CHAPTER 3

Monotonicity Formula and Type I Singularities

In all this chapter $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ is the mean curvature flow of an n–dimensional, compact hypersurface in the maximal interval of smooth existence $[0,T)$.

As before we will use the notation $\varphi_t = \varphi(\,\cdot\,,t)$ and $\widetilde{\mathcal{H}}^n$ will be the n–dimensional Hausdorff measure in $\mathbb{R}^{n+1}$ *counting multiplicities*.

3.1. The Monotonicity Formula for Mean Curvature Flow

We show the fundamental monotonicity formula for mean curvature flow, discovered by Huisken in **[40]** and then generalized by Hamilton in **[37, 38]**.

LEMMA 3.1.1. *Let $f : \mathbb{R}^{n+1} \times I \to \mathbb{R}$ be a smooth function. By a little abuse of notation, we denote by $\int_M f\, d\mu_t$ the integral $\int_M f(\varphi(p,t),t)\, d\mu_t(p)$.*
Then the following formula holds

$$\frac{d}{dt}\int_M f\, d\mu_t = \int_M (f_t - \mathrm{H}^2 f + \mathrm{H}\langle \nabla f \,|\, \nu\rangle)\, d\mu_t\,.$$

PROOF. Straightforward computation. □

If $u_t = -\Delta^{\mathbb{R}^{n+1}} u$ is a solution of the backward heat equation in $\mathbb{R}^{n+1}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M u\, d\mu_t &= \int_M (u_t - \mathrm{H}^2 u + \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t \\ &= -\int_M (\Delta^{\mathbb{R}^{n+1}} u + \mathrm{H}^2 u - \mathrm{H}\langle \nabla u \,|\, \nu\rangle)\, d\mu_t\,.\end{aligned} \tag{3.1.1}$$

LEMMA 3.1.2. *If $\psi : M \to \mathbb{R}^{n+1}$ is a smooth isometric immersion of an n–dimensional Riemannian manifold (M,g), for every smooth function u defined in a neighborhood of $\psi(M)$ we have,*

$$\Delta_g(u(\psi)) = (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle \nu \,|\, (\nabla u)(\psi)\rangle\,,$$

where $(\nabla^2_{\nu\nu} u)(\psi(p))$ is the second derivative of u in the normal direction $\nu(p) \in \mathbb{R}^{n+1}$ at the point $\psi(p)$.

PROOF. Let $p \in M$ and choose normal coordinates at p. Set $\widetilde{u} = u \circ \psi$, then

$$\begin{aligned}\Delta_g \widetilde{u} &= \nabla^2_{ii}(u \circ \psi) \\ &= \nabla_i\left(\frac{\partial u}{\partial y_\alpha}\frac{\partial \psi^\alpha}{\partial x_i}\right) \\ &= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}\frac{\partial \psi^\alpha}{\partial x_i}\frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha}\frac{\partial^2 \psi^\alpha}{\partial x_i^2} \\ &= \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}\frac{\partial \psi^\alpha}{\partial x_i}\frac{\partial \psi^\beta}{\partial x_i} + \frac{\partial u}{\partial y_\alpha} h_{ii}\nu^\alpha \\ &= (\Delta^{\mathbb{R}^{n+1}} u)(\psi) - (\nabla^2_{\nu\nu} u)(\psi) + \mathrm{H}\langle \nu \,|\, (\nabla u)(\psi)\rangle\,,\end{aligned}$$

where we used the Gauss–Weingarten relations (1.1.1). □

It follows that, substituting $\Delta^{\mathbb{R}^{n+1}}u$ in formula (3.1.1) and using the previous lemma, if the function u is positive we get

$$\begin{aligned}\frac{d}{dt}\int_M u\,d\mu_t &= -\int_M \big(\Delta_{g(t)}(u(\varphi_t)) + \nabla^2_{\nu\nu}u + \mathrm{H}^2 u - 2\mathrm{H}\langle\nabla u\,|\,\nu\rangle\big)\,d\mu_t\\ &= -\int_M \big(\nabla^2_{\nu\nu}u + \mathrm{H}^2 u - 2\mathrm{H}\langle\nabla u\,|\,\nu\rangle\big)\,d\mu_t\\ &= -\int_M \left|\mathrm{H} - \frac{\langle\nabla u\,|\,\nu\rangle}{u}\right|^2 u\,d\mu_t + \int_M \left(\frac{|\nabla^\perp u|^2}{u} - \nabla^2_{\nu\nu}u\right)d\mu_t\,,\end{aligned}$$

