4.5. Embedded Closed Curves in the Plane

In the special case of the evolution of an embedded closed curve in the plane, it is possible to exclude at all the type II singularities. This, together with the case of convex, compact, hypersurfaces (as we have seen in the proof of Theorem 3.4.9) are the only known cases in which this can be done in general.

By Theorem 4.4.6 and embeddedness, any blow up limit must a unit multiplicity grim reaper. We apply now a very geometric argument by Huisken in **[71]** in order to exclude also such possibility (see also **[64]** for another similar quantity).

Given the smooth flow γ_t of an initial embedded closed curve in some interval $[0, T)$, we know that the curve stays embedded during the flow so we can see every γ_t as a subset of $\mathbb{R}^2$. At every time $t \in [0, T)$, for every pair of points p and q in γ_t we define $d_t(p, q)$ to be the *geodesic* distance in γ_t of p and q, $|p - q|$ the standard distance in $\mathbb{R}^2$ and L_t the length of γ_t.
We consider the function $\Phi_t : \gamma_t \times \gamma_t \to \mathbb{R}$ defined as

$$\Phi_t(p, q) = \begin{cases} \frac{\pi|p-q|}{L_t} / \sin \frac{\pi d_t(p,q)}{L_t} & \text{if } p \neq q, \\ 1 & \text{if } p = q\,, \end{cases}$$

which is a perturbation of the quotient between the extrinsic and the intrinsic distance of a pair of points on γ_t.
Since γ_t is smooth and embedded for every time, the function Φ_t is well defined and positive. Moreover, it is easy to check that even if d_t is not C^1 at the pairs of points such that $d_t(p, q) = L_t/2$, the function Φ_t is C^2 in the open set $\{p \neq q\} \subset \gamma_t \times \gamma_t$ and continuous on $\gamma_t \times \gamma_t$.
By compactness, the following minimum there exists,

$$E(t) = \min_{p,q \in \gamma_t} \Phi_t(p, q)\,.$$

We call this quantity *Huisken's embeddedness ratio*.
Since the evolution is smooth it is easy to see that the function $E : [0, T) \to \mathbb{R}$ is continuous.

REMARK 4.5.1. The quantity E can be defined also for nonembedded closed curves, but in such case $E = 0$, indeed its positivity is equivalent to embeddedness.

LEMMA 4.5.2 (Huisken **[71]**). *The function $E(t)$ is monotone increasing in every time interval where $E(t) < 1$.*

PROOF. We start differentiating in time $\Phi_t(p, q)$,

$$\begin{aligned}
\frac{d}{dt}\Phi_t(p,q) = &\, \frac{\pi}{L_t}\frac{\langle p-q\,|\,k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|}\Big/\sin\frac{\pi d_t(p,q)}{L_t} \\
&+ \left(\frac{\pi|p-q|}{L_t^2}\int_{\gamma_t} k^2\,ds\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t} \\
&- \frac{\pi^2|p-q|}{L_t^2}\cos\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t} k^2\,ds - \int_q^p k^2\,ds\right)\Big/\sin^2\frac{\pi d_t(p,q)}{L_t} \\
= &\left[\frac{\langle p-q\,|\,k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{1}{L_t}\int_{\gamma_t} k^2\,ds\right. \\
&\left. - \frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\left(\frac{d_t(p,q)}{L_t}\int_{\gamma_t} k^2\,ds - \int_q^p k^2\,ds\right)\right]\Phi_t(p,q) \\
= &\left[\frac{\langle p-q\,|\,k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{1}{L_t}\left(1 - \frac{\pi d_t(p,q)}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\right)\int_{\gamma_t} k^2\,ds\right. \\
&\left. + \frac{\pi}{L_t}\cot\frac{\pi d_t(p,q)}{L_t}\int_q^p k^2\,ds\right]\Phi_t(p,q)
\end{aligned}$$

where s is the arclength and k the curvature of γ_t. It is easy to see that being the function E the minimum of a family of uniformly locally Lipschitz functions, it is also locally Lipschitz, hence differentiable almost everywhere. Then, to prove the statement it is enough to show that

