LEMMA 2.3.4. *We have, for $k \geq 0$, denoting by ∇^k the k–th iterated covariant derivative,*

$$\frac{\partial}{\partial t}\nabla^k h_{ij} = \Delta\nabla^k h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A}\,.$$

PROOF. We work by induction on $k \in \mathbb{N}$. The case $k = 0$ is given by equation (2.3.2), we then suppose that the formula holds for $k-1$. We have, by the previous lemma,

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k h_{ij} &= \nabla\frac{\partial}{\partial t}\nabla^{k-1}h_{ij} + \nabla^{k-1}\mathrm{A} * \nabla\mathrm{A} * \mathrm{A}\\ &= \nabla\Big(\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k-1\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}\Big)\\ &\quad + \nabla^{k-1}\mathrm{A} * \nabla\mathrm{A} * \mathrm{A}\\ &= \nabla\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}\,.\end{aligned}$$

Interchanging now the Laplacian and the covariant derivative and recalling that $\mathrm{Riem} = \mathrm{A} * \mathrm{A}$, we have the conclusion, as all the extra terms we get are of the form $\mathrm{A} * \mathrm{A} * \nabla^k\mathrm{A}$ and $\mathrm{A} * \nabla\mathrm{A} * \nabla^{k-1}\mathrm{A}$. □

PROPOSITION 2.3.5. *The following formula holds,*

$$\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 = \Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}\,. \tag{2.3.5}$$

PROOF. We compute

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 &= 2g\Big(\nabla^k\mathrm{A}, \frac{\partial}{\partial t}\nabla^k\mathrm{A}\Big) + \nabla^k\mathrm{A} * \nabla^k\mathrm{A} * \mathrm{A} * \mathrm{A}\\ &= 2g\Big(\nabla^k\mathrm{A}, \Delta\nabla^k\mathrm{A} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}\Big)\\ &\quad + \nabla^k\mathrm{A} * \nabla^k\mathrm{A} * \mathrm{A} * \mathrm{A}\\ &= 2g\left(\nabla^k\mathrm{A}, \Delta\nabla^k\mathrm{A}\right) + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}\\ &= \Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}\,.\end{aligned}$$

□

2.4. Consequences of Evolution Equations

Let us see some consequences of the application of the maximum principle to the evolution equations for the curvature.
Suppose that we have a mean curvature flow of a compact hypersurface M in the time interval $[0, T)$, we have seen that

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,.$$

First we deal with the so called *mean convex* hypersurfaces that play a major role in the subject. A hypersurface is mean convex if $\mathrm{H} \geq 0$ everywhere. We will see in the next proposition that this property is preserved by the mean curvature flow. Mean convexity is a significant generalization of convexity, for instance, it is enough general to allow the neckpinch behavior described in Section 1.4, in particular, mean convex hypersurfaces do not necessarily shrink to a point at the singular time.

PROPOSITION 2.4.1. *Assume that the initial, compact hypersurface satisfies* $\mathrm{H} \geq 0$. *Then, under the mean curvature flow, the minimum of* H *is increasing, hence* H *is positive for every positive time.*

PROOF. Arguing by contradiction, suppose that in an interval $(t_0, t_1) \subset \mathbb{R}^+$ we have $\mathrm{H}_{\min}(t) < 0$ and $\mathrm{H}_{\min}(t_0) = 0$ ($\mathrm{H}_{\min}$ is obviously continuous in time and $\mathrm{H}_{\min}(0) \geq 0$).
Let $|\mathrm{A}|^2 \leq C$ in such interval, then

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2$$

implies

$$\frac{\partial \mathrm{H}_{\min}}{\partial t} \geq C\mathrm{H}_{\min}$$

for almost every $t \in (t_0, t_1)$.
Integrating this differential inequality in $[s,t] \subset (t_0, t_1)$ we get $\mathrm{H}_{\min}(t) \geq e^{C(t-s)}\mathrm{H}_{\min}(s)$, then sending $s \to t_0^+$ we conclude $\mathrm{H}_{\min}(t) \geq 0$ for every $t \in (t_0, t_1)$ which is a contradiction.

Since then $\mathrm{H} \geq 0$ we get

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2 \geq \Delta \mathrm{H} + \mathrm{H}^3/n\,.$$

With the notation of Theorem 2.1.1, we let $u = -\mathrm{H}$, $X = 0$ and $b(x) = x^3/n$, then, if $\mathrm{H}_{\min}(0) = 0$ the ODE solution $h(t)$ is always zero, so if at some positive time $\mathrm{H}_{\min}(\tau) = 0$, we have that $\mathrm{H}(\,\cdot\,,\tau)$ is constant equal to zero on M, but there are no compact hypersurfaces with zero mean curvature. Hence, $\mathrm{H}_{\min}$ is always increasing during the flow and H is positive on all M at every positive time. □

Actually, this proposition can be slightly improved as follows.