where $\nabla^\perp u$ denotes the projection on the normal space to M of the gradient of u.
Then, assuming that $u : \mathbb{R}^{n+1}\times[0,\tau)\to\mathbb{R}$ is a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}}u$ for some $\tau > 0$, the following formula easily follows,

$$\begin{aligned}\frac{d}{dt}\Big[\sqrt{4\pi(\tau-t)}\int_M u\,d\mu_t\Big] = &-\sqrt{4\pi(\tau-t)}\int_M |\mathrm{H} - \langle\nabla\log u\,|\,\nu\rangle|^2 u\,d\mu_t\\ &-\sqrt{4\pi(\tau-t)}\int_M \Big(\nabla^2_{\nu\nu}u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)}\Big)\,d\mu_t\end{aligned} \tag{3.1.2}$$

in the time interval $[0,\min\{\tau,T\})$.
As we can see, the right hand side consists of a nonpositive quantity and a term which is nonpositive if $\frac{\nabla^2_{\nu\nu}u}{u} - \frac{|\nabla^\perp u|^2}{u^2} + \frac{1}{2(\tau-t)} = \nabla^2_{\nu\nu}\log u + \frac{1}{2(\tau-t)}$ is nonnegative.
Setting $v(x,s) = u(x,\tau-s)$, the function $v : \mathbb{R}^{n+1}\times(0,\tau]\to\mathbb{R}$ is a positive solution of the standard *forward* heat equation in all $\mathbb{R}^{n+1}$ and setting $t = \tau - s$ we have $\nabla^2_{\nu\nu}\log u + \frac{1}{2(\tau-t)} = \nabla^2_{\nu\nu}\log v + \frac{1}{2s}$.
This last expression is exactly the Li–Yau–Hamilton 2–form $\nabla^2\log v + g/(2s)$ for positive solutions of the heat equation on a compact manifold (N,g), evaluated on $\nu\otimes\nu$ (see **[37]**).
In the paper **[37]** (see also **[59]**) Hamilton generalized the Li–Yau differential Harnack inequality in **[56]** (concerning the nonnegativity of $\Delta\log v + \frac{\dim N}{2s}$) showing that, under the assumptions that (N,g) has parallel Ricci tensor ($\nabla\mathrm{Ric} = 0$) and nonnegative sectional curvatures, the 2–form $\nabla^2\log v + g/(2s)$ is nonnegative definite (Hamilton's matrix Li–Yau–Harnack inequality).
In particular, in $\mathbb{R}^{n+1}$ equipped with the canonical flat metric such hypotheses clearly hold and $\nabla^2_{\nu\nu}\log u + \frac{1}{2(\tau-t)} = \Big(\nabla^2\log v + g^{\mathbb{R}^{n+1}}_{\mathrm{can}}/(2s)\Big)(\nu\otimes\nu) \geq 0$. Hence, assuming the boundedness in space of v (equivalently of u), the monotonicity formula implies that $\sqrt{4\pi(\tau-t)}\int_M u\,d\mu_t$ is nonincreasing in time. We resume this discussion in the following theorem by Hamilton **[37, 38]**.

THEOREM 3.1.3 (Huisken's Monotonicity Formula – Hamilton's Extension in $\mathbb{R}^{n+1}$). *Assume that for some $\tau > 0$ we have a positive smooth solution of the backward heat equation $u_t = -\Delta^{\mathbb{R}^{n+1}}u$ in $\mathbb{R}^{n+1}\times[0,\tau)$, bounded in space for every fixed $t\in[0,\tau)$, then*

$$\frac{d}{dt}\Big[\sqrt{4\pi(\tau-t)}\int_M u\,d\mu_t\Big] \leq -\sqrt{4\pi(\tau-t)}\int_M |\mathrm{H} - \langle\nabla\log u\,|\,\nu\rangle|^2 u\,d\mu_t$$

in the time interval $[0,\min\{\tau,T\})$.

REMARK 3.1.4. In the original paper of Hamilton the compactness of the ambient space is required (the proof is based on the maximum principle), in order to extend his result to $\mathbb{R}^{n+1}$ we assumed the boundedness in space of u, see Appendix **??** for details.

Choosing in particular a *backward* heat kernel of $\mathbb{R}^{n+1}$, that is,

$$u(x,t) = \rho_{x_0,\tau}(x,t) = \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{(n+1)/2}}$$

in formula (3.1.2), we get the standard Huisken's monotonicity formula, as the Li–Yau–Hamilton expression $\nabla^2_{\nu\nu}u - \frac{|\nabla^\perp u|^2}{u} + \frac{u}{2(\tau-t)}$ is identically zero in this case.