$\frac{dE(t)}{dt} > 0$ for every time t such that this derivative exists. We will do that as usual, by Hamilton's trick (Lemma 2.1.3).
Let (p, q) be a minimizing pair at a differentiability time t and suppose that $E(t) < 1$. By the very definition of Φ_t, it must be $p \neq q$.
We set $\alpha = \pi d_t(p, q)/L_t$ and notice that $\alpha \cot \alpha < 1$ as $\alpha \in (0, \pi/2]$. Moreover, $\int_{\gamma_t} k^2 \, ds \geq \left(\int_{\gamma_t} k \, ds\right)^2 / L_t \geq 4\pi^2/L_t$. Then, we have

$$\frac{d}{dt} E(t) \geq \left[\frac{\langle p - q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p - q|^2} + \frac{4\pi^2}{L_t^2}(1 - \alpha \cot \alpha) + \frac{\pi}{L_t} \cot \alpha \int_q^p k^2 \, ds\right] E(t)$$

that is,

$$\frac{d}{dt} \log E(t) \geq \frac{\langle p - q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p - q|^2} + \frac{4\pi^2}{L_t^2}(1 - \alpha \cot \alpha) + \frac{\pi}{L_t} \cot \alpha \int_q^p k^2 \, ds \,, \tag{4.5.1}$$

for any minimizing pair (p, q).
Assume that the curve is parametrized counterclockwise by arclength and that p and q are like in Figure 1.

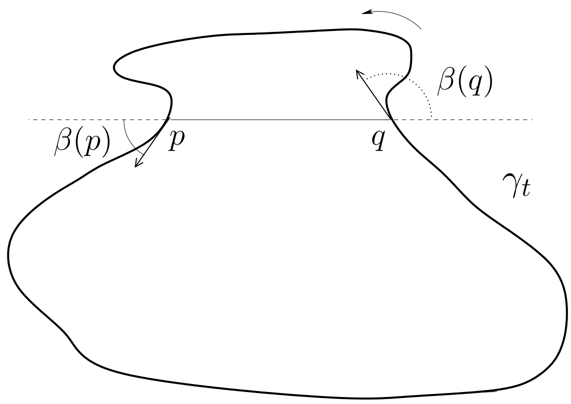

FIGURE 1.

We set $p(s) = \gamma_t(s_1 + s)$ with $p = \gamma_t(s_1)$, then, by minimality we have

$$0 = \frac{d}{ds} \Phi_t(p(s), q)\Big|_{s=0} = \frac{\pi}{L_t} \frac{\langle p - q \,|\, \tau(p)\rangle}{|p - q| \sin \frac{\pi d_t(p,q)}{L_t}} - \frac{\pi |p - q|}{L_t \sin^2 \frac{\pi d_t(p,q)}{L_t}} \cdot \frac{\pi \cos \frac{\pi d_t(p,q)}{L_t}}{L_t}$$

where we denoted by $\tau(p)$ the oriented unit tangent vector to γ_t at p.
By this equality we get

$$\cos \beta(p) = \frac{\langle p - q \,|\, \tau(p)\rangle}{|p - q|} = \frac{\pi |p - q|}{L_t \sin \frac{\pi d_t(p,q)}{L_t}} \cos \frac{\pi d_t(p, q)}{L_t} = E(t) \cos \alpha$$

where $\beta(p) \in [0, \pi/2]$ is the angle between the vectors $p - q$ and $\tau(p)$.
Repeating this argument for the point q we get

$$\cos \beta(q) = -E(t) \cos \alpha$$

where, as before, $\beta(q)$ is the angle between $q - p$ and $\tau(q)$, see Figure 1. Clearly, it follows that $\beta(p) + \beta(q) = \pi$.
Notice that if one of the intersections of the segment $[p, q]$ with the curve is tangential, we would have $E(t) \cos \alpha = 1$ which is impossible as we assumed that $E(t) < 1$. Moreover, by the relation $\cos \beta(p) = E(t) \cos \alpha < \cos \alpha$ it follows that $\beta(p) > \alpha$.