PROPOSITION 2.4.2. *If the initial, compact hypersurface satisfies* $|\mathrm{A}| \leq \alpha\mathrm{H}$ *for some constant* α, *then* $|\mathrm{A}| \leq \alpha\mathrm{H}$ *for every positive time.*

PROOF. We know that $\mathrm{H} > 0$ for every positive time, hence also $|\mathrm{A}| > 0$ for every positive time which implies that it is smooth as $|\mathrm{A}|^2$.
Let $[0, T)$ be the interval of smooth existence of the flow. Computing the evolution equation of the function $f = |\mathrm{A}| - \alpha\mathrm{H}$, we get

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{2|\mathrm{A}|}(\Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4) - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2) \\
&= \Delta|\mathrm{A}| + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla \mathrm{A}|^2) + |\mathrm{A}|^3 - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2) \\
&= \Delta f + |\mathrm{A}|^2 f + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla \mathrm{A}|^2) \\
&\leq \Delta f + |\mathrm{A}|^2|f|\,,
\end{aligned}$$

as the term $|\nabla|\mathrm{A}||^2 - |\nabla \mathrm{A}|^2$ is nonpositive.
Hence, choosing any $T' < T$, if C is the maximum of $|\mathrm{A}|^2$ on $M \times [0, T']$, we have $\partial_t f \leq \Delta f + C|f|$ on $M \times [0, T']$. By the maximum principle 2.1.1, as $f_{\max}(0) \leq 0$, we conclude $f \leq 0$ on $M \times [0, T']$.
By the arbitrariness of $T' < T$, the thesis follows. □

COROLLARY 2.4.3. *If* $H > 0$ *for the initial, compact,* n*–dimensional hypersurface, then there exists* $\alpha_0 > 0$ *such that* $\alpha_0|\mathrm{A}|^2 \leq \mathrm{H}^2 \leq n|\mathrm{A}|^2$ *everywhere on* M *for every time.*
If the initial hypersurface has positive scalar curvature, then the same holds for every positive time.

PROOF. The first claim is immediate by the compactness of M and the previous proposition (the second inequality is algebraic).
Recalling that the scalar curvature is equal to $\mathrm{H}^2 - |\mathrm{A}|^2$, positive scalar curvature implies that $\mathrm{H} > 0$ (H cannot change sign on M and there is always a point where it is positive, as M is compact) and $\mathrm{H}^2/|\mathrm{A}|^2 > 1$, the second part of this corollary is also a consequence of Proposition 2.4.2. □

COROLLARY 2.4.4. *Assume that the initial, compact hypersurface has* $\mathrm{H} \geq 0$*, then, if* A *is not bounded as* $t \to T$ *then* H *is also not bounded.*

PROOF. Immediate consequence of Proposition 2.4.1 and the estimate of the previous corollary. ☐

Now we consider the evolution equation of $|\mathrm{A}|^2$ which implies

$$\frac{\partial}{\partial t}|\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max}\,.$$

Notice that $|\mathrm{A}|^2_{\max}$ is always positive, otherwise at some time t we would have $\mathrm{A} = 0$ identically on M, which would imply that M is a hyperplane in $\mathbb{R}^{n+1}$ in contradiction with the compactness hypothesis of M. Hence, we can divide both members by $|\mathrm{A}|^2_{\max}$ obtaining the following differential inequality for the locally Lipschitz function $1/|\mathrm{A}|^2_{\max}$, holding at almost every time $t \in [0, T)$,

$$-\frac{d}{dt}\frac{1}{|\mathrm{A}|^2_{\max}} \leq 2\,.$$

Integrating in time in any interval $[t, s] \subset [0, T)$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} - \frac{1}{|\mathrm{A}(\,\cdot\,, s)|^2_{\max}} \leq 2(s - t)\,.$$

Suppose now that A is not bounded in $[0, T)$, that is, there exists a sequence of times $s_i \nearrow T$ such that $|\mathrm{A}(\,\cdot\,, s_i)|^2_{\max} \to +\infty$. Substituting these times s_i in the previous inequality and sending $i \to \infty$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} \leq 2(T - t)\,.$$

EXERCISE 2.4.5. Show that the only compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature are the spheres. What can be said about a compact hypersurface in $\mathbb{R}^{n+1}$ with constant $|\mathrm{A}|$?

In other words, we proved the following.

PROPOSITION 2.4.6. *If the second fundamental form* A *during the mean curvature flow of a compact hypersurface is not bounded as* $t \to T < +\infty$*, then it must satisfy the following lower bound for its blow up rate*

$$\max_{p \in M} |\mathrm{A}(p, t)| \geq \frac{1}{\sqrt{2(T - t)}}$$

for every $t \in [0, T)$.
Hence,

$$\lim_{t \to T} \max_{p \in M} |\mathrm{A}(p, t)| = +\infty\,.$$

EXERCISE 2.4.7. Assume that the initial, compact hypersurface has $\mathrm{H} > 0$, then the maximal time of smooth existence of the flow can be estimated as $T_{\max} \leq \frac{n}{2\mathrm{H}^2_{\min}(0)}$.