THEOREM 3.1.5 (Huisken's Monotonicity Formula). *For every $x_0 \in \mathbb{R}^{n+1}$ and $\tau > 0$ we have (see* **[40]***)*

$$\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0\,|\,\nu\rangle}{2(\tau-t)}\right|^2 d\mu_t$$

in the time interval $[0, \min\{\tau, T\})$.
Hence, the integral $\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,d\mu_t$ *is nonincreasing during the flow in* $[0, \min\{\tau, T\})$.

EXERCISE 3.1.6. Show that for every $x_0 \in \mathbb{R}^{n+1}$, $\tau > 0$ and a smooth function $v : M \times [0, T) \to \mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,v\,d\mu_t = &-\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0\,|\,\nu\rangle}{2(\tau-t)}\right|^2 v\,d\mu_t\\ &+\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,(v_t - \Delta_{g(t)}v)\,d\mu_t\,,\end{aligned}$$

in the time interval $[0, \min\{\tau, T\})$.
In particular if $v : M \times [0, T) \to \mathbb{R}$ is a smooth solution of $v_t = \Delta_{g(t)}v$,

$$\frac{d}{dt}\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\,v\,d\mu_t = -\int_M \frac{e^{-\frac{|x-x_0|^2}{4(\tau-t)}}}{[4\pi(\tau-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x - x_0\,|\,\nu\rangle}{2(\tau-t)}\right|^2 v\,d\mu_t$$

in $[0, \min\{\tau, T\})$.

3.2. Type I Singularities and the Rescaling Procedure

In the previous chapter we showed that the curvature must blow up at the maximal time T with the following lower bound

$$\max_{p\in M}|\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}\,.$$

DEFINITION 3.2.1. Let T be the maximal time of existence of a mean curvature flow. If there exists a constant $C > 1$ such that we have the upper bound

$$\max_{p\in M}|\mathrm{A}(p,t)| \leq \frac{C}{\sqrt{2(T-t)}}\,,$$

we say that the flow is developing at time T a *type I singularity*.
If such a constant does not exist, that is,

$$\limsup_{t\to T}\max_{p\in M}|\mathrm{A}(p,t)|\sqrt{T-t} = +\infty$$

we say that we have a *type II singularity*.

In this chapter we deal exclusively with type I singularities and the monotonicity formula will be the main tool for the analysis. The next chapter will be devoted to type II singularities.
From now on, we assume that there exists some constant $C_0 > 1$ such that

$$\frac{1}{\sqrt{2(T-t)}} \leq \max_{p\in M}|\mathrm{A}(p,t)| \leq \frac{C_0}{\sqrt{2(T-t)}}\,, \tag{3.2.1}$$

for every $t \in [0, T)$.
Let $p \in M$ and $0 \leq t \leq s < T$, then

$$|\varphi(p,s) - \varphi(p,t)| = \left|\int_t^s \frac{\partial\varphi(p,\xi)}{\partial t}\,d\xi\right| \leq \int_t^s |\mathrm{H}(p,\xi)|\,d\xi \leq \int_t^s \frac{C_0\sqrt{n}}{\sqrt{2(T-\xi)}}\,d\xi \leq C_0\sqrt{n(T-t)}$$

which implies that the sequence of functions $\varphi(\,\cdot\,, t)$ converges as $t \to T$ to some function $\varphi_T : M \to \mathbb{R}^{n+1}$. Moreover, as the constant C_0 is independent of $p \in M$, such convergence is uniform and the limit function φ_T is continuous. Finally, passing to the limit in the above inequality, we get

$$|\varphi(p,t)-\varphi_T(p)|\leq C_0\sqrt{n(T-t)}\,. \tag{3.2.2}$$

In all the chapter we will denote $\varphi_T(p)$ by $\widehat{p}$.

DEFINITION 3.2.2. Let $\mathcal{S}$ be the set of points $x\in\mathbb{R}^{n+1}$ such that there exists a sequence of pairs $(p_i,t_i)\in M\times[0,T)$ with $t_i\nearrow T$ and $\varphi(p_i,t_i)\to x$.
We call $\mathcal{S}$ the set of *reachable* points.

We have seen in Proposition 2.2.6 that $\mathcal{S}$ is compact and that $x\in\mathcal{S}$ if and only if, for every $t\in[0,T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center x intersects $\varphi(M,t)$. We show now that $\mathcal{S}=\{\widehat{p}\,|\,p\in M\}$.
Clearly $\{\widehat{p}\,|\,p\in M\}\subset\mathcal{S}$, suppose that $x\in\mathcal{S}$ and $\varphi(p_i,t_i)\to x$, then, by inequality (3.2.2) we have $|\varphi(p_i,t_i)-\widehat{p}_i|\leq C_0\sqrt{n(T-t_i)}$, hence, $\widehat{p}_i\to x$ as $i\to\infty$. As the set $\{\widehat{p}\,|\,p\in M\}$ is closed it follows that it must contain the point x.