We look now at the second variation of Φ_t at the same minimizing pair of points (p, q). With the same notation, if $p = \gamma_t(s_1)$ and $q = \gamma_t(s_2)$ we set $p(s) = \gamma_t(s_1 + s)$ and $q(s) = \gamma_t(s_2 - s)$. After a straightforward computation, one gets

$$\begin{aligned}
0 \leq\ & \frac{d^2}{ds^2}\Phi_t(p(s), q(s))\Big|_{s=0} \\
=\ & \frac{\pi}{L_t}\left(\frac{\langle p-q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|} + \frac{4\pi^2|p-q|}{L_t^2}\right)\Big/\sin\frac{\pi d_t(p,q)}{L_t} \\
=\ & \left[\frac{\langle p-q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}\right]E(t)\,.
\end{aligned}$$

Hence, getting back to inequality (4.5.1) we have

$$\begin{aligned}
\frac{d}{dt}\log E(t) \geq\ & \frac{\langle p-q \,|\, k(p)\nu(p) - k(q)\nu(q)\rangle}{|p-q|^2} + \frac{4\pi^2}{L_t^2}(1 - \alpha\cot\alpha) + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds \\
\geq\ & -\frac{4\pi^2}{L_t^2}\alpha\cot\alpha + \frac{\pi}{L_t}\cot\alpha\int_q^p k^2\,ds \\
=\ & \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right),
\end{aligned}$$

so it remains to show that this last expression is positive. As

$$\int_p^q k^2\,ds \geq \left(\int_p^q k\,ds\right)^2 \Big/ d_t(p,q)$$

and noticing that $\int_p^q k\,ds$ is the angle between the tangent vectors $\tau(p)$ and $\tau(q)$ we have $\left(\int_p^q k\,ds\right)^2 = 4\beta(p)^2 < 4\alpha^2$, as we concluded before.
Thus,

$$\begin{aligned}
\frac{d}{dt}\log E(t) \geq\ & \frac{\pi\cot\alpha}{L_t}\left(\int_q^p k^2\,ds - \frac{4\pi}{L_t}\alpha\right) \\
>\ & \frac{\pi\cot\alpha}{L_t}\left(\frac{4\alpha^2}{d_t(p,q)} - \frac{4\pi}{L_t}\alpha\right) \\
=\ & 0
\end{aligned}$$

recalling that $\alpha = \pi d_t(p,q)/L_t$. □

REMARK 4.5.3. By its definition and this lemma, the function E is always nondecreasing. Actually, to be more precise, by means of a simple geometric argument it can be proved that if $E(t) = 1$ the curve γ_t must be a circle. Hence, in any other case E is strictly increasing in time.

An immediate consequence is that for every initial embedded, closed curve in $\mathbb{R}^2$, there exists a positive constant C depending on the initial curve such that on all $[0, T)$ we have $E(t) \geq C$. The same conclusion holds for any rescaling of such curves as the function E is scaling invariant by construction.

REMARK 4.5.4. This lemma also provide an alternative proof of the fact that an initial embedded, closed curve stays embedded. Indeed, it cannot develop a self–intersection during its curvature flow, otherwise E would get zero.

We can then exclude type II singularities in the curvature flow of embedded closed curves. Any blow up limit flow γ^∞ is given (up to rigid motions) by a grim reaper, that is, the translating graph Γ of the function $y = -\log\cos x$ in the interval $(-\pi/2, \pi/2)$. Assuming that $\gamma_0^\infty = \Gamma$, we consider the following four points $p_1 = (-x_1, -\log\cos x_1)$, $q_1 = (x_1, -\log\cos x_1)$, $p_2 = (-x_2, -\log\cos x_2)$ and $q_2 = (x_2, -\log\cos x_2)$ belonging to Γ, for $0 < x_1 < x_2 < \pi/2$ such that $-\log\cos x_2 > \pi/2 - 3\log\cos x_1$.

As the rescaled curves γ_0^k converge locally in C^1 to Γ, for any $\varepsilon > 0$ such that $x_2 + \varepsilon < \pi/2$ and k is large enough the curve γ_0^k will be C^1–close to Γ in the open rectangle $R_\varepsilon = (-x_2 - \varepsilon, x_2 + \varepsilon) \times (-\varepsilon, -\log\cos x_2 + \varepsilon)$, hence there will be a pair of points $(p, q) \in \gamma_0^k$ arbitrarily close to (p_1, q_1) and another pair $(\widetilde{p}, \widetilde{q}) \in \gamma_0^k$ arbitrarily close to (p_2, q_2). As $k \to \infty$, the geodesic distance $d_{\gamma_0^k}(p, q)$ on the closed curve γ_0^k between p and q is definitely given by the length of the part of the curve which is close to the vertex of Γ, indeed, this latter is smaller than $\pi - 2\log\cos x_1$, when k is large enough, instead the *other part* of the curve has a length which is at least the sum of the Euclidean distances $|\widetilde{p} - p| + |\widetilde{q} - q|$ which is definitely larger than $2(\log\cos x_1 - \log\cos x_2)$ and this last quantity is larger than $\pi - 4\log\cos x_1$, by construction.