PROPOSITION 2.4.8. *If the second fundamental form is bounded in the interval* $[0, T)$ *with* $T < +\infty$*, then all its covariant derivatives are also bounded.*

PROOF. By Proposition 2.3.5 we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 &= \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r \in \mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \\ &\leq \Delta|\nabla^k \mathrm{A}|^2 + P(|\mathrm{A}|, \dots, |\nabla^{k-1}\mathrm{A}|)|\nabla^k \mathrm{A}|^2 + Q(|\mathrm{A}|, \dots, |\nabla^{k-1}\mathrm{A}|)\,,\end{aligned}$$

where P and Q are smooth functions independent of time (actually they are polynomials in their arguments). Notice that in the arguments of P, Q there is not $\nabla^k \mathrm{A}$, indeed, in the terms $\nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}$ there can be only one or two occurrences of $\nabla^k \mathrm{A}$, since $p + q + r = k$ and $p, q, r \in \mathbb{N}$. If there are two, suppose that $r = k$, then necessarily $p = q = 0$ and we estimate

$|\mathrm{A} * \mathrm{A} * \nabla^k \mathrm{A} * \nabla^k \mathrm{A}| \leq |\mathrm{A}|^2 |\nabla^k \mathrm{A}|^2$, if there is only one this means that $p, q, r < k$ and we again estimate $|\nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}| \leq |\nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A}|^2/2 + |\nabla^k \mathrm{A}|^2/2$.

Reasoning by induction on k, being the case $k = 0$ in the hypotheses, we assume that all the covariant derivatives of A up to the order $k-1$ are bounded, hence also $P(|\mathrm{A}|, \dots, |\nabla^{k-1}\mathrm{A}|)$ and $Q(|\mathrm{A}|, \dots, |\nabla^{k-1}\mathrm{A}|)$ are bounded, thus

$$\frac{\partial}{\partial t} |\nabla^k \mathrm{A}|^2 \leq \Delta |\nabla^k \mathrm{A}|^2 + C|\nabla^k \mathrm{A}|^2 + D\,.$$

By the maximum principle, this implies

$$\frac{d}{dt} |\nabla^k \mathrm{A}|^2_{\max} \leq C|\nabla^k \mathrm{A}|^2_{\max} + D\,,$$

and since the interval $[0, T)$ is bounded, the quantity $|\nabla^k \mathrm{A}|^2_{\max}$ is also bounded, as one can obtain an easy exponential estimate for the function $u(t) = |\nabla^k \mathrm{A}|^2_{\max}$, integrating the ordinary differential inequality $u' \leq Cu + D$, holding for almost every time $t \in [0, T)$. □

PROPOSITION 2.4.9. *If the second fundamental form is bounded in the interval $[0, T)$ with $T < +\infty$, then T cannot be a singular time for the mean curvature flow of a compact hypersurface $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$.*

PROOF. By the previous proposition we know that all the covariant derivatives of A are bounded by constants depending on T and the geometry of the initial hypersurface. As H is bounded, we have

$$|\varphi(p, t) - \varphi(p, s)| \leq \int_s^t |\mathrm{H}(p, \xi)|\, d\xi \leq C(t - s)$$

for every $0 \leq s \leq t < T$, then the maps $\varphi_t = \varphi(\,\cdot\,, t)$ uniformly converge to a continuous limit map $\varphi_T : M \to \mathbb{R}^{n+1}$ as $t \to T$.
We fix now a vector $v = \{v^i\} \in T_pM$,

$$\frac{d}{dt} \log |v|^2_g = \frac{\frac{\partial g_{ij}}{\partial t} v^i v^j}{|v|^2_g} = \frac{-2\mathrm{H} h_{ij} v^i v^j}{|v|^2_g} \leq C \frac{|\mathrm{A}|^2 |v|^2_g}{|v|^2_g} \leq C$$

then, for every $0 \leq s \leq t < T$

$$\left| \log \frac{|v|^2_{g(t)}}{|v|^2_{g(s)}} \right| \leq \int_s^t \left| \frac{d}{d\xi} \log |v|^2_{g(\xi)} \right| d\xi \leq C(t - s)$$

which implies that the metrics $g(t)$ are all equivalent and the norms $|\cdot|_{g(t)}$ uniformly converge, as $t \to T$, to an equivalent norm $|\cdot|_T$ which is continuous. As the parallelogram identity passes to the limit, it must hold also for $|\cdot|_T$, hence this latter comes from a metric tensor g_T which can be obtained by polarization. Moreover, since g_T is equivalent to all the other metrics, it is also positive definite.
Another consequence of such equivalence is that we are free to use any of these metrics in doing our estimates.

By the evolution equation for the Christoffel symbols, we see that

$$\left|\Gamma^k_{ij}(t)\right| \leq \left|\Gamma^k_{ij}(0)\right| + \int_0^t \left| \frac{\partial}{\partial \xi} \Gamma^k_{ij}(\xi) \right| d\xi \leq C + \int_0^T |\mathrm{A} * \nabla \mathrm{A}|\, d\xi \leq C + DT\,,$$

for some constants depending only on the initial hypersurface. Thus, the Christoffel symbols are equibounded in time, after fixing a local chart. This implies for every tensor S,

$$\left| \left| \frac{\partial S}{\partial x_i} \right| - |\nabla_i S| \right| \leq C|S|\,,$$

that is, the derivatives in coordinates differ by the relative covariant ones by equibounded terms.

