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Tautness of codimension-1 foliations in dimension 3 and transversality with embedded surfaces(**)

Abstract

Let $(M,\mathcal{F})$ be an orientable compact 3-manifold equipped with a codimension-1 transversely orientable foliation, and let $S \neq S^2$ be a compact orientable surface π_1-injectively embedded in M.

This work consists of two parts. First we study the tautness of $\mathcal{F}$; we define the notions of generalized Reeb component and non-taut component, and we show that the absence of non-taut components is equivalent to tautness.

Then we study the problem of transversality of S respect to $\mathcal{F}$, and we prove that if $\mathcal{F}$ does not contain generalized Reeb components then, either S is isotope to a leaf of $\mathcal{F}$ or, up to isotopy, S can be made transverse to $\mathcal{F}$ except at $-\chi(S)$ saddles (i.e. isolated tangency points with index -1).

Foliazioni tese di codimensione 1 in dimensione 3 e trasversalità rispetto a superfici embedded

Sunto

Sia $(M,\mathcal{F})$ una varietà compatta orientabile di dimensione 3 con una foliazione trasversalmente orientabile di codimensione 1, e sia $S \neq S^2$ una superficie compatta orientabile embedded in M in modo che l'embedding induca un omomorfismo iniettivo di $\pi_1(S)$.

Questo lavoro si svolge in due tempi. Prima di tutto studiamo $\mathcal{F}$, definiamo la nozione di componente di Reeb generalizzata, quella di componente non tesa e mostriamo come l'assenza di componenti non tese equivale al fatto che $\mathcal{F}$ sia tesa.

Quindi studiamo il problema della trasversalità di S rispetto a $\mathcal{F}$ e dimostriamo che, se $\mathcal{F}$ non contiene componenti di Reeb generalizzate, allora o S è isotopa a una foglia di $\mathcal{F}$ oppure con un'isotopia S può essere messa trasversa a $\mathcal{F}$ tranne che in $-\chi(S)$ selle (cioè punti isolati di tangenza con indice -1).

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Introduction

Around 1972 Roussarie [Rou] and Thurston [T1] simultaneously proved the following result:

Theorem 1 *Let M be an orientable compact 3-manifold and let $\mathcal{F}$ be a transversely orientable foliation of codimension 1 on M without Reeb components. If $\partial M \neq \emptyset$ then each component of ∂M is required to be a leaf of $\mathcal{F}$ or to be transverse to $\mathcal{F}$.*

Let φ be an embedding of a compact orientable surface $S \neq S^2$ in M such that $\varphi_* : \pi_1(S) \to \pi_1(M)$ is an injective homomorphism. Then φ is isotopic to an embedding φ' which is transverse to $\mathcal{F}$ except at a finite number of saddle and circle tangencies.

This theorem plays a fundamental role in the theory of Thurston norm [T2]. Roussarie proved in [Rou] the following:

(*) If S is the torus T^2, and if $\mathcal{F}$ does not contain components of type I and II (Reeb-type foliations constructed starting from an annulus instead of a disc), then the circle tangencies can be isotoped away.

Moreover, in [T2] we find the following statement:

(**) If each leaf intersects a closed transverse curve, then the circle tangencies can be isotoped away.

This fact is often quoted in literature with hints like "...The same techniques work in the present case ..." ([T2]) or "...it is well known to experts that ..." ([G]), but we couldn't find its proof (the most common references are [Rou], [T1], [T2]). Moreover note that the hypotheses in (**) are stronger than the hypotheses in (*) because, by Theorem 1.1.5 (Sullivan), the first ones are equivalent to tautness of $\mathcal{F}$ whereas it is not difficult to find a non-taut foliation satisfying the second ones. The purpose of this work is to give a complete proof of (**) and to specify its hypotheses. Namely, in section 2 we define the notion of generalized Reeb component and in section 3 we prove the following:

Theorem 2 *Let V be a compact orientable smooth 3-manifold. Let $\mathcal{F}$ be a codimension-one, transversely orientable foliation of class at least C^2 on V not containing any generalized Reeb component. If $\partial V \neq \emptyset$, then each component of ∂V is required to be transverse to $\mathcal{F}$ or to be a leaf of $\mathcal{F}$.*

Let S be a closed orientable and connected surface of genus ≥ 1 and let $\varphi : S \to V$ be a smooth embedding (at least C^2) such that $\varphi_* : \pi_1(S) \to \pi_1(V)$ is an injective homomorphism.

Then φ is isotopic either to an embedding which is transverse to $\mathcal{F}$ except at a finite number of saddle tangencies, or to an embedding whose image is contained in a leaf of $\mathcal{F}$.

Working on this problem, it is natural to compare the hypotheses

1) (V,$\mathcal{F}$) does not contain generalized Reeb components;

2) $\mathcal{F}$ is taut.

Remark. The notion of 'generalized Reeb component' (GRC) exists in literature, but its definition varies. E.g. in [G] GRCs are "bundles over S^1 with fiber a compact surface S with boundary. The boundary tori are leaves and the interior leaves are homeomorphic to int(S) and nearly tangent to S except near the ends which spiral in the same direction about the boundary tori"; in [ET] a GRC in a compact manifold V is "a submanifold $N \subset V$ of maximal dimension bounded by tori, such that the orientation of these tori as leaves is the same as (or simultaneously opposite to) their orientation as boundary components of N".

In this paper we study only the orientable case and we choose a non-standard terminology. Namely, we introduce the name 'non-taut component' to mean what GRC means in [ET], and we use 'GRC' in the sense of [G].

In section 2 we study these objects and give some examples. We show how the absence of non-taut components is equivalent to tautness (Theorem 2.1.6). We see that a GRC is also a non-taut component but not conversely, so the hypothesis 2) is stronger than 1). We also show how the geometric nature of the noyion of GRC (and not tautness) plays a fundamental role in the proof of Theorem 2.