We define now a tool which will be fundamental in the sequel.

DEFINITION 3.2.3. For every $p\in M$, we define the *heat density* function

$$\theta(p,t)=\int_M\frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$$

and the *limit heat density* function as

$$\Theta(p)=\lim_{t\to T}\theta(p,t)\,.$$

As M is compact, we can also define the following *maximal heat density*

$$\sigma(t)=\max_{x_0\in\mathbb{R}^{n+1}}\int_M\frac{e^{-\frac{|x-x_0|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t \tag{3.2.3}$$

and its limit $\Sigma=\lim_{t\to T}\sigma(t)$.

Clearly, $\theta(p,t)\leq\sigma(t)$, for every $p\in M$ and $t\in[0,T)$ and $\Theta(p)\leq\Sigma$ for every $p\in M$.
The function Θ is well defined as the limit exists finite since $\theta(p,t)$ is monotone nonincreasing in t and positive. Moreover, the functions $\theta(\,\cdot\,,t)$ are all continuous and monotonically converging to Θ, hence this latter is upper semicontinuous and nonnegative.

The function $\sigma:[0,T)\to\mathbb{R}$ is also positive and monotone nonincreasing, being the maximum of a family of nonincreasing smooth functions, hence the limit Σ is well defined and finite. Moreover, such family is uniformly locally Lipschitz (look at the right hand side of the monotonicity formula), hence also σ is locally Lipschitz, then by Hamilton's trick 2.1.3, at every differentiability time $t\in[0,T)$ of σ we have

$$\sigma'(t)=-\int_M\frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\left|\mathrm{H}+\frac{\langle x-x_t\,|\,\nu\rangle}{2(T-t)}\right|^2\,d\mu_t \tag{3.2.4}$$

where $x_t\in\mathbb{R}^{n+1}$ is any point where the maximum defining $\sigma(t)$ is attained, that is,

$$\sigma(t)=\int_M\frac{e^{-\frac{|x-x_t|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t\,.$$

REMARK 3.2.4. Notice that we did not define $\sigma(t)$ as the maximum of $\theta(\,\cdot\,,t)$

$$\max_{p\in M}\int_M\frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[4\pi(T-t)]^{n/2}}\,d\mu_t$$

which is *taken among* $p\in M$. Clearly, this latter can be smaller than $\sigma(t)$.

We rescale now the moving hypersurfaces around $\widehat{p} = \lim_{t\to T} \varphi(p,t)$, following Huisken **[40]**,

$$\widetilde{\varphi}(q,s) = \frac{\varphi(q,t(s)) - \widehat{p}}{\sqrt{2(T-t(s))}} \qquad s = s(t) = -\frac{1}{2}\log(T-t)$$

and we compute the evolution equation for $\widetilde{\varphi}(q,s)$ in the time interval $\left[-\frac{1}{2}\log T, +\infty\right)$,

$$\begin{aligned}\frac{\partial\widetilde{\varphi}(q,s)}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1}\frac{\partial}{\partial t}\left(\frac{\varphi(q,t)-\widehat{p}}{\sqrt{2(T-t)}}\right)\\ &= \sqrt{2(T-t)}\,\frac{\partial\varphi(q,t)}{\partial t} + \frac{\varphi(q,t)-\widehat{p}}{\sqrt{2(T-t)}}\\ &= \sqrt{2(T-t)}\,\mathrm{H}(q,t)\nu(q,t) + \widetilde{\varphi}(q,s)\\ &= \widetilde{\mathrm{H}}(q,s)\widetilde{\nu}(q,s) + \widetilde{\varphi}(q,s)\,,\end{aligned}$$

where $\widetilde{\mathrm{H}}$ is the mean curvature of the rescaled hypersurfaces $\widetilde{\varphi}_s = \widetilde{\varphi}(\,\cdot\,,s)$.
As $|\widetilde{\mathrm{A}}| = \sqrt{2(T-t)}\,|\mathrm{A}| \leq C_0 < +\infty$, all the hypersurfaces $\widetilde{\varphi}_s$ have equibounded curvatures, moreover,

$$|\widetilde{\varphi}(p,s)| = \left|\frac{\varphi(p,t(s))-\widehat{p}}{\sqrt{2(T-t(s))}}\right| \leq \frac{C_0\sqrt{2n(T-t(s))}}{\sqrt{2(T-t(s))}} = C_0\sqrt{n}$$

which implies that at every time $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ the open ball of radius $C_0\sqrt{2n}$ centered at the origin of $\mathbb{R}^{n+1}$ intersects the hypersurface $\widetilde{\varphi}(\,\cdot\,,s)$. More precisely, the point $\widetilde{\varphi}(p,s)$ belongs to the interior of such ball.