In the rest of the proof, by simplicity, we will denote by ∂ the coordinate derivatives and by ∇ the covariant ones.

As the time derivative of the Christoffel symbols is a tensor of the form $\mathrm{A} * \nabla \mathrm{A}$, we have

$$|\partial_t \partial^s_{l_1\dots l_s} \Gamma^k_{ij}| = |\partial^s_{l_1\dots l_s} \partial_t \Gamma^k_{ij}| = |\partial^s_{l_1\dots l_s} \mathrm{A} * \nabla \mathrm{A}|\,,$$

hence, by an induction argument on the order s and integration as above, one can show that $\left|\partial^s_{l_1\dots l_s} \Gamma^k_{ij}\right| \leq C$ for every $s \in \mathbb{N}$.
Then, again by induction, the following formula (where we avoid indicating the indices) relating the iterated covariant and coordinate derivatives of a tensor S, holds

$$|\,|\nabla^s S| - |\partial^s S|\,| \leq \sum_{i=1}^{s} \sum_{j_1+\dots+j_i+k\leq s-1} \left|\partial^{j_1}\Gamma \dots \partial^{j_i}\Gamma \partial^k S\right| \leq C \sum_{k=1}^{s-1} \left|\partial^k S\right|\,.$$

This implies that if a tensor has all its covariant derivatives bounded, also all the coordinate derivatives are bounded. In particular this holds for the tensor A, that is, $\left|\partial^k \mathrm{A}\right| \leq C_k$. Moreover, by induction, as $\nabla^k g = 0$ all the coordinate derivatives of the metric tensor g are equibounded.

We already know that $|\varphi|$ is bounded and $|\partial\varphi| = 1$, then by the Gauss–Weingarten relations (1.1.1)

$$\partial^2\varphi = \Gamma\partial\varphi + \mathrm{A}\nu\,, \qquad \partial\nu = \mathrm{A} * \partial\varphi\,,$$

we get

$$\begin{aligned}
|\partial^k\varphi| &= \left|\sum_{i=0}^{k-2} \binom{k-2}{i} \partial^{k-2-i}\Gamma\partial^{i+1}\varphi + \sum_{i=0}^{k-2} \binom{k-2}{i} \partial^{k-2-i}\mathrm{A}\partial^i\nu\right| \\
&\leq C\sum_{i=0}^{k-2} |\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2} |\partial^{i-1}(\mathrm{A} * \partial\varphi)| + C \\
&= C\sum_{i=0}^{k-2} |\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2} \Big|\sum_{p+q+r=i-1} \partial^p\mathrm{A} * \partial^q g * \partial^{r+1}\varphi\Big| + C \\
&\leq C\sum_{i=0}^{k-2} |\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}\sum_{r=0}^{i-1} |\partial^{r+1}\varphi| + C \\
&\leq C\sum_{i=0}^{k-2} |\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2} |\partial^{i}\varphi| + C \\
&\leq C\sum_{i=0}^{k-1} |\partial^{i}\varphi|
\end{aligned}$$

where we estimated with a constant all the occurrences of $\partial^k\mathrm{A}$ and $\partial^k g$. Hence, we obtain by induction that $|\partial^k\varphi| < C_k$ for constants C_k independent of time $t \in [0, T)$. By the Ascoli–Arzelà theorem we can conclude that $\varphi_T : M \to \mathbb{R}^{n+1}$ is a smooth immersion and the convergence $\varphi(\,\cdot\,, t) \to \varphi_T$ is in C^∞.

Moreover, with the same argument, repeatedly differentiating the evolution equation $\partial_t\varphi = \mathrm{H}\nu$ one gets also uniform boundedness of the time derivatives of the map φ, that is $|\partial^s_t\partial^k_x\varphi| \leq C_{s,k}$. Hence the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ can be extended smoothly to the boundary of the domain of φ with the map φ_T.

By means of the short time existence Theorem 1.5.1 we can now "restart" the flow with the immersion φ_T, obtaining a smooth extension of the map φ which is in contradiction with the fact that T was the maximal time of smooth existence. $\square$

OPEN PROBLEM 2.4.10. Recently the condition of bounded second fundamental form was weakened by Le and Sesum **[85]** to a lower bound on A and an integral bound on H.
An interesting open problem is whether actually a uniform bound only on the mean curvature H is sufficient to exclude singularities during the flow (see **[84]**).

Thus, we conclude this section stating the following slightly improved version of Theorem 1.5.1.

THEOREM 2.4.11. *For any initial, compact, smooth hypersurface immersed in $\mathbb{R}^{n+1}$ there exists a unique mean curvature flow which is smooth in a maximal time interval $[0, T_{\max})$.*
Moreover, $T_{\max}$ is finite and

$$\max_{p\in M} |\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T_{\max}-t)}}$$

for every $t \in [0, T_{\max})$.

Notice that it follows that the maximal time of smooth existence of the flow can be estimated from below as $T_{\max} \geq \frac{1}{2|\mathrm{A}(\,\cdot\,,0)|^2_{\max}}$.