Our proof of Theorem 2 will also allow us to replace the general assumption (absence of GRCs) by a weaker (optimal) assumption which depends on th surface S we aim to isotope. More precisely we will see that

- GRCs, defined as in 2.3.1, have a complexity hierarchy (essentially the Euler characteristic of the fiber).

- Theorem 2 hlods true assuming $(V, \mathcal{F})$ not to contain GRCs of complexity smaller than a certain bound which depends on $\chi(S)$.

It is may be worth remarking that Theorem 1 does not seem to display the hierarchy just mentioned. Assuming $\mathcal{F}$ to contain no (genuine) Reeb component, one proves the result for all surface S.

Finally we note that many results of this paper are already well known to experts; some results of section 2 can be seen as corollaries of work of Goodman [Go], but we preferred to give independent proofs; the techniques used to prove Theorem 2 are generalizations of the techniques used by Roussarie in [Rou].

1 Some preliminary facts

1.1 Some definitions

Let M be an orientable smooth (at least C^2) 3-manifold and let $\mathcal{F}$ be a smooth foliation of codimension 1 on M.

Definition 1.1.1 A *transverse section* Σ is an interval, smoothly embedded in M, which is transverse to $\mathcal{F}$ at every point.

Definition 1.1.2 A *transversal* is a smooth embedding $\varphi : [0,1] \to M$, transverse to $\mathcal{F}$ at every point.

A transversal φ is *closed* if $\varphi(0)=\varphi(1)$ or if $\varphi(0)$ and $\varphi(1)$ are in ∂V.

Definition 1.1.3 $\mathcal{F}$ is *transversely orientable* if its normal in some Riemannian metric can be so oriented at every point that the orientation depends continuously on the point.

If $\mathcal{F}$ is transversely oriented, then for every transversal γ the *positive* orientation of γ is well defined as the one concordant with the orientation of $\mathcal{F}$.

Definition 1.1.4 $\mathcal{F}$ is *taut* if and only if a closed transversal exists which meets all the leaves of $\mathcal{F}$.

Theorem 1.1.5 *(Sullivan [Su]) $\mathcal{F}$ is taut if and only if every leaf has a closed transversal passing through it.*

The proof of this theorem is not difficult in the compact case.

Definition 1.1.6 (Novikov connected components [N]) If $\mathcal{F}$ is transversely oriented and A and B are two different leaves of $\mathcal{F}$, we say $A > B$ if and only if a positive transversal exists from A to B. We set $A > A\ \forall A$.

We say that A and B lie on the same connected component if and only if $A > B$ and $B > A$.

We say $A \succ B$ if and only if every closed transversal meeting A meets also B.

Theorem 1.1.7 (Classification of the leaves)

Let F be a leaf of $\mathcal{F}$ and let Σ be a transverse section such that $F \cap \Sigma \neq \emptyset$. Then we have one and only one of the following situations:

1) $\Sigma \cap F$ is a discrete set.

2) The closure in Σ of $\Sigma \cap F$ contains an open set of Σ.
3) The interior of the closure in Σ of $\Sigma \cap F$ is the empty set.
Moreover:
- 1) $\Longleftrightarrow$ F, with the intrinsic topology, is embedded in M
- 2) $\Longleftrightarrow$ The interior of the closure of F in M is not empty $\Longleftrightarrow$ $F \subset$ $\mathrm{int}(\overline{F})$.
We say:
1) F is embedded.
2) F is locally dense.
3) F is exceptional.

For the proof see [CL]. For the terminology of example 2.2.2 we also refer to [CL]

1.2 The starting point

Let V be a smooth 3-manifold with a transversely orientable foliation $\mathcal{F}$. Let R be a compact orientable smooth 2-manifold different from the disc D^2 and from the sphere S^2 and let φ be a smooth embedding of R in V such that:
i) If $\partial R \neq \emptyset$, then for every component γ of ∂R, either $\varphi(\gamma) \subset \partial V$ or $\varphi(\gamma) \subset \mathrm{int}(V)$.
ii) $\varphi(\mathrm{int}(R)) \subset \mathrm{int}(V)$ and φ is transverse to ∂V.
iii) φ is transverse to $\mathcal{F}$ in a neighborhood of ∂R and for every component γ of ∂R, either $\varphi(\gamma)$ is transverse to $\mathcal{F}$ or it is contained in a leaf of $\mathcal{F}$.

Definition 1.2.1 One such φ is called *reduced* if the set on which it is not transverse to $\mathcal{F}$ consists of:

1) $|\chi(R)|$ saddle tangencies which, by Hopf theorem, are the only singularities of the foliation $\varphi^*(\mathcal{F})$ ($\chi(R)$ is the Euler characteristic of R).

2) Some circle tangencies $\gamma^1 \dots \gamma^l$, $l \geq 0$, embedded by φ into different leaves L_i. Each γ^i has a neighborhood $M^i \simeq \gamma^i \times [-1,1]_y \times [-1,1]_z$, where $\gamma^i = \gamma^i \times \{0\} \times \{0\}$, $\mathcal{F} = \{\{z = z_0\} : z_0 \in [-1,1]\}$ and $\varphi(S) = \gamma^i \times \{z = -y^2\}$. Moreover every γ^i is homotopically non-trivial in the leaf L_i.

The following result of Roussarie is our starting point.

Theorem 1.2.2 *[Rou] Let $(V, \mathcal{F})$ and R be as above. Assume $\mathcal{F}$ not to contain S^2-leaves and $(V, \mathcal{F})$ not to contain Reeb components. If $\partial V \neq \emptyset$ then each component of ∂V is required to be a leaf of $\mathcal{F}$ or to be transverse to*

$\mathcal{F}$. Let φ be an embedding of R in V satisfying the above conditions i),ii),iii) and such that $\varphi_* : \pi_1(R) \to \pi_1(V)$ is injective.

Then φ is isotopic to a reduced embedding.

Remark. By Reeb's global stability theorem ([R]), if $\mathcal{F}$ contains an S^2-leaf, then V is diffeomorphic to a bundle over S^1 or D^1 with fiber S^2; in particular $\pi_1(V) = \{0\}$ or $\pi_1(V) = \mathbb{Z}$. Therefore, since φ_* is injective, if $\pi_1(R)$ is large enough then $\mathcal{F}$ cannot contain a S^2-leaf.

