APPENDIX E

Abresch and Langer Classification of Homothetically Shrinking Closed Curves

(In collaboration with Annibale Magni)

We briefly discuss here the classification of Abresch and Langer in [**1**] of the homothetically shrinking closed curves in the plane, that is, satisfying the structural equation $k + \langle x \,|\, \nu \rangle = 0$.

We have seen in Proposition 3.4.1 that among the curves with $k + \langle x \,|\, \nu \rangle = 0$ the only embedded, complete and connected ones are the lines through the origin and the unit circle.
If we now do not assume the embeddedness, we have to deal either with a smooth, complete immersion of $\mathbb{S}^1$ or of $\mathbb{R}$ possibly with self–intersections. In the first case, the curve is closed and compact, in the second case we can see that the initial part of the analysis in the proof of Proposition 3.4.1 still holds, hence, either the curve is a line through the origin of $\mathbb{R}^2$ or the curvature is everywhere positive and bounded from above, which implies that the whole curve is bounded. Then, if we only assume an estimate of the length of the curve (which holds by Lemma 3.2.7 and boundedness if the curve is a blow up limit), by the completeness it follows that the curve is closed.

Hence, we concentrate only on closed curves. As we said, $k > 0$, the equations (3.4.1) hold and the quantity $k_\theta^2 + k^2 - \log k^2$ is constant along the curve, equal to some constant E which must be larger than one (otherwise we are dealing with the unit circle). Again, the curve is symmetric with respect to the critical points of the curvature, which are all nondegenerate, isolated and finite. Hence, the curvature $k(\theta)$ is oscillating between its maximum and its minimum with some period $T > 0$. If we exclude the unit circle, such period must be an integer fraction (at least of a factor 2 by the four vertex theorem, see the proof of Proposition 3.4.1) of an integer multiple (at least 2, otherwise we are dealing with the unit circle) of 2π, that is, $T = 2m\pi/n$ with $n, m \geq 2$.

Notice that there are two parameters here around, the rotation number of the closed curve and the number of critical points of the curvature.

Suppose that $k_{\min} < k_{\max}$ are these two consecutive critical values of k, it follows that they are two distinct positive zeroes of the function $E + \log k^2 - k^2$ when $E > 1$ with $0 < k_{\min} < 1 < k_{\max}$. We have that the change $\Delta\theta$ in the angle θ along the piece of curve delimited by two consecutive points where the curvature assumes the values $k_{\min}$ and $k_{\max}$, is equal to the semiperiod $T/2$. Then, the analysis reduces to understanding what are the admissible T.
Such quantity $\Delta\theta$ is given by the integral

$$I(E) = \int_{k_{\min}}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}}\,.$$

Abresch and Langer (and also Epstein and Weinstein) by studying the behavior of this integral were able to classify *all* the immersed closed curves in $\mathbb{R}^2$ satisfying the structural equation $k + \langle \gamma \,|\, \nu \rangle = 0$. These form a family of curves indexed by two parameters called *Abresch–Langer curves*, see [**1**] for a detailed description.
We now state and partially prove the main properties of the integral $I(E)$ needed in such analysis.
It should be noticed that, by the discussion about the period T, the last statement in the next proposition implies Proposition 3.4.1.

PROPOSITION E.1.1. *The function $I : (1, +\infty) \to \mathbb{R}$ satisfies*

(1) $\lim_{E\to 1^+} I(E) = \pi/\sqrt{2}$,
(2) $\lim_{E\to +\infty} I(E) = \pi/2$,
(3) $I(E)$ *is monotone nonincreasing.*

As a consequence $I(E) > \pi/2$.

PROOF. Notice that the study of the quantity $I(E)$ is equivalent to the study of the semiperiod for the one–dimensional Hamiltonian system with Hamiltonian function given by $H(k_\theta, k) = (k_\theta^2 + k^2 - \log k^2)/2$.

(1) The global minimum $1/2$ of the strictly convex potential $V(k) = (k^2 - \log k^2)/2$ is assumed at $k = 1$ and the limiting value for the period of the Hamiltonian system when $E \to 1^+$ is equal to the period of the corresponding linearized system (see [**45**, Chapter 12]). The linearized Hamiltonian is $H_L(\hat{k}_\theta, \hat{k}) = \hat{k}_\theta^2/2 + \hat{k}^2 + 1/2$ for the new variable $\hat{k} = k-1$, which gives the equation $\hat{k}_{\theta\theta} = -2\hat{k}$ for $\hat{k}$. As any solution of this last ODE is clearly $\sqrt{2}\pi$–periodic, its semiperiod is equal to $\pi/\sqrt{2}$.

(2) As $0 < k_{\min} < 1 < k_{\max}$ for $E > 1$, we can write

$$I(E) = \int_{k_{\min}}^{1} \frac{dk}{\sqrt{E - k^2 + \log k^2}} + \int_{1}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}} = I_-(E) + I_+(E)\,.$$

We want to prove that $\lim_{E\to+\infty} I_-(E) = 0$ and $\lim_{E\to+\infty} I_+(E) = \pi/2$.
Introducing the variable $w = k/k_{\min}$ the first integral becomes

$$I_-(E) = k_{\min} \int_{1}^{1/k_{\min}} \frac{dw}{\sqrt{k_{\min}^2(1-w^2) + \log w^2}}\,.$$

Notice that, given a real number $0 < \alpha < 1$, it is always possible to find $\widetilde{k}(\alpha)$ such that $|k_{\min}(1 - w^2)| \leq \alpha |\log w^2|$ with $w \in [1, 1/k_{\min}]$ and $k_{\min} \leq \widetilde{k}$. Fixing then such an α, we have

$$\begin{aligned}
0 \leq I_-(E) &\leq \frac{k_{\min}}{\sqrt{1-\alpha}} \int_1^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}} \\
&\leq \frac{k_{\min}}{\sqrt{1-\alpha}} \Big(\int_1^n \frac{dw}{\sqrt{2\log w}} + \int_n^{1/\sqrt{k_{\min}}} \frac{dw}{\sqrt{2\log w}} + \int_{1/\sqrt{k_{\min}}}^{1/k_{\min}} \frac{dw}{\sqrt{2\log w}} \Big) \\
&\leq k_{\min}(C_1 + C_2/\sqrt{k_{\min}} + o_{k_{\min}}(1)/k_{\min})\,,
\end{aligned}$$

hence, the claim on $I_-(E)$ follows.
Regarding $I_+(E)$, we proceed in a similar way by changing again the integration variable to $w = k/k_{\max}$. In this way we obtain

$$\begin{aligned}
\lim_{E\to+\infty} I_+(E) = \lim_{E\to+\infty} &\int_{1/k_{\max}}^{1} \frac{dw}{\sqrt{1 - w^2 + \frac{2\log w}{k_{\max}^2}}} \\
= \lim_{E\to+\infty} &\int_0^1 \chi_{[1/k_{\max},1]} \frac{dw}{\sqrt{1 - w^2 + \frac{2\log w}{k_{\max}^2}}} \\
= \pi/2\,,&
\end{aligned}$$

where in the last equality we applied the dominated convergence theorem.

(3) See the original paper of Abresch and Langer [**1**] or the general result by Zevin and Pinsky in [**128**]. □