Mathematical Methods 2017/01/13

Solve the following exercises in a fully detailed way, explaining and justifying any step.

(1) (5 points) Compute $\displaystyle\int_{\mathbb{R}} \frac{x^2}{x^4+16}$.

(2) (6 points) Define $f_n(x) = \dfrac{(2-x)x^n}{1+2^n}$. Discuss the convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ on sub-intervals of $\mathbb{R}$.

(3) (5 points) Let V be a real vector space and endowed with an inner product $\langle \cdot, \cdot \rangle$. Define the norm of a vector and show that it induces a distance on V. In the specific case when $V = C^\infty(0, 2\pi)$ and $\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)dt$, compute the norm of $\sin(x)$ and the distance between $\sin(x)$ and 1.

SOLUTIONS

(1) The function $f(x) = \dfrac{x^2}{x^4+16}$ is continuous and positive, hence measurable and summable on bounded intervals. Moreover, for big x, f is bounded by a constant times $1/x^2$. Since $1/x^2$ is summable so is f and by monotone convergence

$$\int_{\mathbb{R}} f(x) = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$$

To compute that integral we use the residue method. Let C_R be the upper semicircle of radius R centered at zero and counterclockwise oriented. C_R can be parametrized by Re^{it} with $t \in [0, \pi]$. Let $I_R = [-R, R]$ and γ_R be the concatenation of I_R and C_R.

As a function of a complex variable, f is holomorphic except at four simple poles: the zeroes of $x^4 + 16$, which are $\pm\sqrt{2}(1 \pm i)$. Let $z_0 = \sqrt{2} + \sqrt{2}i, z_1 = -\sqrt{2} + \sqrt{2}i, z_2 = -\sqrt{2} - \sqrt{2}i, z_3 = \sqrt{2} - \sqrt{2}i$.

The index of γ_R at z_0, z_1 is 1 and that at z_2, z_3 is zero, because γ_R is counterclockwise oriented, z_0, z_1 are inside the region bounded by γ_R while z_2, z_3 lies outside.

The residue of f at z_i is

$$\lim_{z\to z_i} \frac{(z - z_i)z^2}{z^4 + 16} = \lim_{z\to z_i} \frac{z^2}{4z^3} = \frac{1}{4z_i}$$

(by de l'hopital's rule or, if you prefers, because $z^4 + 16 = z^4 - z_i^4 = (z - z_i)(z^3 + z^2 z_i + z z_i^2 + z_i^3)$). So by residue theorem we have

$$\int_{\gamma_R} f(z)dz = 2\pi i\left(\frac{1}{4z_1} + \frac{1}{4z_0}\right) = \frac{1}{2}\pi i\left(\frac{\bar{z}_0}{|z_0|^2} + \frac{\bar{z}_1}{|z_1|^2}\right) = \frac{1}{2}\pi i\left(\frac{\sqrt{2} - \sqrt{2}i - \sqrt{2} - \sqrt{2}i}{4}\right) = \frac{\pi}{2\sqrt{2}}$$

To conclude we have to check that the integral of f over γ_R is the requested integral:

$$\int_{\mathbb{R}} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \int_{\gamma_R} f(z)dz - \lim_{R\to\infty} \int_{C_R} f(x)dx$$

and $|\int_{C_R} f(x)dx| \leq \int_{C_R} |f(x)| \leq \int_{C_R} \frac{1}{R^2} = \pi/R \to 0$ as $R \to \infty$.

(2) For $|x| > 2$ we have $|f_n(x)| = |(x-2)|\left(\frac{|x|}{2}\right)^n \frac{1}{1+1/2^n} \to \infty$ because $|x/2| > 1$. Therefore if $x > 2$ then $f_n(x)$ goes to infinite and for $x < -2$ f_n oscillates: $f_{2k}(x) \to \infty$ and $f_{2k+1}(x) \to -\infty$. In any case, for $|x| > 2$ there is no pointwise convergence to a real-valued function.

If $|x| < 2$ then $|f_n(x)| = |(x-2)|\left(\frac{|x|}{2}\right)^n \frac{1}{1+1/2^n} \to 0$ because $|x/2| < 1$. Therefore in $(-2,2)$ the sequence pointwise converges to zero.