Idea of proof: First of all we put φ in a generic position with respect to $\mathcal{F}$, i.e. in such a way that φ has only a finite number of points of tangency. Each of these points locally looks either like a maximum (or minimum) or like a saddle; since R is different from D^2 and S^2, $\chi(R) \geq 0$ and then, by the Hopf theorem (see for example [M]), there are as many maxima and minima (centers) as saddles.

We begin by "flattening out" R near any center. This eventually leads either to a cancellation with a saddle or to the appearence of a circle tangency, and allows to establish Theorem 1.2.2.

1.3 The scheme of the proof

In order to prove Theorem 2, first of all we observe that, since the surface S is of genus ≥ 1, then $\pi_1(S)$ is large enough (in the sense of the above remark). Moreover the conditions i) and iii) are trivially satisfied since $\partial S = \emptyset$, and we can easily obtain ii). Therefore we can apply Theorem 1.2.2.

Once we have a reduced embedding, we can locally modify it near the circle tangencies in such a way as to obtain an embedding which has some *annulus tangencies* (or *contact annuli*) replacing the circle ones, i.e. we make the embedding adhere a little to the leaves L_i.

In the cases in which the holonomy of L_i is trivial, we try to displace leaf by leaf the annuli, keeping their boundaries on the image of φ. The displacement of an annulus goes on until we find one of the following situations:

- We run into a leaf with non-trivial holonomy.
- We run into a saddle tangency.
- We run into a different contact annulus.

We study all the possibilities and case by case we define a standard move by which either we make the annulus transverse to $\mathcal{F}$ or we go on with the displacement.

When we go on, it happens that the contact surfaces (the annuli) become complicated. So we must be able to displace generic compact orientable surfaces.

By well-ordering the *contact configurations*, we conclude the proof by induction. The inductive step is the Proposition 3.1.1, i.e. by a finite number of applications of Proposition 3.1.1, we come to a minimal configuration in which either we make φ transverse to $\mathcal{F}$ (except at $-\chi(S)$ saddles), or the image of φ is contained in a leaf of $\mathcal{F}$.

The displacement of contact surfaces is not at all trivial and here we fundamentally use the absence of GRCs and the injectivity of φ_* (Theorem 2.3.5).

2 Non-taut foliations in dimension 3, orientable case

2.1 Non-taut components

In this part we examine transversely orientable foliations of codimension 1 on orientable 3-manifolds. We define the *non-taut components* and show how such components are the kernel of 'non-tautness'.

Definition 2.1.1 A *non-taut component* is a connected compact orientable 3-manifold V with a codimension-1, transversely orientable foliation $\mathcal{F}$ such that:

i) $\emptyset \neq \partial V =$ Union of leaves.

ii) If we fix an orientation for V and one for $\mathcal{F}$, and we consider on ∂V the two orientations (as boundary of V and as union of leaves of $\mathcal{F}$), then either these orientations agree on all components of ∂V, or they disagree on all components of ∂V.

iii) Every leaf of $\mathrm{int}(V)$ admits a closed transversal passing through it.

Example 2.1.2 Let us consider a compact surface $S \subset \mathbb{R}^2$, $\partial S \neq \emptyset$. Let $W = S \times \mathbb{R}$ and $f = \frac{1}{\mathrm{dist}(s,\partial S)} : \mathrm{int}(S) \mapsto \mathbb{R}$. We define $\mathcal{F}$ on W as follows: the leaves of $\mathcal{F}$ are the components of $\partial S \times \mathbb{R}$ and the graphs of the functions $f_c = (f+c)$, as c varies in $\mathbb{R}$ (see figure 1-a). Now let $V = W/\mathbb{Z}$ (where $\mathbb{Z}$ acts on $\mathbb{R}$ by translations). Clearly $\mathcal{F}$ is $\mathbb{Z}$-invariant, so it remains well-defined on V. It is easy to verify that $(V, \mathcal{F})$ is a non-taut component.

Example 2.1.3 Let S be the pant in $\mathbb{R}^2$, $S = \overline{B((0,0),4)} \setminus (B((2,0),1) \cup B((-2,0),1))$ and $(W, \mathcal{F})$ be as in Example 2.1.2. Let $\varphi \in \mathrm{Diff}^+(S)$ be the diffeomorphism induced by the rotation of angle π around the origin in $\mathbb{R}^2$ and let $V = W/G_\varphi$ when $G_\varphi(x,t) = (\varphi(x), t+1)$ (see figure 1-b). In this case $\mathcal{F}$ is G_φ-invariant and it is easy to verify that the result is a non-taut component whose boundary consists of two tori.

This construction can be generalized by considering any orientable compact surface S (with non empty boundary), any $\varphi \in Diff^+(S)$ and a foliation $\mathcal{F}$ which is G_φ-invariant. The number of boundary tori of the resulting 3-manifold depends on φ.

Remark. A Reeb component is a non-taut component.