Hence, when k is large enough, the Huisken's embeddedness ratio for the rescaled curve γ_0^k is not larger than

$$\begin{aligned}
\frac{\pi|p-q|}{L} \Big/ \sin\frac{\pi d_{\gamma_0^k}(p,q)}{L} &\leq \frac{\pi(\pi+2\varepsilon)}{L} \Big/ \sin\frac{\pi d_{\gamma_0^k}(p,q)}{L} \\
&\leq \frac{\pi(\pi+2\varepsilon)}{L} \Big/ \frac{2d_{\gamma_0^k}(p,q)}{L} \\
&= \frac{\pi(\pi+2\varepsilon)}{2d_{\gamma_0^k}(p,q)} \\
&\leq \frac{\pi^2}{d_{\gamma_0^k}(p,q)}\,,
\end{aligned}$$

where L is the total length of the curve γ_0^k and we used the inequality $\sin x \geq 2x/\pi$ holding for every $x \in [0, \pi/2]$.
Finally, again by the C^1–convergence of γ_0^k to Γ in R_ε, we can also assume that $d_{\gamma_0^k}(p, q)$ is larger than $-\log\cos x_1$.

Now we consider a sequence of pairs $x_1^i < x_2^i$ as above such that $x_1^i \to \pi/2$, then we have a sequence of rescaled curves $\gamma_0^{k_i}$ such that the associated Huisken's embeddedness ratio tends to zero, as $d_{\gamma_0^{k_i}}(p, q) \to +\infty$ when $i \to \infty$.
This is in contradiction with the fact that the function E is scaling invariant and uniformly bounded from below by some positive constant C for all the curves of the flow.
As this argument does not change if we apply to Γ any rigid motion, in presence of a type II singularity in the embedded case, we would have a contradiction with the conclusion of Theorem 4.4.6.

THEOREM 4.5.5. *Type II singularities cannot develop during the curvature flow of an embedded, closed curve in $\mathbb{R}^2$.*

Collecting together Theorem 3.5.1 about type I singularities and this last proposition, we obtain Theorem 3.4.8 by Gage and Hamilton and the following theorem due to Grayson **[52]**, whose original proof is more geometric and direct, showing that the intervals of negative curvature vanish in finite time before any singularity. We underline that the success of the line of proof we followed is due to the bound from below on Huisken's embeddedness ratio implied by Lemma 4.5.2.
Modifying a little such quantity, Andrews and Bryan **[12]** were even able to give a simple and direct proof without passing through the classification of singularities.

THEOREM 4.5.6 (Grayson's Theorem). *Let γ_t be the curvature flow of a closed, embedded, smooth curve in the plane, in the maximal interval of smooth existence $[0, T)$.*
Then, there exists a time $\tau < T$ such that γ_τ is convex.
As a consequence, the result of Gage and Hamilton 3.4.8 applies and subsequently the curve shrinks smoothly to a point as $t \to T$.

REMARK 4.5.7. This result, extended by Grayson to curves moving inside general surfaces, allowed him to have a proof of the *three geodesics theorem* on the sphere **[54]** (first outlined by Lusternik and Schnirelman in **[91]**).

We add a final remark in this case of embedded closed curves.
Letting $A(t)$ be the area enclosed by γ_t which moves by curvature, we have

$$\frac{d}{dt}A(t) = -\int_{\gamma_t} k\,ds = -2\pi\,,$$

hence, as the evolution is smooth till the curve shrinks to a point at time $T > 0$ and clearly $A(t)$ goes to zero, we have $A(0) = 2\pi T$. That is, the maximal time of existence is exactly equal to the initially enclosed area divided by 2π.

4.5.1. An Alternative Proof of Grayson's Theorem. Ideas and techniques are related to the unpublished work of Ilmanen [**80**].

In the special case of embedded curves in the plane, one can avoid the use of Hamilton's Harnack inequality in order to deal with type II singularities. By means of Huisken's monotonicity formula we can produce a homothetic blow up limit also in the type II case.