2.5. Convexity Invariance

Corollary 2.4.3 is a consequence of a more general invariance property of the elementary symmetric polynomials of the curvatures, as we are going to show.
We recall that the *elementary symmetric polynomial* of degree k of $\lambda_1, \dots, \lambda_n$ is defined as

$$\mathrm{S}_k = \sum_{1\leq i_1<i_2<\dots<i_k\leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$$

for $k = 1, \dots, n$. In particular, if λ_i are the eigenvalues of the second fundamental form A we have $\mathrm{S}_1 = \mathrm{H}$, S_2 is the scalar curvature and $|\mathrm{A}|^2 = \mathrm{S}_1^2 - 2\mathrm{S}_2$.
It is not difficult to show that

$$\begin{aligned} \lambda_1 \geq 0, \dots, \lambda_n \geq 0 \quad &\Longleftrightarrow \quad \mathrm{S}_1 \geq 0, \dots, \mathrm{S}_n \geq 0\,, \\ \lambda_1 > 0, \dots, \lambda_n > 0 \quad &\Longleftrightarrow \quad \mathrm{S}_1 > 0, \dots, \mathrm{S}_n > 0\,. \end{aligned} \tag{2.5.1}$$

These polynomials enjoy various concavity properties, see **[73, 91]**.

PROPOSITION 2.5.1. *Let $\Gamma_k \subset \mathbb{R}^n$ denote the connected component of $\{\mathrm{S}_k > 0\}$ containing the positive cone. Then $\mathrm{S}_l > 0$ in Γ_k for all $l = 1, \dots, k$ and the quotient $\mathrm{S}_{k+1}/\mathrm{S}_k$ is concave on Γ_k.*

The above properties remain unchanged if we regard the polynomials S_k as functions of the Weingarten operator h^i_j instead of the principal curvatures, as we have the following algebraic result, see [**9**, Lemma 2.22] or [**73**, Lemma 2.11].

PROPOSITION 2.5.2. *Let $f(\lambda_1, \dots, \lambda_n)$ be a symmetric convex (concave) function of its arguments and let $F(A) = f(\text{eigenvalues of } A)$ for any $n \times n$ symmetric matrix A whose eigenvalues belong to the domain of f. Then F is convex (concave).*

We are now ready to derive the evolution equations of some relevant quantities and to apply the maximum principle to obtain some invariance properties.

PROPOSITION 2.5.3. *Let $F(h^i_j)$ be a homogeneous function of degree one. Let φ be a mean curvature flow of a compact, n–dimensional hypersurface with $\mathrm{H} > 0$ and such that h^i_j belongs everywhere to the domain of F. Then,*

$$\frac{\partial}{\partial t}\frac{F}{\mathrm{H}} - \Delta\frac{F}{\mathrm{H}} = \frac{2}{\mathrm{H}}\Big\langle \nabla\mathrm{H}\,\Big|\,\nabla\frac{F}{\mathrm{H}}\Big\rangle - \frac{1}{\mathrm{H}}\frac{\partial^2 F}{\partial h^i_j \partial h^k_l}\, g^{pq}\nabla_p h^i_j \nabla_q h^k_l\,.$$

As a consequence, if F is concave (convex), any pinching of the form $F \geq \alpha\mathrm{H}$ ($F \leq \alpha\mathrm{H}$) is preserved during the flow by the maximum principle, as the last term is then nonnegative (nonpositive).

PROOF. A straightforward computation using formula (2.3.3) in Proposition 2.3.1 and Euler's theorem on homogeneous functions yields

$$\begin{aligned} \frac{\partial}{\partial t}\frac{F}{\mathrm{H}} &= \frac{1}{H}\frac{\partial F}{\partial h^i_j}(\Delta h^i_j + |\mathrm{A}|^2 h^i_j) - \frac{F}{\mathrm{H}^2}(\Delta\mathrm{H} + |\mathrm{A}|^2\mathrm{H}) \\ &= \Delta\frac{F}{\mathrm{H}} + \frac{2}{\mathrm{H}}\Big\langle \nabla\mathrm{H}\,\Big|\,\nabla\frac{F}{\mathrm{H}}\Big\rangle - \frac{1}{\mathrm{H}}\frac{\partial^2 F}{\partial h^i_j \partial h^k_l}\, g^{pq}\nabla_p h^i_j \nabla_q h^k_l\,. \end{aligned}$$

□

In particular, the previous proposition can be applied to the quantity $F = \mathrm{S}_{k+1}/\mathrm{S}_k$, provided $\mathrm{S}_k \neq 0$. This leads to the following result, which generalizes Corollary 2.4.3.

PROPOSITION 2.5.4. *Let the initial, compact hypersurface satisfy* $\mathrm{S}_k > 0$ *everywhere for a given* $k \in \{1, \dots, n\}$ *and let* $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ *be its evolution by mean curvature. Then, for any* $i = 2, \dots, k$ *there exists* α_i *such that* $\mathrm{S}_i \geq \alpha_i \mathrm{H}^i > 0$ *for every* $p \in M$ *and* $t \in [0, T)$.