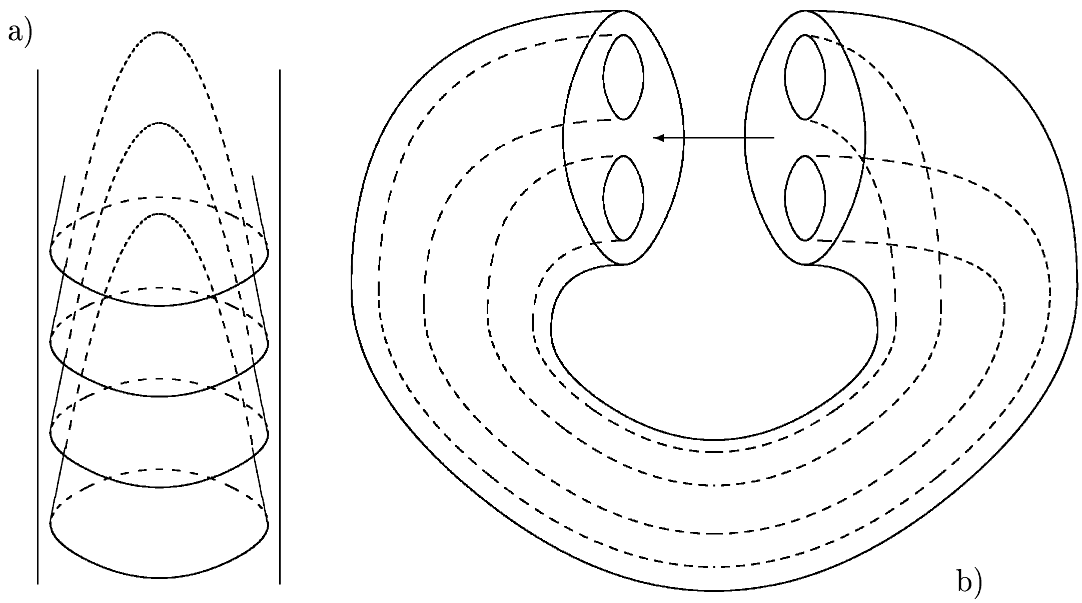


Figure 1: a): The foliation $\mathcal{F}$ defined on W in example 2.1.2, b): The manifold V of example 2.1.3.

In order to study these components we start justifying their name.

Proposition 2.1.4 *A non-taut component* $(V, \mathcal{F})$ *is not taut.*

Proof. By Theorem 1.1.5 it suffices to prove that V has a leaf without closed transversals. Fix a metric and an orientation for V. It follows by condition ii) of definition 2.1.1 that the trajectories positively normal to $\mathcal{F}$ on ∂V are all pointing inwards V or all pointing outwards V and so no transversal passing through a leaf of ∂V can be closed.$\square$

Remark. By condition iii) of definition 2.1.1 it follows that if V is a non-taut component, then $\mathcal{F}_{|\mathrm{int}(V)}$ is taut.

The boundaries of the non-taut components in Examples 2.1.2 and 2.1.3 consist of toric leaves. This is not accidental.

Proposition 2.1.5 *If V is a non-taut component, then ∂V is a union of tori. (See also [Go]).*

Proof. Firstly we fix a Riemannian metric on V and an orientation of $(V, \mathcal{F})$ so that the leaves of ∂V are oriented like ∂V; so the unitary vector field η positively normal to $\mathcal{F}$ is well-defined. Since η is nowhere zero and points outward on ∂V, it follows by Hopf's theorem that $\chi(V) = 0$ (see for example [M]). Moreover $2\chi(V) = \chi(\partial V)$, because if we mirror V in its boundary we get a closed manifold, whose χ is 0 and can be computed as $2\chi(V) - \chi(\partial V)$. So $\chi(\partial V) = 0$.

Suppose now that ∂V contains S^2 leaves. Then by Reeb's stability Theorem [R] every leaf must be an S^2, in particular each leaf in ∂V is an S^2 and then $\chi(\partial V) > 0$ (remember that $\partial V \neq \emptyset$) which is a contradiction. Then ∂V does not contain S^2 leaves. It follows that each component of ∂V has zero Euler characteristic and therefore it is a torus.□

The following theorem provides a characterization, via non-taut components, of foliations which are not taut.

Theorem 2.1.6 *Let M be closed orientable 3-manifold and let $\mathcal{F}$ be a transversely orientable foliation on M of codimension 1. Then $(M, \mathcal{F})$ is taut if and only if it does not contain any non-taut component.*

Proof. First of all we fix an orientation for $(M, \mathcal{F})$. By Proposition 2.1.4 if $(M, \mathcal{F})$ contains a non-taut component then it cannot be taut. Conversely, suppose $\mathcal{F}$ is not taut. Then by Theorem 1.1.5 there exists a leaf F without closed transversals passing through it. Let us consider the set $X = \cup\{B \text{ leaf of } \mathcal{F} \mid F > B\}$ (i.e. $X = \cup\{B \text{ leaf of } \mathcal{F} \mid \exists \text{ a positive transversal from } F \text{ to } B\}$).

We observe that from the definition it follows that X is saturated in $\mathcal{F}$ (i.e. it is a union of leaves) and it is an open subset (and then a submanifold) of M. Moreover if $x \in B \subset X$ and γ is a positive transversal from F to B, then we can modify γ near B so that x becomes the ending point of γ, therefore $X = \{x \in M | \exists \text{ a positive tranversal from } F \text{ to } x\}$.

We set $W = \overline{X}$, W is saturated in $\mathcal{F}$. Consider now a foliated local chart for $\mathcal{F}$ on a neighborhood $U \approx B^2 \times (-1, 1)$. If $x, y \in U$, we say that "y lies on the positive (negative) side of x" if and only if a positive (negative) transversal from x to y exists in U. If $x \in X$ and y lies on the positive side of x, then we can find a positive transversal in M from F to y, therefore $y \in X$. So for each $x \in X$ the positive side of x is contained in X.

Now let $x \in W \setminus X$ and U be as above. In U, X lies only on the positive side of x. In fact, suppose on the contrary that there exists $y \in X$ which lies on the negative side of x. Then $x \in X$ and this contradicts the fact that $x \in W \setminus X$. Moreover if y lies on the positive side of x then $y \in X$ (see figure 2) in fact, since $x \in W$ we can find $y' \in X$ near x and such that y lies on the positive side of y', thus, by the above argument, $y \in X$. It follows that the foliated local chart becomes also a smooth chart for W which therefore is a compact submanifold of M.

Recall now that F has no closed transversals passing through it, so $F \subset W \setminus X$. In particular $\partial W \neq \emptyset$.

