Mathematical Methods 2017/01/27

Solve the following exercises in a fully detailed way, explaining and justifying any step.

(1) (6 points) Compute $\displaystyle\int_0^{2\pi} \frac{1}{\cos(x)^2+16}$.

(2) (5 points) Solve, via Fourier series, the differential equation

$$x'' + x' + 2x = \sin(t) + \cos(2t).$$

where the unknown function $x(t)$ is defined on $\mathbb{R}$ and required to be periodic of period 2π.

(3) (5 points) State the dominated convergence theorem. Provide an example of application of that theorem and an example where the theorem is not applicable.

SOLUTIONS

(1). The function $1/(\cos(x)^2+16)$ is continuous on $[0, 2\pi]$ and so the integral exists and it is finite. By changing variable

$$z = e^{ix} \qquad dz = ie^{ix}dx \qquad \cos(x) = (z + \frac{1}{z})/2$$

and letting γ be the unit circle in $\mathbb{C}$ oriented counter clockwise, the requested integral becomes

$$\int_\gamma \frac{1}{iz((\frac{z+\frac{1}{z}}{2})^2+16)}dz = \int_\gamma \frac{4z}{i((z^2+1)^2+64z^2)}dz$$

The function $\frac{4z}{i((z^2+1)^2+64z^2)}$ is holomorphic except at 4 simple poles: the zeroes of $(z^2+1)^2+64z^2$, which are $i(\pm 4 \pm \sqrt{17})$. The poles $\pm i(4+\sqrt{17})$ lies outside the region bounded by γ so the index of gamma at such poles is zero. The poles $\pm i(-4+\sqrt{17})$ are inside the unit disk, so the index of γ at such poles is 1 because γ is counter clockwise oriented. By residue theorem

$$\int_\gamma \frac{4z}{i((z^2+1)^2+64z^2)}dz = 2\pi i(Res(\frac{4z}{i((z^2+1)^2+64z^2)}, i(4-\sqrt{17}))+Res(\frac{4z}{i((z^2+1)^2+64z^2)}, i(\sqrt{17}-4))$$

we have

$$\begin{aligned}(z^2+1)^2+64z^2 &= (z-i(4+\sqrt{17})(z+i(4+\sqrt{17}))(z-i(4-\sqrt{17}))(z+i(4-\sqrt{17}) = \\ &= (z^2+(4+\sqrt{17})^2)(z-i(4-\sqrt{17}))(z+i(4-\sqrt{17})\end{aligned}$$

whence

$$Res(\frac{4z}{i((z^2+1)^2+64z^2)}, i(4-\sqrt{17})) = \frac{4i(4-\sqrt{17})}{i(-(4-\sqrt{17})^2+(4+\sqrt{17})^2)(2i(4-\sqrt{17}))} = \frac{1}{8i\sqrt{17}}$$

$$Res(\frac{4z}{i((z^2+1)^2+64z^2)}, -i(4-\sqrt{17})) = \frac{-4i(4-\sqrt{17})}{i(-(4-\sqrt{17})^2+(4+\sqrt{17})^2)(-2i(4-\sqrt{17}))} = \frac{1}{8i\sqrt{17}}$$

Therefore the requested integral is

$$2\pi i(\frac{1}{8i\sqrt{17}} + \frac{1}{8i\sqrt{17}}) = \frac{\pi}{2\sqrt{17}}.$$

(2). Let $x = b_0 + \sum_{n=1}^{\infty} a_n \sin(nt) + b_n \cos(nt)$ be the Fourier series of x. Then the Fourier series of x' is

$$\sum_{n=1}^{\infty} na_n \cos(nt) - nb_n \sin(nt)$$

and that of x'' is

$$\sum_{n=1}^{\infty} -n^2 a_n \sin(nt) - n^2 b_n \cos(nt).$$

Thus, the Fourier series of $2x + x' + x''$ is

$$2b_0 + \sum_{n=1}^{\infty} (2a_n - nb_n - n^2 a_n) \sin(nt) + (2b_n + na_n - n^2 b_n) \cos(nt)$$

The function $\sin(t) + \cos(2t)$ is its Fourier series. In order to impose the equality we must have

- $2b_0 = 0$
- $2a_1 - b_1 - a_1 = 1$
- $2b_1 + a_1 - b_1 = 0$
- $2a_2 - 2b_2 - 4a_2 = 0$
- $2b_2 + 2a_2 - 4b_2 = 1$
- $2a_n - nb_n - n^2 a_n = 2b_n + na_n - n^2 b_n = 0$ for $n \geq 3$

The system $\begin{cases} 2a_1 - b_1 - a_1 = 1 \\ 2b_1 + a_1 - b_1 = 0 \end{cases}$ has solution $a_1 = -b_1 = 1/2$.

The system $\begin{cases} 2a_2 - 2b_2 - 4a_2 = 0 \\ 2b_2 + 2a_2 - 4b_2 = 1 \end{cases}$ has solution $a_2 = -b_2 = 1/4$.

For $n \geq 3$, the system $\begin{cases} 2a_n - nb_n - n^2 a_n = 0 \\ 2b_n + na_n - n^2 b_n = 0 \end{cases}$ has solution $a_n = b_n = 0$

So we must have

$$x = \frac{1}{2}(\sin(t) - \cos(t)) + \frac{1}{4}(\sin(2t) - \cos(2t)).$$

Let's check that this solves the initial equation:

$$x' = \frac{1}{2}(\cos(t) + \sin(t) + \cos(2t) + \sin(2t))$$

$$x'' = \frac{1}{2}(-\sin(t) + \cos(t) - 2\sin(2t) + 2\cos(2t))$$

hence

$$2x + x' + x'' = \sin(t) - \cos(t) + \frac{1}{2}(\sin(2t) - \cos(2t)) +$$

$$+ \frac{1}{2}(\cos(t) + \sin(t) + \cos(2t) + \sin(2t)) + \frac{1}{2}(-\sin(t) + \cos(t) - 2\sin(2t) + 2\cos(2t)) = \sin(t) + \cos(2t)$$

(3). Theorem: Let $\Omega \subset \mathbb{R}^k$ be a measurable set (w.r.t. the Lebesgue measure). Let $(f_n)_{n\in\mathbb{N}} : \Omega \to \mathbb{C}$ be a sequence of measurable functions. Suppose that there is a function $f : \Omega \to \mathbb{C}$ such that f_n pointwise converges to f.

If there is $g : \Omega \to \mathbb{R}$ summable such that $|f_n(x)| \leq g(x)$ for any $x \in \Omega$ and $n \in \mathbb{N}$, Then f is summable and

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx.$$

Trivial example: $\Omega = [0,1] \subset \mathbb{R}$, $f_n(x) = 0$, $f = g = 0$.

Less trivial example: $\Omega = [0,\infty)$, $f_n(x) = e^{-nx^2}$, $g(x) = e^{-x^2}$. g is summable (with integral $\sqrt{\pi}/2$, we did it at lesson!). The pointwise limit of f_n is the function $f(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}$. Thus $\lim_n \int_0^\infty f_n = 0$.

Non-Example: $\Omega = (0,\infty)$, $f_n(x) = ne^{-nx}$. The pointwise limit of f_n is the function $f(x) = 0$. But

$$\int_0^\infty ne^{-nx}dx = -e^{-nx}|_0^\infty = 1 \not\to 0 = \int_0^\infty f(x)dx.$$