Then, we rescale also the monotonicity formula in order to get information on these hypersurfaces. In the following $\widetilde{\mu}_s = \frac{\mu_t}{[2(T-t)]^{n/2}}$ will be the canonical measure associated to the rescaled hypersurface $\widetilde{\varphi}_s$ which, by means of equation (2.3.1), satisfies

$$\frac{d}{ds}\widetilde{\mu}_s = (n - \widetilde{\mathrm{H}}^2)\widetilde{\mu}_s\,,$$

as

$$\begin{aligned}\frac{\partial}{\partial s}\widetilde{\mu}_s &= \left(\frac{ds}{dt}\right)^{-1}\frac{\partial}{\partial t}\left(\frac{\mu_t}{[2(T-t)]^{n/2}}\right)\\ &= n\left(\frac{\mu_t}{[2(T-t)]^{n/2}}\right) + \frac{1}{[2(T-t)]^{n/2-1}}\frac{\partial}{\partial t}\mu_t\\ &= n\widetilde{\mu}_s - \frac{1}{[2(T-t)]^{n/2-1}}\mathrm{H}^2\mu_t\\ &= n\widetilde{\mu}_s - \widetilde{\mathrm{H}}^2\widetilde{\mu}_s\,.\end{aligned}$$

PROPOSITION 3.2.5 (Rescaled Monotonicity Formula). *We have*

$$\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s = -\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s \leq 0 \tag{3.2.5}$$

which integrated becomes

$$\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{s_1} - \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{s_2} = \int_{s_1}^{s_2}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,ds\,.$$

In particular,

$$\int_{-\frac{1}{2}\log T}^{+\infty}\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,ds \leq \int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{-\frac{1}{2}\log T} \leq C < +\infty\,,$$

for a uniform constant $C = C(\mathrm{Area}(\varphi_0), T)$ independent of $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ and $p \in M$.

PROOF. Keeping in mind that $y = \frac{x-\widehat{p}}{\sqrt{2(T-t)}}$ and $s = -\frac{1}{2}\log(T-t)$ we have,

$$\begin{aligned}
\frac{d}{ds}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s &= \left(\frac{ds}{dt}\right)^{-1}\frac{d}{dt}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s\\
&= 2(T-t)\frac{d}{dt}\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\,d\mu_t\\
&= -2(T-t)\int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4(T-t)}}}{[2(T-t)]^{n/2}}\left|\mathrm{H} + \frac{\langle x-\widehat{p}\,|\,\nu\rangle}{2(T-t)}\right|^2 d\mu_t\\
&= -2(T-t)\int_M e^{-\frac{|y|^2}{2}}\left|\frac{\widetilde{\mathrm{H}}}{\sqrt{2(T-t)}} + \frac{\langle y\,|\,\widetilde{\nu}\rangle}{\sqrt{2(T-t)}}\right|^2 d\widetilde{\mu}_s\\
&= -\int_M e^{-\frac{|y|^2}{2}}\left|\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 d\widetilde{\mu}_s\,.
\end{aligned}$$

The other two statements trivially follow. □

As a first consequence, we work out an upper estimate on the volume of the rescaled hypersurfaces in the balls of $\mathbb{R}^{n+1}$.
Fix a radius $R > 0$, if $B_R = B_R(0) \subset \mathbb{R}^{n+1}$, then we have

$$\begin{aligned}
\widetilde{\mathcal{H}}^n(\widetilde{\varphi}(M,s)\cap B_R) &= \int_M \chi_{B_R}(y)\,d\widetilde{\mu}_s \qquad (3.2.6)\\
&\leq \int_M \chi_{B_R}(y)e^{\frac{R^2-|y|^2}{2}}\,d\widetilde{\mu}_s\\
&\leq e^{R^2/2}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_s\\
&\leq e^{R^2/2}\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{-\frac{1}{2}\log T}\\
&\leq \widehat{C}e^{R^2/2}
\end{aligned}$$

where the constant $\widehat{C}$ is independent of R and s.