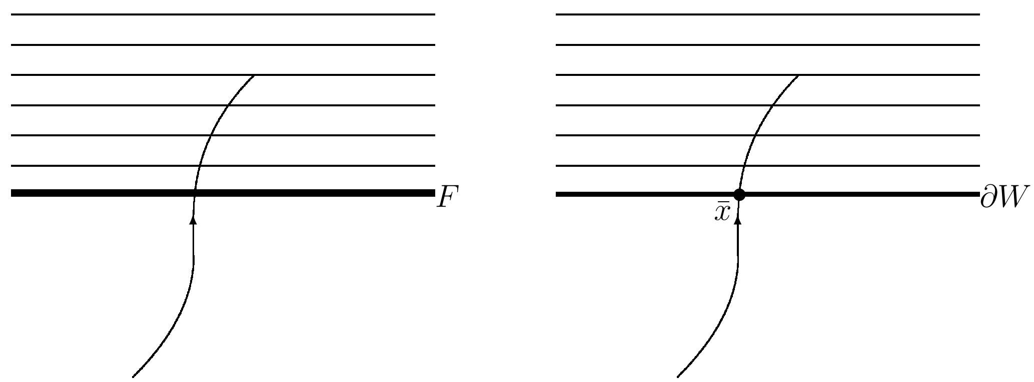


Figure 2: Foliated local charts near ∂W.

Saying that for every $x \in \partial W$ the set X lies only on the positive side of x is equivalent to saying that the vector field positively normal to $\mathcal{F}$ points inward at all boundary points of W. In order to see this suppose on the contrary that there exists a leaf $A \subset \partial W$ whose positively normal vectors point outward of W. Let $B \subset X$ be a leaf which passes near A into a foliated local chart. Since $B \subset X$, we can find a positive transversal from F to B. By modifying such a transversal in a neighborhood of B and into the local chart, we find a positive transversal from F to A (see figure 3). It follows that $A \subset X$ and thus A can't be contained in ∂W.

Remark. The argument of Proposition 2.1.5 also works for W, so ∂W is a union of tori.

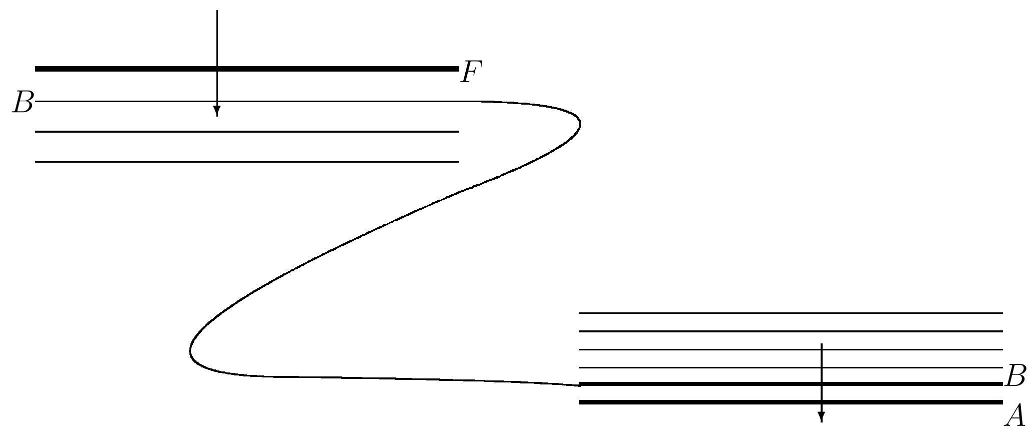


Figure 3: How to find a positive transversal from F to A.

Let us consider now the set $\mathcal{K}$ of non-empty submanifolds V of M such that $V \subset W$, V is compact of dimension 3, V is saturated in $\mathcal{F}$ and the positive transversals at every boundary point of V are pointing into V. We order $\mathcal{K}$ by inclusion. Since $W \in \mathcal{K}$, $\mathcal{K} \neq \emptyset$. Let now $\{V_i\}$ be a chain in $\mathcal{K}$ and let $V = \bigcap\limits_i V_i$. We claim that $V \neq \emptyset$. In order to see it, consider a sequence $x_i \in \partial V_i$, if necessary by passing to a subsequence, we can suppose that x_i converges to x. Consider now a foliated local chart on a neighborhood U of x of the type $U \approx B^2 \times (-1, 1)$. Since the positive transversals at every boundary point of each V_i point into V_i, then the positive side of ∂V_i in U must be contained in V_i and so $x \in V$.

Obviously V is saturated in $\mathcal{F}$ and it is a submanifold of W, i.e. $V \in \mathcal{K}$ and thus $\mathcal{K}$ is inductive.

Let V_0 be a minimal element of $\mathcal{K}$. V_0 with the foliation induced by $\mathcal{F}$ is a non-taut component. In fact the conditions i) and ii) of definition 2.1.1 are easy to verify; regarding condition iii) observe that if $\text{int}(V_0)$ contains a leaf without closed transversals passing through it, we can define a new W as above. Such a W is easily checked to be strictly contained in V_0, which contradicts the minimality of V_0.$\square$

Corollary 2.1.7 *Let M be a closed orientable 3-manifold and let $\mathcal{F}$ be a codimension 1 transversely orientable foliation on M. If $\mathcal{F}$ does not contain toric leaves, $\mathcal{F}$ is taut.*

The proof immediately follows from Theorem 2.1.6 and Proposition 2.1.5.

Corollary 2.1.8 *Let $(M, \mathcal{F})$ be as above, if L is a closed leaf of genus > 1, then L is met by a closed transversal.*

Proof. (See also [Go]) Suppose the contrary. Then starting from L, we can construct a manifold W as in the proof of Theorem 2.1.6. Now L is a component of ∂W which is a union of tori. This gives a contradiction. □

2.2 Some examples

Let $(V, \mathcal{F})$ be a non-taut component. It follows from Definition 2.1.1, and in particular from condition *iii*), that int(V) consists of only one component in the sense of Novikov [1]. In fact by theorem 1.1 in [N], if more components are present, then there exists in int(V) at least one leaf without closed transversals passing through it. Therefore, again by [N], there exists a leaf A such that $A \succ B \ \ \forall B \subset \text{int}(V)$ [2].
There are three cases:

1) A is embedded.

2) A is an exceptional leaf.

