeneous space

- (Borel-Harish-Chandra) If $G$ is semi simple the diagonal imbedding of $G(\mathbb{Q})$ in $G(\mathbb{A})$ is discrete and has finite covolume:

$$\text{vol}(G(\mathbb{A})/G(\mathbb{Q})) < \infty$$

- Harmonic analysis on $G(\mathbb{A})/G(\mathbb{Q})$ is the key (according to Langlands) to non-abelian class field theory.
The Langlands philosophy

- There exists a natural correspondence between representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of degree $n$ and the unitary representations that occur in the decomposition of $L^2(G(\mathbb{A})/G(\mathbb{Q}))$.

- Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. 

Number fields and function fields

- One can replace $\mathbb{Q}$ by a number field, namely, a finite extension of $\mathbb{Q}$.

- A function field is a finite extension of $F(T)$ where $F$ is a finite field and $T$ is an indeterminate. The Langlands philosophy makes sense when the ground field is a function field instead of $\mathbb{Q}$.

- (Lafforge) The Langlands correspondence can be proved when the ground field is a function field.

- One can also ask what happens when we take the ground field to be a function field over $\mathbb{C}$. This is the geometric Langlands correspondence.

- (Witten) The geometric Langlands is related to the gauge theories of quantum physics.