If $x = 2$ then $f_n(x) = 0$ for any n and so the limit is 0. If $x = -2$ then the sequence $f_n(-2) = 4(-1)^n \frac{2^n}{1+2^n}$ oscillates between -4 and 4. So $f_n(-2)$ does not converge pointwise.

In conclusion, the sequence pointwise converges to zero on $(-2,2]$ and has no real limit elsewhere.

Let's check the uniform convergence. We have to compute $||f_n - 0||$ by using the L^∞-norm (i.e. the sup-norm). Since f is smooth, we can search for its extremal points by finding the zeroes of its derivative.

$$f_n' = \frac{2nx^{n-1} - (n+1)x^n}{1+2^n} = \frac{x^{n-1}}{1+2^n}(2n - x(n+1))$$

which vanishes at $x = 0$ and $x = \frac{2n}{n+1}$. Note that $\frac{2n}{n+1} \to 2$ as $n \to \infty$. Therefore the extremal values of f_n in $(-2,2]$ are the max and the min of $\{f_n(-2), f_n(2), f_n(0), f_n(\frac{2n}{n+1})\}$ (do not forget the extremes of the interval!!!). Since $|f_n(-2)| \to 4 \neq 0$ we have no uniform convergence on $(-2,2]$.

On the other hand, in any other interval $[a,b]$ contained in $(-2,2]$ (that is to say $a > -2, b \leq 2$) the extremal values of f are the sup and the min of

$$\{f_n(a), f_n(b), f_n(0), f_n(\frac{2n}{n+1})\}$$

Since $f_n(a), f_n(b), f_n(0) \to 0$ (because we have pointwise convergence to zero in $(-2,2]$) we have only to check the value

$$f_n(\frac{2n}{n+1}) = \frac{(2-\frac{2n}{n+1})(\frac{2n}{n+1})^n}{1+2^n} = ((2-\frac{2n}{n+1})(\frac{n}{n+1})^n)\frac{2^n}{1+2^n} = \frac{1}{n+1}\left(\frac{n}{1+n}\right)^n \frac{2^n}{1+2^n}$$

which is bounded by $\frac{1}{n+1}$ which goes to zero.

Therefore, for any $[a,b] \subset (-2,2]$ the sequence f_n uniformly converges to zero.

As for the L^p convergence, note that since f_n is bounded on $(-2,2]$, the uniform convergence to zero on any $[a,b] \subset (-2,2]$ implies the L^p convergence to zero in $(-2,2]$.

(3) The norm of a vector v is defined as $||v|| = \sqrt{\langle v, v\rangle}$. The induced distance on V is given by $d(v, w) = ||v - w||$. We check now that $||v - w||$ is a distance.

(1) (Positiveness) $d(v, w)$ is positive by definition and $d(v, w) = 0$ only if $||v - w|| = 0$, which is the case only if $v - w = 0$, that is to say if $v = w$.
(2) (Symmetry) $||v|| = || - v||$ for any v, so $d(v, w) = ||v - w|| = ||w - v|| = d(w, v)$.
(3) (Triangular inequality) For any $u, v, w \in V$ we have $d(v, w) = ||v - w|| = ||v - u + u - w|| = \sqrt{\langle v - u + u - w, v - u + u - w\rangle} = \sqrt{\langle v - u, v - u\rangle + 2\langle v - u, u - w\rangle + \langle u - w, u - w\rangle} = \sqrt{||v - u||^2 + 2\langle v - u, u - w\rangle - ||u - w||^2} \leq \sqrt{||v - u||^2 + 2||v - u||||u - w|| + ||u - w||^2} = \sqrt{(||v - u|| + ||u - w||)^2} = ||v - u|| + ||u - w|| = d(v, u) + d(u, w)|$ where the inequality follows from the Cauchy-Schwarz inequality.

The norm of $\sin x$ is $\sqrt{\int_0^{2\pi} \sin^2(x)} = \sqrt{\pi}$. The distance from $\sin x$ and 1 is $|| \sin(x) - 1|| = \sqrt{\int_0^{2\pi} (\sin x - 1)^2} = \sqrt{\int \sin^2 - 2 \sin x + 1} = \sqrt{\pi + 2\pi - 2 \int_0^{2\pi} \sin x} = \sqrt{3\pi}$.