PROOF. We assume that the hypersurface M is connected, otherwise we argue component by component.
For every pair of of points p and q in M, the set of principal curvatures at p and the set of principal curvatures at q belong to the same connected component of $\{\mathrm{S}_k > 0\} \subset \mathbb{R}^n$, seeing S_k as a map from $\mathbb{R}^n$ to $\mathbb{R}$ (connect with an arc the two points). Then, as the initial hypersurface is compact, there exists a point $p \in M$ where all the principal curvatures are positive (consider a tangent sphere containing the hypersurface), hence, the set of principal curvatures at all the points of M belongs to the connected component Γ_k of the positive cone defined in Proposition 2.5.1. Hence, for every $i = 1, \dots, k$ we have $\mathrm{S}_i > 0$ everywhere on the initial hypersurface. In particular $\mathrm{H} = \mathrm{S}_1 > 0$ and, by compactness, we have $\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1}$ for suitable constants $\beta_i > 0$, for any $i = 2, \dots, k$.
We know from Proposition 2.4.2 that $\mathrm{H} > 0$ everywhere on M for every $t \in [0, T)$. Then we can consider the quotient $\mathrm{S}_2/\mathrm{H}^2 = \mathrm{S}_2/(\mathrm{H}\mathrm{S}_1)$ which is well defined for every t and it is greater than β_2 at time $t = 0$. By Proposition 2.5.3 its minimum is nondecreasing, hence $\mathrm{S}_2 \geq \beta_2 \mathrm{H}^2$ for every $t \in [0, T)$.
We now apply the same procedure to the quotient $\mathrm{S}_3/(\mathrm{H}\mathrm{S}_2)$ to conclude that it is greater than β_3 for every $t \in [0, T)$, then in general $\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1}$ for $i = 2, \dots, k$.
Multiplying together all these inequalities we get

$$\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1} \geq \beta_i \beta_{i-1} \mathrm{H}^2 \mathrm{S}_{i-2} \geq \cdots \geq \beta_i \beta_{i-1} \cdots \beta_2 \mathrm{H}^i$$

and the claim follows by setting $\alpha_i = \beta_i \beta_{i-1} \cdots \beta_2$. □

COROLLARY 2.5.5. *If the initial, compact hypersurface is strictly convex, it remains strictly convex under the mean curvature flow.*

PROOF. Strict convexity is equivalent to the set of conditions $\mathrm{S}_1, \dots, \mathrm{S}_n > 0$ on the eigenvalues of the second fundamental form, by relations (2.5.1) and these conditions are preserved under the mean curvature flow, by the previous proposition. □

REMARK 2.5.6. By Hamilton's strong maximum principle for tensors in [**56**, Section 8] (Theorem C.1.3 in Appendix C), if an initial, compact hypersurface is only convex (not necessarily strictly convex), then it becomes immediately strictly convex. Even more precisely, in this case, the smallest eigenvalue of the second fundamental form on all M increases in time.
Indeed, the Weingarten operator is nonnegative definite for every positive time and satisfies (see Proposition 2.3.1)

$$\frac{\partial}{\partial t} h_i^j = \Delta h_i^j + |\mathrm{A}|^2 h_i^j \,,$$

then by Theorem C.1.3 its rank (hence the rank of A) is constant in some time interval $(0, \delta)$, moreover, the null space is invariant under parallel transport and in time. Then, supposing that such rank m is less than the dimension n of the hypersurface, we have an $(n - m)$–dimensional subspace $N_p \subset T_p M$ at every point $p \in M$, invariant under parallel transport, where $\mathrm{A}_p(v, v) = 0$ for every $v \in N_p$.
If $v \in T_p M$ is a vector in the null space, any geodesic γ in M starting at p is also a geodesic in $\mathbb{R}^{n+1}$ as $\dot\gamma$ remains always in the null space of A and

$$\nabla^{\mathbb{R}^{n+1}}_{\dot\gamma} \dot\gamma = \nabla^M_{\dot\gamma} \dot\gamma + \mathrm{A}(\dot\gamma, \dot\gamma)\nu = 0 \,.$$

Hence, all the $(n - m)$–dimensional null space (as an affine subspace of $\mathbb{R}^{n+1}$) is contained in M, this is in contradiction with the compactness of M.

REMARK 2.5.7. If the initial hypersurface is not convex, it is not true that the smallest eigenvalue of A increases, think of Angenent's homothetically shrinking torus we mentioned in Section 1.4 (see [**17**]).

Notice that the results about the strict monotonicity of geometric quantities during the flow are valid when the initial hypersurface is compact and can fail otherwise. For instance, an evolving cylinder does not become immediately strictly convex.