3) A is locally dense.

We expose now an example for case 3) and one for case 2), showing in this way that all the cases are possible (examples 2.1.2 and 2.1.3 are part of case 1)).

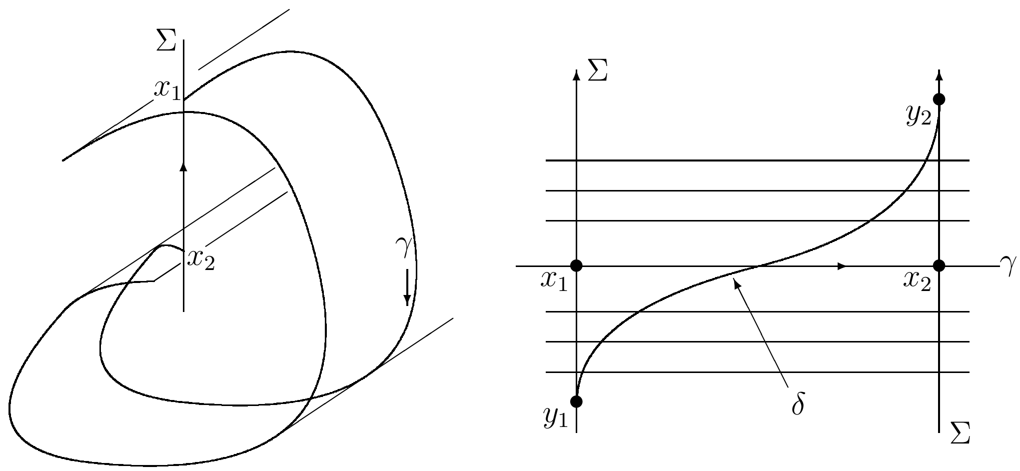
Case 3).

Let Σ be a transverse section passing through A (i.e. a D^1 transverse to $\mathcal{F}$ and intersecting A). Let x_1 and x_2 be points of $\Sigma \cap A$ and γ be a path from x_1 to x_2 in A (see figure 4).

Let us consider a foliated local chart on a neighborhood of γ. Here we can find in Σ two points y_i near x_i $(i = 1, 2)$ such that in Σ we have $(y_2, y_1) \subset (x_2, x_1)$. Then we can construct a path δ from y_1 to y_2 such that δ intersects only A and the leaves intersected by Σ (see figure 4). We can now close δ with Σ so as to obtain a closed transversal passing through A and intersecting only leaves intersected by Σ.

Since $A \succ B \ \ \forall$ leaf B, Σ must intersect every leaf in int(V). Moreover, since this argument works for every transverse section, it follows that each leaf in int(V) is locally dense. In this case we say that $(V, \mathcal{F})$ is a *dense component*.

[1] See definition 1.1.6.
[2] See definition 1.1.6



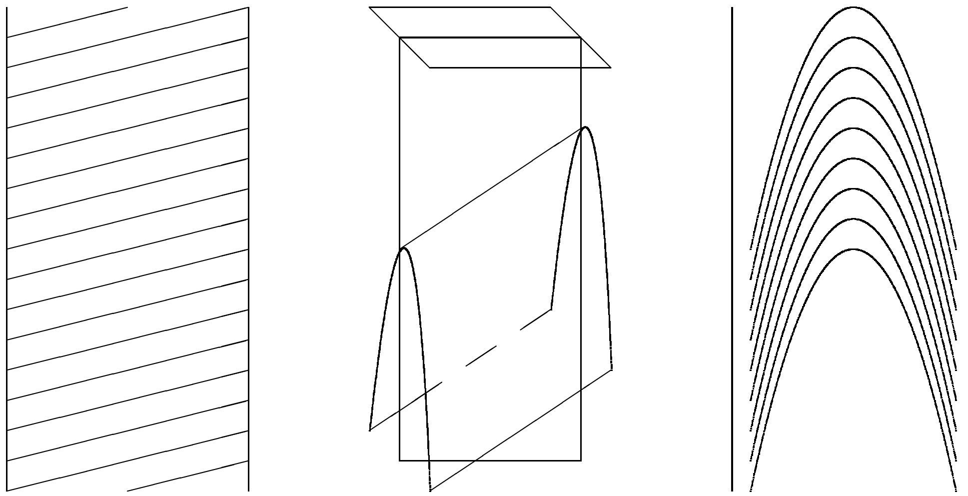
Situation into the manifold. Local chart.

Figure 4

Example 2.2.1 This example is a 3-dimensional version of the foliation defined on the torus $S^1 \times S^1$ with each leaf dense. Let us consider $\mathbb{R}^2$ foliated by parallel lines, whose angular coefficient is irrational, and let us make the quotient by $\mathbb{Z}$ (where $\mathbb{Z}$ acts by translations on the first coordinate of $\mathbb{R}^2$). We obtain a foliation on $S^1 \times \mathbb{R}$. Now we define a foliation on $S^1 \times \mathbb{R} \times (-1, 1)$ by imposing that on each fence $p \times \mathbb{R} \times (-1, 1)$ ($p \in S^1$) the leaves are the graphs of the functions $f_t : (-1, 1) \mapsto \mathbb{R}$, $f_t(x) = \dfrac{1}{1 - x^2} + t$. We extend the foliation to $S^1 \times \mathbb{R} \times [-1, 1]$ by adding the leaves $S^1 \times \mathbb{R} \times \{1\}$ and $S^1 \times \mathbb{R} \times \{-1\}$. By making the quotient by $\mathbb{Z}$ (this time $\mathbb{Z}$ acts by translations on $\mathbb{R}$) we obtain a foliation $\mathcal{F}$ on $V = T^2 \times [-1, 1]$ and it is easy to check that $(V, \mathcal{F})$ is a dense non-taut component (in figure 5 S^1 is pictured as an interval).

Example 2.2.2 *(Case 2))* This example is based on Sacksteder's example of foliation with exceptional leaves on $V_2 \times S^1$, where V_2 is a closed oriented surface of genus 2 (see [CL]).