As underlined in Remark 3.2.23, White's Theorem 3.2.22 holds in general, without assuming any blow up rate on the curvature, hence at a singularity time $T > 0$ we have that $\Sigma > 1$ (recall Definition 3.2.3). Moreover, the estimates in Lemma 3.2.7 are also independent of the type I hypothesis.
Then, rescaling the curves around the moving points x_t like in Remark 3.3.9, we have

$$\sigma(0) - \Sigma = \int_{-\frac{1}{2}\log T}^{+\infty}\int_{\gamma_r} e^{-\frac{|y|^2}{2}}\,\left|\widetilde{k} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,ds\,dr < +\infty\,.$$

Clearly, since we are not assuming the type I hypothesis, the curvatures $\widetilde{k}$ of the rescaled curves $\widetilde{\gamma}_r$ are not bounded, but by this formula it follows that for every family of disjoint intervals $(a_i, b_i) \subset [-\frac{1}{2}\log T, +\infty)$ such that $\sum_{i\in\mathbb{N}}(b_i - a_i) = +\infty$ we can find a sequence $r_i \in (a_i, b_i)$ such that $r_i \nearrow +\infty$,

$$\lim_{i\to\infty}\frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}}\,\left|\widetilde{k} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2\,ds = 0$$

and

$$\lim_{i\to\infty}\frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}}\,ds = \lim_{i\to\infty}\sigma(t(r_i)) = \Sigma\,. \tag{4.5.2}$$

By the estimate (3.2.7) on the local length, it follows that the sequence of curves $\widetilde{\gamma}_{r_i}$ has curvatures *locally* equibounded in L^2. Hence, we can extract a subsequence which converges in C^1_{loc} to a limit curve $\widetilde{\gamma}_\infty$. Such limit curve satisfies $\widetilde{k} + \langle x\,|\,\widetilde{\nu}\rangle = 0$, as the integral $\int_{\widetilde{\gamma}} e^{-\frac{|y|^2}{2}}\left|\widetilde{k} + \langle y\,|\,\widetilde{\nu}\rangle\right|^2 ds$ is lower semicontinuous under C^1_{loc}–convergence. Moreover, by a bootstrap argument $\widetilde{\gamma}_\infty$ is smooth.
By the energy argument in the proof of Proposition 3.4.1 and the length estimate in Lemma 3.2.7, this limit curve is either a line (with possible integer multiplicity) or it is bounded, hence closed and the convergence is actually in C^1. As the initial curve was embedded, the Huisken's embeddedness ratio E is uniformly bounded from below on the sequence of rescaled curves, this implies that also $\widetilde{\gamma}_\infty$ is embedded. Indeed, if it has self–intersections or multiplicities the quantity E must approach zero, in the case of a closed limit curve because of the C^1–convergence, in the case that the limit curve is a line by means of the same argument used to exclude the grim reaper in the proof of Theorem 4.5.5.
Hence, by the classification theorem 3.4.1 we conclude that there are only two possibilities for $\widetilde{\gamma}_\infty$, either a line through the origin of $\mathbb{R}^2$ or the unit circle, both with unit multiplicity.
Since the second point of Lemma 3.2.7 implies that

$$\lim_{i\to\infty}\frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_{r_i}} e^{-\frac{|y|^2}{2}}\,ds = \frac{1}{\sqrt{2\pi}}\int_{\widetilde{\gamma}_\infty} e^{-\frac{|y|^2}{2}}\,ds\,,$$

and the first limit is equal to $\Sigma > 1$ by equation (4.5.2), we conclude that $\widetilde{\gamma}_\infty$ is the unit circle. Moreover, the curvatures of the converging sequence of curves are equibounded in L^2 (not only locally).

Fixing $i \in \mathbb{N}$ and letting $\rho = r - r_i < 1$, as $r = -\frac{1}{2}\log 2(T-t)$, recalling the formulas in Remark 2.3.2 we compute the evolution of the following quantity,

$$\begin{aligned}
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds =& 2(T-t)\frac{d}{dt}\int_{\gamma_t}\sqrt{2(T-t)}\,k^2\,ds+\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\\
&+2(T-t)\rho\frac{d}{dt}\int_{\gamma_t}(\sqrt{2(T-t)})^3\,k_s^2\,ds\\
=&-\sqrt{2(T-t)}\int_{\gamma_t}k^2\,ds+(\sqrt{2(T-t)})^3\int_{\gamma_t}(2kk_{ss}+k^4)\,ds\\
&+\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds-3\big(\sqrt{2(T-t)}\big)^3\rho\int_{\gamma_t}k_s^2\,ds\\
&+\big(\sqrt{2(T-t)}\big)^5\rho\int_{\gamma_t}(2k_sk_{sss}+7k^2k_s^2)\,ds\\
=&\int_{\widetilde{\gamma}_r}\Big[-\widetilde{k}^2+2\widetilde{k}\widetilde{k}_{ss}+\widetilde{k}^4+\widetilde{k}_s^2-3\rho\widetilde{k}_s^2+2\rho\widetilde{k}_s\widetilde{k}_{sss}+7\rho\widetilde{k}^2\widetilde{k}_s^2\Big]\,ds\,,
\end{aligned}$$