REMARK 3.2.6. As

$$\int_M e^{-\frac{|y|^2}{2}}\,d\widetilde{\mu}_{-\frac{1}{2}\log T} = \int_M \frac{e^{-\frac{|x-\widehat{p}|^2}{4T}}}{(2T)^{n/2}}\,d\mu_0 \leq \frac{\mathrm{Area}(\varphi_0)}{(2T)^{n/2}}\,,$$

we can choose the constant $\widehat{C}$ to be independent also of $p \in M$.

Another consequence is the following key technical lemma which is necessary in order to take the limits of integrals of functions on the sequences of rescaled hypersurfaces.

LEMMA 3.2.7 (Stone **[71]**). *The following estimates hold.*

(1) *There is a uniform constant $C = C(n, \mathrm{Area}(\varphi_0), T)$ such that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2}\log T, +\infty\right)$,*

$$\int_M e^{-|y|}\,d\widetilde{\mu}_s \leq C\,.$$

(2) *For any $\varepsilon > 0$ there is a uniform radius $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ such that, for any $p \in M$ and for all $s \in \left[-\frac{1}{2}\log T, +\infty\right)$,*

$$\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\,d\widetilde{\mathcal{H}}^n \leq \varepsilon\,,$$

that is, the family of measures $e^{-|y|^2/2}\,d\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}_s(M)$ is tight *(see* **[19]***).*

PROOF. By the rescaled monotonicity formula (3.2.5) we have that, for any $p \in M$ and for all $s \in \Big[-\frac{1}{2}\log T, +\infty\Big)$,

$$\int_M e^{-|y|^2/2}\, d\widetilde{\mu}_s \leq \int_M e^{-|y|^2/2}\, d\widetilde{\mu}_{-\frac{1}{2}\log T}\,.$$

According to Remark 3.2.6, the right hand integral may be estimated by a constant depending only on T and $\mathrm{Area}(\varphi_0)$, not on $p \in M$. Hence, we have the following estimates for all $p \in M$ and for all $s \in \Big[-\frac{1}{2}\log T, +\infty\Big)$,

$$\int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_1 \tag{3.2.7}$$

and

$$\int_{\widetilde{\varphi}_s(M)\cap B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_2 \tag{3.2.8}$$

where C_1 and C_2 are constants depending only on n, T and $\mathrm{Area}(\varphi_0)$.
Then, we compute for any p and s,

$$\begin{aligned}
\frac{d}{ds}\int_M e^{-|y|}\, d\widetilde{\mu}_s &= \int_M \Big\{n - \widetilde{\mathrm{H}}^2 - \frac{1}{|y|}\langle y \,|\, \widetilde{\mathrm{H}}\widetilde{\nu} + y\rangle\Big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&\leq \int_M \big\{n - \widetilde{\mathrm{H}}^2 - |y| + |\widetilde{\mathrm{H}}|\big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&< \int_M \big\{n + 1 - |y|\big\} e^{-|y|}\, d\widetilde{\mu}_s \\
&\leq (n+1)\left\{\int_{\widetilde{\varphi}_s(M)\cap B_{n+1}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n - \int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n\right\}.
\end{aligned}$$

But then, by inequality (3.2.7) we see that we must have either

$$\frac{d}{ds}\int_M e^{-|x|}\, d\widetilde{\mu}_s < 0\,,$$

or

$$\int_{\widetilde{\varphi}_s(M)\setminus B_{2n+2}(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C_1\,.$$

Hence, in view of inequality (3.2.8), it follows that either

$$\frac{d}{ds}\int_M e^{-|y|}\, d\widetilde{\mu}_s < 0\,,$$

or

$$\int_M e^{-|y|}\, d\widetilde{\mu}_s \leq C_1 + C_2\,,$$

which implies

$$\int_M e^{-|y|}\, d\widetilde{\mu}_s \leq \max\left\{(C_1 + C_2), \int_M e^{-|y|}\, d\widetilde{\mu}_{-\frac{1}{2}\log T}\right\} = C_3$$

for any p and s.
The proof of part (1) of the lemma follows by noticing that the integral quantity on the right hand side can clearly be estimated by a constant depending on T and $\mathrm{Area}(\varphi_0)$ but not on $p \in M$.