Let a, b, c, d be the generators of $\pi_1(V_2)$ so that the unique nontrivial relation between these elements is $aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1$ and let G be the free subgroup of $\pi_1(V_2)$ generated by a and c. Given $f, g \in diff^+(S^1)$ we can define a homomorphism $\varphi : \pi_1(V_2) \mapsto diff^+(S^1)$ setting $\varphi(a) = f, \varphi(c) = g, \varphi(b) = \varphi(d) = 1$. φ is well-defined, since $\varphi(aba^{-1}b^{-1}cdc^{-1}d^{-1}) = f \circ f^{-1} \circ g \circ g^{-1} = 1$.



$S^1 \times \mathbb{R}$. Leaf in $S^1 \times \mathbb{R} \times [-1, 1]$. Fence $p \times \mathbb{R} \times [-1, 1]$.

Figure 5: Constructing a dense component.

Let $\mathcal{F}$ be the suspension of φ, the fiber bundle where $\mathcal{F}$ is defined is homeomorphic to $V_2 \times S^1$. The leaves of $\mathcal{F}$ are transversal to the fiber S^1 and in one-to-one correspondence with the orbits of φ (i.e. the orbits of $\varphi(\pi_1(V_2))$ which acts obviously on S^1). The exceptional and minimal sets of $\mathcal{F}$ are in one-to-one correspondence respectively with the exceptional and minimal sets of φ. These correspondences can be visualized considering a fiber S^1, a leaf $F \in \mathcal{F}$ and finally $F \cap S^1$.

Considering S^1 as $[0, 1]/0 \sim 1$, we define the diffeomorphisms:

– $f(x) = x + \frac{1}{3} \qquad (\text{mod } 1)$

– $g(x) = \begin{cases} \frac{x}{3} & x \in [1, \frac{1}{2}] \\ 3x - \frac{5}{3} & x \in [\frac{2}{3}, \frac{5}{6}] \end{cases}$

$g(1) = 1,\ g'(1) = \frac{1}{3},\ g^{(k)}(1) = 0\ k \geq 2,\ g \in C^\infty(S^1)$ and such that its graph has the form of figure 6:

It is known ([CL]) that such a φ produces a foliation with exceptional leaves. For instance, the orbit of $\frac{1}{3}$ corresponds to an exceptional leaf, and an exceptional set for φ is the set K obtained by constructing three copies of a Cantor set in $[0, \frac{1}{6}], [\frac{1}{3}, \frac{1}{2}], [\frac{2}{3}, \frac{5}{6}]$ (the orbit of $\frac{1}{3}$ is dense in K).

We'll see that the leaves which are $\succ$-maximal are exceptional. First of all we observe that if a leaf F passes through $(\frac{5}{6}, 1)$ (i.e. F corresponds to

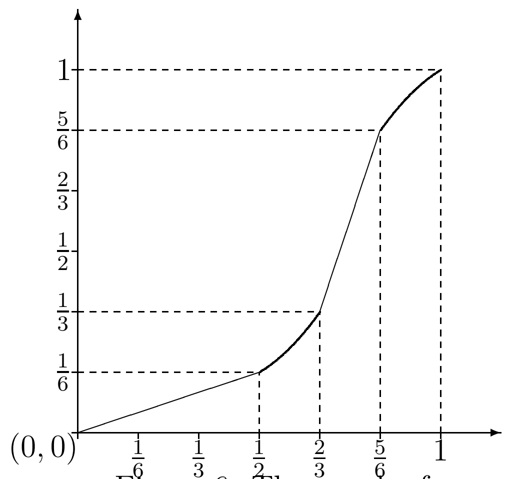


Figure 6: The graph of g.

the orbit of a point in $(\frac{5}{6}, 1))$ then it can't be $\succ$-maximal. In fact, $g((\frac{5}{6}, 1)) \subset (\frac{5}{6}, 1)$ and so, as in case 3), we can find a closed transversal γ passing through F and intersecting only leaves passing through $(\frac{5}{6}, 1)$. In particular γ doesn't intersect the leaf passing through $\frac{1}{3}$ and so F is not $\succ$-maximal.

Let now $x \notin K$. We'll see that the orbit of x contains points in $(\frac{5}{6}, 1)$ and it follows that maximal leaves correspond to orbits in K and so are exceptional. If $x \in (\frac{1}{6}, \frac{1}{3}) \cup (\frac{1}{2}, \frac{2}{3}) \cup (\frac{5}{6}, 1)$, then applying f we find a point (in the orbit of x) in $(\frac{5}{6}, 1)$. If $x \in (0, \frac{1}{6}) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{2}{3}, \frac{5}{6})$, then applying f we find a point in $(\frac{2}{3}, \frac{5}{6})$. The Cantor set can be defined by induction starting from $C_0 = [0, 1]$ and constructing C_{i+1} from C_i removing each central third from each interval forming C_i. If $x \in (\frac{2}{3}, \frac{5}{6})$ and x lies in one of the intervals removed at the step i to obtain the Cantor in $(\frac{2}{3}, \frac{5}{6})$, we have that $g(x) \in (\frac{1}{2}, \frac{2}{3})$ or $g(x)$ lies in one of the intervals removed at the step $i - 1$ to obtain the Cantor in $(\frac{1}{3}, \frac{1}{2})$ or the Cantor in $(\frac{2}{3}, \frac{5}{6})$. Therefore, since $f((\frac{1}{3}, \frac{1}{2})) = (\frac{2}{3}, \frac{5}{6})$, with a finite number of applications of f and g we find a point of the orbit of x in $(\frac{5}{6}, 1)$.

The pair $(V_2 \times S^1, \mathcal{F})$ is not yet a non taut-component just because $\partial V_2 = \emptyset$.