where we used the formula $\partial_t k_s = \partial_s\partial_t k + k^2 k_s = k_{sss} + 4k^2 k_s$.
By integration by parts and Peter–Paul inequality, we have

$$\int_{\widetilde{\gamma}_r}\widetilde{k}^2\widetilde{k}_s^2\,ds=\frac{1}{3}\int_{\widetilde{\gamma}_r}\partial_s(\widetilde{k}^3)\widetilde{k}_s\,ds=-\frac{1}{3}\int_{\widetilde{\gamma}_r}\widetilde{k}^3\widetilde{k}_{ss}\,ds\leq\frac{1}{6}\int_{\widetilde{\gamma}_r}\widetilde{k}^6+\widetilde{k}_{ss}^2\,ds$$

and

$$\begin{aligned}
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\leq&\int_{\widetilde{\gamma}_r}\Big[-\widetilde{k}_s^2+\widetilde{k}^4-\widetilde{k}^2-3\rho\widetilde{k}_s^2-2\rho\widetilde{k}_{ss}^2+7\rho(\widetilde{k}^6+\widetilde{k}_{ss}^2)/6\Big]\,ds\\
\leq&\int_{\widetilde{\gamma}_r}(-\widetilde{k}_s^2+\widetilde{k}^4+3\rho\widetilde{k}^6)\,ds\,.
\end{aligned}$$

Now, the following interpolation inequalities for any closed curve in the plane of length L (see Aubin [**20**, page 93])

$$\|\widetilde{k}\|_{L^4}^4\leq C\|\widetilde{k}_s\|_{L^2}\|\widetilde{k}\|_{L^2}^3+\frac{C}{L}\|\widetilde{k}\|_{L^2}^4\quad\text{ and }\quad\|\widetilde{k}\|_{L^6}^6\leq C\|\widetilde{k}_s\|_{L^2}^2\|\widetilde{k}\|_{L^2}^4+\frac{C}{L^2}\|\widetilde{k}\|_{L^2}^6$$

imply, by means of Young inequality,

$$\int_{\widetilde{\gamma}_r}\widetilde{k}^4\,ds\leq\frac{1}{2}\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds+\frac{1}{2}\Big(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\Big)^3+\Big(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\Big)^3+\frac{C}{L^3(\widetilde{\gamma}_r)}$$

and

$$3\rho\int_{\widetilde{\gamma}_r}\widetilde{k}^6\,ds\leq\Big(\rho\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\Big)^3+2\Big(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\Big)^3+\frac{C}{L^2(\widetilde{\gamma}_r)}\Big(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\Big)^3.$$

Hence, as we know that $L(\widetilde{\gamma}_r)\geq\int_{\widetilde{\gamma}_r}e^{-\frac{|y|^2}{2}}\,ds\geq\sqrt{2\pi}$ and $\rho<1$, we conclude

$$\begin{aligned}
\frac{d}{dr}\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\leq&\int_{\widetilde{\gamma}_r}(-\widetilde{k}_s^2+\widetilde{k}_s^2/2)\,ds+C\Big(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\Big)^3+C\\
&+\Big(\rho\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\Big)^3+C\Big(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\Big)^3\\
\leq&C\Big(\int_{\widetilde{\gamma}_r}\widetilde{k}^2\,ds\Big)^3+\Big(\rho\int_{\widetilde{\gamma}_r}\widetilde{k}_s^2\,ds\Big)^3+C\\
\leq&C\Big(\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+\rho\widetilde{k}_s^2)\,ds\Big)^3+C\,,
\end{aligned}$$

for a constant C independent of $r\geq r_i$ and $i\in\mathbb{N}$.
Integrating this differential inequality for the quantity $Q_i(r)=\int_{\widetilde{\gamma}_r}(\widetilde{k}^2+(r-r_i)\widetilde{k}_s^2)\,ds$ in the interval $[r_i, r_i+2\delta]$ it is easy to see that if $\delta>0$ is small enough, we have $Q_i(r)\leq C(\delta, Q_i(r_i))=$