PROPOSITION 2.5.8. *If for a constant $\alpha \in \mathbb{R}$ there holds $\mathrm{A} \geq \alpha \mathrm{H} g$ (as forms) for the initial, compact hypersurface, this condition is preserved during the mean curvature flow.*

PROOF. We consider the function $f = h_{ij}v^iv^j - \alpha \mathrm{H} g_{ij}v^iv^j$ where $v^i(p,t)$ is a time dependent smooth vector field such that $\partial v^i/\partial t = \mathrm{H}h^i_k v^k$,

$$\begin{aligned}
\frac{\partial f}{\partial t} =& \frac{\partial h_{ij}}{\partial t}v^iv^j + 2h_{ij}v^i\frac{\partial v^j}{\partial t} - \alpha\frac{\partial \mathrm{H}}{\partial t}g_{ij}v^iv^j + 2\alpha\mathrm{H}^2h_{ij}v^iv^j - 2\alpha\mathrm{H}g_{ij}v^i\frac{\partial v^j}{\partial t}\\
=& (\Delta h_{ij} - 2\mathrm{H}h^2_{ij} + |\mathrm{A}|^2h_{ij})v^iv^j + 2\mathrm{H}h^2_{ij}v^iv^j - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)g_{ij}v^iv^j\\
=& (\Delta h_{ij} + |\mathrm{A}|^2h_{ij})v^iv^j - \alpha\Delta\mathrm{H}g_{ij}v^iv^j - \alpha\mathrm{H}|\mathrm{A}|^2g_{ij}v^iv^j\\
=& \Delta(h_{ij}v^iv^j - \alpha\mathrm{H}g_{ij}v^iv^j) + |\mathrm{A}|^2(h_{ij} - \alpha\mathrm{H}g_{ij})v^iv^j - 4(\nabla_kh_{ij} - \alpha\nabla_k\mathrm{H}g_{ij})v^i\nabla^kv^j\\
&- 2(h_{ij} - \alpha\mathrm{H}g_{ij})\nabla_kv^i\nabla^kv^j - 2(h_{ij} - \alpha\mathrm{H}g_{ij})v^i\Delta v^j\\
=& \Delta f + |\mathrm{A}|^2 f - 4(\nabla_kh_{ij} - \alpha\nabla_k\mathrm{H}g_{ij})v^i\nabla^kv^j\\
&- 2(h_{ij} - \alpha\mathrm{H}g_{ij})\nabla_kv^i\nabla^kv^j - 2(h_{ij} - \alpha\mathrm{H}g_{ij})v^i\Delta v^j\,.
\end{aligned}$$

Let $\mu(t)$ be the smallest value of $h_{ij}(q,t)v^iv^j - \alpha\mathrm{H}g_{ij}(q,t)v^iv^j$ for t fixed, $q \in M$ and $v \in T_qM$ a unit tangent vector of (M, g_t).
Being μ a locally Lipschitz function, it is differentiable at almost every time, moreover by the hypotheses, we have $\mu(0) \geq 0$.
We suppose that there exists an open interval of time (t_0, t_1) where μ is negative and $\mu(t_0) = 0$. Let $\tilde{t} \in (t_0, t_1)$ be a differentiability point of μ, then there exists a point $p \in M$ and a unit vector $v \in T_pM$ such that

$$\mu(\tilde{t}) = h_{ij}(p,\tilde{t})v^iv^j - \alpha\mathrm{H}(p,\tilde{t})g_{ij}(p,\tilde{t})v^iv^j \leq h_{ij}(q,\tilde{t})w^iw^j - \alpha\mathrm{H}(q,\tilde{t})g_{ij}(q,t)w^iw^j$$

for every $q \in M$ and $w \in T_qM$ of unit norm. We extend the unit vector $v \in T_pM$ in space to a vector field that we still call v with the following properties,

- $g_{\tilde{t}}(v(q), v(q)) \leq 1$ for every $q \in M$,
- $\nabla_{g_{\tilde{t}}} v(p) = 0$,
- $\Delta_{g_{\tilde{t}}} v(p) = 0$.

This can be done as follows: we choose local normal coordinates around $p \in M$ (the point p "goes" to the origin), then, the last two conditions are fulfilled by a local vector field w if

$$\frac{\partial w^i}{\partial x_j}(0) = 0 \qquad \text{and} \qquad \Delta w^i(0) + \frac{\partial \Gamma^i_{jk}}{\partial x_j}(0,\tilde{t})w^k(0) = 0\,,$$

where this Laplacian is the standard Laplacian of $\mathbb{R}^n$. Hence, the field with coordinates

$$w^i(x) = v^i(0) - \frac{x_1^2}{2}\frac{\partial \Gamma^i_{j1}}{\partial x_j}(0,\tilde{t})v^k(0)$$

satisfies them as $w(0) = v(0)$. It is now easy to check that the normalized unit vector field $v = w/|w|$ is locally defined in a neighborhood of the point p and satisfies all the three conditions above. Then, we consider a smooth function with compact support contained where such unit vector field v is defined, with modulus not larger than one and equal to one in a neighborhood of the point p. The product of the vector field v with such function gives a global smooth vector field on the whole manifold M with the above properties.
Now, we extend v also in time in the interval (t_0, t_1) by solving the ODE $\partial v^i/\partial t = \mathrm{H}h^i_k v^k$ and we consider the associated function f as above.

