Mathematical Methods 2017/01/13

Solve the following exercises in a fully detailed way, explaining and justifying any step.

(1) (5 points) Compute \( \int_{\mathbb{R}} \frac{x^2}{x^4 + 16} \).

(2) (6 points) Define \( f_n(x) = \frac{(2 - x)x^n}{1 + 2^n} \). Discuss the convergence of the sequence \( \{f_n\}_{n \in \mathbb{N}} \) on sub-intervals of \( \mathbb{R} \).

(3) (5 points) Let \( V \) be a real vector space and endowed with an inner product \( \langle \cdot, \cdot \rangle \). Define the norm of a vector and show that it induces a distance on \( V \). In the specific case when \( V = C^\infty(0, 2\pi) \) and \( \langle f, g \rangle = \int_0^{2\pi} f(t)g(t)dt \), compute the norm of \( \sin(x) \) and the distance between \( \sin(x) \) and 1.

SOLUTIONS

(1) The function \( f(x) = \frac{x^2}{x^4 + 16} \) is continuous and positive, hence measurable and summable on bounded intervals. Moreover, for big \( x \), \( f \) is bounded by a constant times \( 1/x^2 \). Since \( 1/x^2 \) is summable so is \( f \) and by monotone convergence

\[
\int_{\mathbb{R}} f(x) = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx.
\]

To compute that integral we use the residue method. Let \( C_R \) be the upper semicircle of radius \( R \) centered at zero and counterclockwise oriented. \( C_R \) can be parametrized by \( Re^{it} \) with \( t \in [0, \pi] \). Let \( I_R = [-R, R] \) and \( \gamma_R \) be the concatenation of \( I_R \) and \( C_R \).

As a function of a complex variable, \( f \) is holomorphic except at four simple poles: the zeroes of \( x^4 + 16 \), which are \( \pm \sqrt{2}(1 \pm i) \). Let \( z_0 = \sqrt{2} + \sqrt{2}i, z_1 = -\sqrt{2} + \sqrt{2}i, z_2 = -\sqrt{2} - \sqrt{2}i, z_3 = \sqrt{2} - \sqrt{2}i \).

The index of \( \gamma_R \) at \( z_0, z_1 \) is 1 and that at \( z_2, z_3 \) is zero, because \( \gamma_R \) is counterclockwise oriented, \( z_0, z_1 \) are inside the region bounded by \( \gamma_R \) while \( z_2, z_3 \) lies outside.

The residue of \( f \) at \( z_i \) is

\[
\lim_{z \to z_i} \frac{(z - z_i)z^2}{z^4 + 16} = \lim_{z \to z_i} \frac{z^2}{4z^3} = \frac{1}{4z_i}
\]

(by de l’hospital’s rule or, if you prefers, because \( z^4 + 16 = z^4 - z_i^4 = (z - z_i)(z^3 + z_i z + z_i^2 + z_i^3) \)). So by residue theorem we have

\[
\int_{\gamma_R} f(z)dz = 2\pi i \left( \frac{1}{4z_1} + \frac{1}{4z_0} \right) = \frac{1}{2} \pi i (\frac{\bar{z}_0}{|z_0|^2} + \frac{\bar{z}_1}{|z_1|^2}) = \frac{1}{2} \pi i (\frac{\sqrt{2} - \sqrt{2}i - \sqrt{2} - \sqrt{2}i}{4}) = \frac{\pi}{2\sqrt{2}}
\]

To conclude we have to check that the integral of \( f \) over \( \gamma_R \) is the requested integral:

\[
\int_{\mathbb{R}} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx = \int_{\gamma_R} f(z)dz - \lim_{R \to \infty} \int_{C_R} f(x)dx
\]

and \( |\int_{C_R} f(x)dx| \leq \int_{C_R} |f(x)| \leq \int_{C_R} \frac{1}{R^2} = \pi/R \to 0 \) as \( R \to \infty \).
(2) For $|x| > 2$ we have $|f_n(x)| = |(x - 2)| \left(\frac{|x|}{2}\right)^n \frac{1}{1 + 1/2^n} \to \infty$ because $|x/2| > 1$. Therefore if $x > 2$ then $f_n(x)$ goes to infinite and for $x < -2$ $f_n$ oscillates: $f_{2k}(x) \to \infty$ and $f_{2k+1}(x) \to -\infty$. In any case, for $|x| > 2$ there is no pointwise convergence to a real-valued function.