Let now again $p \in M$ and $s \in \Big[-\frac{1}{2}\log T, +\infty\Big)$ arbitrary. Now subdivide $\widetilde{\varphi}_s(M)$ into "annular pieces" $\{\widetilde{M}^k_s\}_{k=0}^{\infty}$ by setting

$$\widetilde{M}^0_s = \widetilde{\varphi}_s(M) \cap B_1(0)\,,$$

and for each $k \geq 1$,

$$\widetilde{M}^k_s = \big\{y \in \widetilde{\varphi}_s(M) \,|\, 2^{k-1} \leq |y| < 2^k\big\}\,.$$

Then, by part (1) of the lemma $\widetilde{\mathcal{H}}^n(\widetilde{M}^k_s) \leq C_3 e^{(2^k)}$ for each k, independently of the choice of p and s. Hence in turn, for each k we have

$$\int_{\widetilde{M}^k_s} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \leq C_3 e^{-\frac{1}{2}(2^{k-1})^2} e^{(2^k)} = C_3 e^{(2^k - 2^{2k-3})}$$

again independently of the choice of p and s.
For any $\varepsilon > 0$ we can find a $k_0 = k_0(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ such that

$$\sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon\,,$$

then, if $R = R(\varepsilon, n, \mathrm{Area}(\varphi_0), T)$ is simply taken to be equal to 2^{k_0-1}, we have

$$\int_{\widetilde{\varphi}_s(M)\setminus B_R(0)} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n = \sum_{k=k_0}^{\infty} \int_{\widetilde{M}^k_s} e^{-|y|^2/2}\, d\widetilde{\mathcal{H}}^n \leq \sum_{k=k_0}^{\infty} C_3 e^{(2^k - 2^{2k-3})} \leq \varepsilon$$

and we are done also with part (2) of the lemma. □

COROLLARY 3.2.8. *If a sequence of rescaled hypersurfaces $\widetilde{\varphi}_{s_i}$ locally smoothly converges (up to reparametrization) to some limit hypersurface $\widetilde{M}_\infty$, we have*

$$\int_{\widetilde{M}_\infty} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq C$$

and

$$\lim_{i\to\infty} \int_M e^{-\frac{|y|^2}{2}}\, d\widetilde{\mu}_{s_i} = \int_{\widetilde{M}_\infty} e^{-\frac{|y|^2}{2}}\, d\widetilde{\mathcal{H}}^n\,,$$

where the constant C is the same of the previous lemma.

PROOF. Actually, it is only sufficient to show that the measures $\widetilde{\mathcal{H}}^n \llcorner \widetilde{\varphi}(M, s_i)$ associated to the hypersurfaces weakly*–converge to the measure $\widetilde{\mathcal{H}}^n \llcorner \widetilde{M}_\infty$. Indeed, for every $R > 0$ we have,

$$\int_{\widetilde{M}_\infty \cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq \liminf_{i\to\infty} \int_{\widetilde{\varphi}(M,s_i)\cap B_R(0)} e^{-|y|}\, d\widetilde{\mathcal{H}}^n \leq \liminf_{i\to\infty} \int_M e^{-|y|}\, d\widetilde{\mu}_{s_i} \leq C$$

by the first part of the lemma above. Sending R to $+\infty$, the first inequality follows.
The second statement is an easy consequence of the estimates in the second part of the lemma. □

Now we want to estimate the covariant derivatives of the rescaled hypersurfaces.

PROPOSITION 3.2.9 (Huisken [**40**]). *For every $k \in \mathbb{N}$ there exists a constant C_k depending only on k, n, C_0 (the constant in formula* (3.2.1)*) and the initial hypersurface such that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$ for every $p \in M$ and $s \in \left[-\frac{1}{2}\log T, +\infty\right)$.*

PROOF. By Proposition 2.3.5 we have for the original flow,

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 = \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,,$$

hence, with a straightforward computation, noticing that $|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} = |\nabla^k \mathrm{A}|^2_g [2(T-t)]^{k+1}$ we get

$$\begin{aligned}\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \leq & -2(k+1)|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\ & + C(n,k) \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} |\widetilde{\nabla}^p \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^q \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^r \widetilde{\mathrm{A}}|_{\widetilde{g}} |\widetilde{\nabla}^k \widetilde{\mathrm{A}}|_{\widetilde{g}}\,.\end{aligned}$$