$C\left(\delta, \int_{\widetilde{\gamma}_{r_i}} \widetilde{k}^2\,ds\right) = C(\delta)$, for every $r \in [r_i, r_i + 2\delta]$, as the curves $\widetilde{\gamma}_{r_i}$ have uniformly bounded curvature in L^2. Hence, if $r \in [r_i + \delta, r_i + 2\delta]$ we have the estimates

$$\int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + \delta\widetilde{k}_s^2)\,ds \leq \int_{\widetilde{\gamma}_r} (\widetilde{k}^2 + (r - r_i)\widetilde{k}_s^2)\,ds \leq C(\delta)$$

which imply

$$\int_{\widetilde{\gamma}_r} \widetilde{k}_s^2\,ds \leq \frac{C(\delta)}{\delta}\,,$$

for every $r \in [r_i + \delta, r_i + 2\delta]$ and a constant $C(\delta)$ independent of $i \in \mathbb{N}$.
We can now find as before a sequence of values $q_i \in [r_i + \delta, r_i + 2\delta]$ such that

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}} \left|\widetilde{k} + \langle y \,|\, \widetilde{\nu}\rangle\right|^2 ds = 0\,.$$

and

$$\lim_{i\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\gamma}_{q_i}} e^{-\frac{|y|^2}{2}}\,ds = \lim_{i\to\infty} \sigma(t(q_i)) = \Sigma > 1\,.$$

As this new sequence of rescaled curves $\widetilde{\gamma}_{q_i}$ also satisfies the length estimate (3.2.7) and has $\widetilde{k}$ and $\widetilde{k}_s$ uniformly bounded in L^2, we can extract a subsequence (not relabeled) that converges in C^2 to a limit curve which is again the unit circle.
Then, definitely the curves $\widetilde{\gamma}_{q_i}$ have positive curvature, hence they are convex. This means that the same holds for γ_t at some time $t \in [0, T)$, which is Grayson's result.

REMARK 4.5.8. Pushing a little forward this analysis, one can actually prove along the same lines also the C^∞–convergence of the *full* sequence of the rescaled curves to the unit circle, as proved by Gage and Hamilton in **[47, 48, 49]**.

REMARK 4.5.9. Actually, the C^1_{loc}–convergence to a line in the case $\Sigma = 1$ allows the possibility to avoid the application of White's theorem. Indeed, the boundedness of the curvature around every $x_0 \in \mathcal{S}$ then follows also by the interior estimates of Ecker and Huisken.

We remark that the interesting point of this line in proving Grayson's theorem is the fact that we did not distinguish between *type I* and *type II* singularities. Indeed, the curvature of the rescaled curves could be unbounded, but the control in L^2_{loc} implies the C^1_{loc}–convergence which is sufficient to obtain the smoothness of the limit curve. In higher dimension the uniform control of the mean curvature in L^2_{loc} is not strong enough to give the C^1_{loc}–convergence of a subsequence of rescaled hypersurfaces, hence, this "unitary" line of analysis is difficult to be pursued in order to get smooth homothetic blow up limits also for type II singularities.
It is anyway possible to produce a "homothetic" blow up limit introducing weak definitions of hypersurfaces (varifolds, currents, see **[80]**), the difficulty is then to show the regularity and the embeddedness of such limit.
Some very interesting unpublished results in this direction were obtained by Ilmanen in dimension two **[80]** (which is, in some sense, the critical case), in particular, assuming the embeddedness and the mean convexity of the surfaces, it can be shown that the convergence and the blow up limits are smooth.

All this discussion underlines the variational nature of the arguments (in particular, the monotonicity formula) in the analysis of type I singularities, against the nonvariational point of view (substantially based on the maximum principle) in dealing with type II ones.

4.6. An Example of Singularity Analysis

We give an example how the results of this and previous chapter can be used to fully understand the singularity formation in some cases (following a suggestion of Or Hershkovits).
We consider a torus of rotation in $\mathbb{R}^3$ such that $\mathrm{H} > 0$, obtained rotating around the z axis a small circle in the xz plane with center on the x axis quite far from the origin. One clearly expects that the torus collapses on a circle.