If $|x| < 2$ then $|f_n(x)| = |(x - 2)| \left(\frac{|x|}{2}\right)^n \frac{1}{1 + 1/2^n} \to 0$ because $|x/2| < 1$. Therefore in $(-2, 2)$ the sequence pointwise converges to zero.

Let’s check the uniform convergence. We have to compute $||f_n - 0||$ by using the $L^\infty$-norm (i.e. the sup-norm). Since $f$ is smooth, we can search for its extremal points by finding the zeroes of its derivative.

$$f'_n = \frac{2nx^{n-1} - (n+1)x^n}{1 + 2^n} = \frac{x^{n-1}}{1 + 2^n} (2n - x(n+1))$$

which vanishes at $x = 0$ and $x = \frac{2n}{n+1}$. Note that $\frac{2n}{n+1} \to 2$ as $n \to \infty$. Therefore the extremal values of $f_n$ in $(-2, 2]$ are the max and the min of $\{f_n(-2), f_n(2), f_n(0), f_n(\frac{2n}{n+1})\}$ (do not forget the extremes of the interval!!!). Since $|f_n(-2)| \to 4 \neq 0$ we have no uniform convergence on $(-2, 2]$.

On the other hand, in any other interval $[a, b]$ contained in $(-2, 2]$ (that is to say $a > -2, b \leq 2$) the extremal values of $f$ are the sup and the min of

$$\{f_n(a), f_n(b), f_n(0), f_n(\frac{2n}{n+1})\}$$

Since $f_n(a), f_n(b), f_n(0) \to 0$ (because we have pointwise convergence to zero in $(-2, 2]$) we have only to check the value

$$f_n(\frac{2n}{n+1}) = \left(\frac{2 - \frac{2n}{n+1}}{1 + 2^n}\right)^n = \left(\frac{2n}{n+1}\right)^n \frac{2n}{1 + 2^n}$$

which is bounded by $\frac{1}{n+1}$ which goes to zero.

Therefore, for any $[a, b] \subset (-2, 2]$ the sequence $f_n$ uniformly converges to zero.

As for the $L^p$ convergence, note that since $f_n$ is bounded on $(-2, 2]$, the uniform convergence to zero on any $[a, b] \subset (-2, 2]$ implies the $L^p$ convergence to zero in $(-2, 2]$. 
(3) The norm of a vector \( v \) is defined as \( ||v|| = \sqrt{\langle v, v \rangle} \). The induced distance on \( V \) is given by \( d(v, w) = ||v - w|| \). We check now that \( ||v - w|| \) is a distance.

(1) (Positiveness) \( d(v, w) \) is positive by definition and \( d(v, w) = 0 \) only if \( ||v - w|| = 0 \), which is the case only if \( v = w \), that is to say if \( v = w \).

(2) (Symmetry) \( ||v|| = ||-v|| \) for any \( v \), so \( d(v, w) = ||v - w|| = ||w - v|| = d(w, v) \).

(3) (Triangular inequality) For any \( u, v, w \in V \) we have

\[
\begin{align*}
d(v, w) &= ||v - w|| = ||v - u + u - w|| = \sqrt{\langle v - u + u - w, v - u + u - w \rangle} = \\
&= \sqrt{\langle v - u, v - u \rangle + 2\langle v - u, u - w \rangle + \langle u - w, u - w \rangle = \\
&= \sqrt{||v - u||^2 + 2||v - u||||u - w|| + ||u - w||^2} \leq \\
&= \sqrt{||v - u||^2 + 2||v - u||||u - w|| + ||u - w||^2} = \sqrt{(||v - u|| + ||u - w||)^2} = \\
&= ||v - u|| + ||u - w|| = d(v, u) + d(u, w) \text{ where the inequality follows from the Cauchy-Schwarz inequality.}
\end{align*}
\]

The norm of \( \sin x \) is \( \sqrt{\int_0^{2\pi} \sin^2(x)} = \sqrt{\pi} \). The distance from \( \sin x \) and 1 is \( ||\sin(x) - 1|| = \sqrt{\int_0^{2\pi} (\sin x - 1)^2} = \sqrt{\int \sin^2 - 2\sin x + 1} = \sqrt{\pi + 2\pi - 2\int_0^{2\pi} \sin x} = \sqrt{3\pi} \).