As $|\widetilde{\mathrm{A}}|_{\widetilde{g}}$ is bounded by the constant C_0, supposing by induction that for $i = 0, \dots, k-1$ we have uniform bounds on $|\widetilde{\nabla}^i \widetilde{\mathrm{A}}|_{\widetilde{g}}$ with constants $C_i = C_i(n, C_0, \varphi_0)$, we can conclude by means of Peter–Paul inequality

$$\frac{\partial}{\partial s}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} \leq \widetilde{\Delta}|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k \widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k$$

for some constants B_k and D_k depending only on n, k, C_0 and the initial hypersurface.
Then,

$$\begin{aligned}\frac{\partial}{\partial s}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \leq\, & \widetilde{\Delta}|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} - 2|\widetilde{\nabla}^{k+1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\ & + B_k\widetilde{\Delta}|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_kB_{k-1}|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\ & - 2B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k + B_kD_{k-1} \\ \leq\, & \widetilde{\Delta}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\ & + B_kB_{k-1}|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + D_k + B_kD_{k-1} \\ \leq\, & \widetilde{\Delta}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) - B_k|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} \\ & + B_kB_{k-1}C^2_{k-1} + D_k + B_kD_{k-1} \\ \leq\, & \widetilde{\Delta}(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \\ & - B_k(|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}} + B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}) \\ & + B_kB_{k-1}C^2_{k-1} + D_k + B_kD_{k-1} + B_k^2C^2_{k-1}\end{aligned}$$

where we used the inductive hypothesis $|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_{k-1}$.
By the maximum principle, the function $|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|^2_{\widetilde{g}}+B_k|\widetilde{\nabla}^{k-1}\widetilde{\mathrm{A}}|^2_{\widetilde{g}}$ is then uniformly bounded in space and time by a constant C_k^2 depending on n, k, C_0 and the initial hypersurface, hence $|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$.
By the inductive hypothesis, the thesis of the proposition follows. □

We are now ready to study the convergence of the rescaled hypersurfaces as $s \to +\infty$.

PROPOSITION 3.2.10. *For every point $p \in M$ and every sequence of times $s_i \to +\infty$ there exists a subsequence (not relabeled) of times such that the hypersurfaces $\widetilde{\varphi}_{s_i}$, rescaled around $\widehat{p}$, locally smoothly converge (up to reparametrization) to some nonempty, smooth, complete limit hypersurface $\widetilde{M}_\infty$ such that $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$ for every $y \in \widetilde{M}_\infty$.*
Any limit hypersurface satisfies $\widetilde{\mathcal{H}}^n(\widetilde{M}_\infty \cap B_R) \leq C_R$ for every ball of radius R in $\mathbb{R}^{n+1}$ and for every $k \in \mathbb{N}$ there are constants C_k such that $|\widetilde{\nabla}^k\widetilde{\mathrm{A}}|_{\widetilde{g}} \leq C_k$.
Moreover, if the initial hypersurface was embedded, $\widetilde{M}_\infty$ is embedded.

PROOF. We give a sketch of the proof, following Huisken [**40**].
By estimate (3.2.6) there is a uniform upper bound on $\mathcal{H}^n(\widetilde{\varphi}(M,s) \cap B_R)$ for each R, independent of s. Moreover, by the uniform control on the norm of the second fundamental form of the rescaled hypersurfaces in Proposition 3.2.9, there is a number $r_0 > 0$ such that, for each $s \in \left[-\frac{1}{2}\log T, +\infty\right)$ and each $q \in M$, if $U^s_{r_0,q}$ is the connected component of $\widetilde{\varphi}_s^{-1}(B_{r_0}(\widetilde{\varphi}_s(q)))$ in M containing q, then $\widetilde{\varphi}_s(U^s_{r_0,q})$ can be written as a graph of a smooth function f over a subset of the ball of radius r_0 in the tangent hyperplane to $\widetilde{\varphi}_s(M) \subset \mathbb{R}^{n+1}$ at the point $\widetilde{\varphi}_s(q)$.
The estimates of Proposition 3.2.9 then imply that all the derivatives of such function f up to the order $\alpha \in \mathbb{N}$ are bounded by constants C_α independent of s.
Following now the method in [**53**] we can see that, for each $R > 0$, a subsequence of the hypersurfaces $\widetilde{\varphi}(M,s) \cap B_R(0)$ must converge smoothly to a limit hypersurface in $B_R(0)$. Then, the existence of a smooth, complete limit hypersurface $\widetilde{M}_\infty$ follows from a diagonal argument, letting $R \to +\infty$. Recalling the fact that every rescaled hypersurface intersects the ball of radius $C_0\sqrt{2n}$ centered at the origin of $\mathbb{R}^{n+1}$, this limit cannot be empty. The estimates on the volume and derivatives of the curvature follow from the analogous properties for the converging sequence.
The fact that $\widetilde{M}_\infty$ satisfies $\widetilde{\mathrm{H}} + \langle y\,|\,\widetilde{\nu}\rangle = 0$ is a consequence of the rescaled monotonicity formula – we will see that in the next lecture – the same for the fact that if the initial hypersurface was embedded, $\widetilde{M}_\infty$ is embedded.

□