Notice that since $\mu(t)$ is negative in (t_0, t_1), the function $f(\,\cdot\,, \tilde{t})$ gets a minimum in space at $p \in M$, indeed, if $f(q, \tilde{t}) < 0$, we have $v(q) \neq 0$ and

$$\begin{aligned} f(p, \tilde{t}) = \mu(\tilde{t}) &\leq \frac{h_{ij}(q, \tilde{t})v^i(q)v^j(q) - \alpha\mathrm{H}(q, \tilde{t})g_{\tilde{t}}(v(q), v(q))}{g_{\tilde{t}}(v(q), v(q))} \\ &= \frac{f(q, \tilde{t})}{g_{\tilde{t}}(v(q), v(q))} \\ &\leq f(q, \tilde{t}) \end{aligned}$$

as $g_{\tilde{t}}(v(q), v(q)) \leq 1$ by construction. Hence, $\Delta f(p, \tilde{t}) \geq 0$ and at the point $(p, \tilde{t})$ we have

$$\frac{\partial f}{\partial t} = \Delta f + |\mathrm{A}|^2 f \geq Cf$$

where $C > 0$ is a constant such that $|\mathrm{A}|^2 \leq C$ on $[0, t_1)$.
By this inequality, given $\varepsilon > 0$, there exists some $t_2 \in (t_0, \tilde{t})$, such that if $\bar{t} \in (t_2, \tilde{t})$ we have

$$f(p, \bar{t}) < f(p, \tilde{t}) - C(\tilde{t} - \bar{t})f(p, \tilde{t}) + \varepsilon(\tilde{t} - \bar{t})\,.$$

Being $v(p, \bar{t})$ still a unit vector, as $\partial g(v, v)/\partial t = -2\mathrm{H}h_{ij}v^iv^j + 2g(\partial v/\partial t, v) = 0$ so the norm of $v(p, t)$ is constant in time, we get

$$\mu(\bar{t}) \leq f(p, \bar{t}) < f(p, \tilde{t}) - C(\tilde{t} - \bar{t})f(p, \tilde{t}) + \varepsilon(\tilde{t} - \bar{t}) = \mu(\tilde{t}) - C(\tilde{t} - \bar{t})\mu(\tilde{t}) + \varepsilon(\tilde{t} - \bar{t})\,.$$

In other words $\frac{\mu(\tilde{t}) - \mu(\bar{t})}{\tilde{t} - \bar{t}} \geq C\mu(\tilde{t}) - \varepsilon$ and being $\tilde{t}$ a differentiability time for μ, passing to the limit as $\bar{t} \nearrow \tilde{t}$, we obtain $\mu'(\tilde{t}) \geq C\mu(\tilde{t}) - \varepsilon$.
Finally, as ε is arbitrarily small, we conclude $\mu'(\tilde{t}) \geq C\mu(\tilde{t})$.
Since this relation holds at every differentiability time $\tilde{t}$ in (t_0, t_1) where $\mu(\tilde{t}) < 0$, hence almost everywhere in (t_0, t_1), we can integrate it in such interval. Recalling that $\mu(t_0) = 0$ by continuity, we conclude that $\mu(t)$ must be identically zero in $[t_0, t_1)$ which is in contradiction with the hypotheses.
Notice the similarities with the proofs of Lemma 2.1.3 and Proposition 2.4.1.

□

EXERCISE 2.5.9. Show that for an initial hypersurface with $\mathrm{H} > 0$ the smallest eigenvalue of the form h_{ij}/H is nondecreasing during the flow.

Finally, further invariance properties for the mean curvature flow can be obtained again by means of Hamilton's maximum principle for tensors [**56**, Sections 4 and 8] (whose proof is a generalization of the argument above), see Appendix C. Let us first recall a definition, we say that an immersed hypersurface is k–convex, for some $1 \leq k \leq n$, if the sum of the k smallest principal curvatures is nonnegative at every point. In particular, one–convexity coincides with convexity, while n–convexity means nonnegativity of the mean curvature H, that is, mean convexity.
Then we mention the following result generalizing Corollary 2.5.5 (see [**75**]).

PROPOSITION 2.5.10. *If an initial, compact hypersurface is k–convex, then it is so for every positive time under the mean curvature flow.*

PROOF. The result follows from Hamilton's maximum principle for tensors, provided we show that the inequality $\lambda_1 + \cdots + \lambda_k \geq \alpha\mathrm{H}$ describes a convex cone in the set of all matrices, and that this cone is invariant under the system of ODE's $dh^i_j/dt = |\mathrm{A}|^2 h^i_j$ for the Weingarten operator.
As

$$(\lambda_1 + \cdots + \lambda_k)(p) = \min_{\substack{e_1, \dots, e_k \in T_pM \\ g_p(e_i, e_j) = \delta_{ij}}} \{\mathrm{A}_p(e_1, e_1) + \cdots + \mathrm{A}_p(e_k, e_k)\}\,,$$

the quantity $\lambda_1 + \cdots + \lambda_k$ is a concave function of the Weingarten operator, being the infimum of a family of linear maps. Therefore the inequality $\lambda_1 + \cdots + \lambda_k \geq \alpha\mathrm{H}$ describes a convex cone of matrices. In addition, the system $dh^i_j/dt = |\mathrm{A}|^2 h^i_j$ changes the Weingarten operator by homotheties, thus leaves any cone invariant. The conclusion follows. □