Let us consider a fiber S^1 with a foliated neighborhood $U \sim D^2 \times S^1$ where the leaves of induced foliation are $D^2 \times S^1$ and let us remove U from $V_2 \times S^1$. We obtain a foliated manifold $(M_1, \mathcal{F}_1)$ such that $\partial M_1 = S^1 \times S^1$. Let us consider now a foliation obtained like in the example 2.1.2 letting S be an annulus $C = S^1 \times [0, 1]$. Such a foliation is defined on $C \times S^1$. We remove from $C \times S^1$ the part $(S^1 \times (\frac{1}{2}, 1]) \times S^1$, so we obtain a foliated manifold $(M_2, \mathcal{F}_2)$ such that $\partial M_2 = S^1 \times S^1$. Finally we can glue M_1 to M_2, since the induced foliations on the respective boundaries are isomorphic. We obtain so a foliated manifold $(V, \mathcal{J})$. We let the reader check that $(V, \mathcal{J})$ is a non-taut component with exceptional leaves.

2.3 Generalized Reeb components and displacements of surfaces

Let $(V, \mathcal{F})$ be a non-taut component and let A be a $\succ$-maximal leaf. If A is embedded, a priori it is not clear if any other leaf in V is embedded or how the other leaves appear.

Definition 2.3.1 (Generalized Reeb components).
A non-taut component V which is topologically homeomorphic to $A \times S^1$, where A is an orientable compact surface with a non empty boundary, and such that every leaf in $\text{int}(V)$ is embedded and homeomorphic to $\text{int}(A)$, is called a *generalized Reeb component (GRC).*

Remark. A classic Reeb component is a GRC; the foliations described in examples 2.1.2 and 2.1.3 are GRCs.

Remark. As the following example shows, the fact that topologically $V \approx A \times S^1$ does not imply that the foliation on V is the product one.

Example 2.3.2 Let us consider the cylinder $C_1 = C \times \mathbb{R} \subset \mathbb{R}^3$, where $C \subset \mathbb{R}^2$ is the annulus delimited by the circles of radius $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and let us consider on C_1 the foliation $\mathcal{F}$ whose leaves are the graphs of the functions from C to $\mathbb{R}$ $f_r(\rho, \theta) = \tan(\rho) + r$ when r varies in $\mathbb{R}$. Let us consider now into the cylinder C_2, done like C_1, the leaves L_n given by the graphs of the functions $g_n(\rho, \theta) = \dfrac{1}{\frac{\pi}{2}^2 - (\rho - \pi)^2} + n$ when $n \in \mathbb{Z}$.

The region between L_n and L_{n+1} is diffeomorphic to C_1 and so we can complete the foliation on C_2 by gluing copies of $(C_1, \mathcal{F})$ between the leaves L_n. The result is a $\mathbb{Z}$-invariant foliation (if $\mathbb{Z}$ acts on $\mathbb{R}$ by translations). The foliated manifold we obtain by making the quotient, which is a GRC, is homeomorphic to $C \times S^1$ but the foliation is not the product one.

Before going on we specify what we mean for “displacement of surfaces”. Let $\mathcal{E}$ be the Reeb foliation on $D^2 \times S^1 \subset \mathbb{R}^3$ and let D be a disc contained in a leaf. Let $f : \partial D \times [0, 1] \cup D \times \{0\} \mapsto D^2 \times S^1$ be a smooth function such that $f(D \times \{0\}) = D$, $\forall\, t \in [0, 1]$ $f(\partial D \times \{t\})$ is contained in a leaf L_t, the trajectories $f(*, t)$ are normal to $\mathcal{E}$ and $f(\partial D \times \{1\})$ is contained in $S^1 \times S^1 = \partial(D^2 \times S^1)$ (see figure 7).

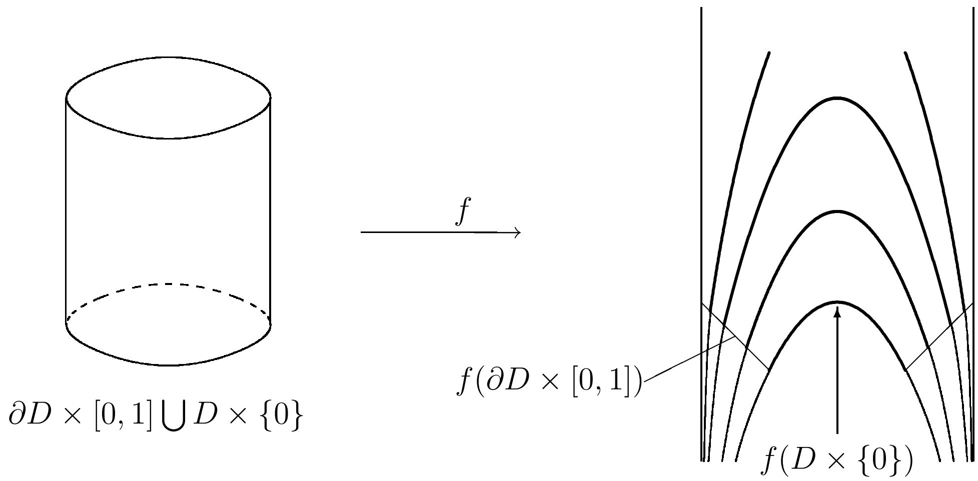


Figure 7: $f : \partial D \times [0,1] \bigcup D \times \{0\} \mapsto D^2 \times S^1$.

We wonder if such a function can be extended to a function $F : D \times [0,1] \mapsto D^2 \times S^1$ in such a way that $F(D \times \{t\}) \subset L_t$ and the trajectories $F(*, t)$ are normal to $\mathcal{E}$, i.e. we wonder if we can *displace* the disc D along the route fixed by f.

In general, the answer is no. Novikov showed in [N] how the presence of Reeb components is the obstruction to extend functions like f.

In this section we examine the generalized problem of displacement of any compact surface A contained in a leaf of a foliation $\mathcal{F}$, along trajectories normal to $\mathcal{F}$, once the route of ∂A is fixed. We see how in this case the obstruction to displace A is the presence in $\mathcal{F}$ of a GRC in which the leaves are homeomorphic to $\mathrm{int}(A)$.

In the following V will be an orientable compact (connected) smooth 3-manifold and $\mathcal{F}$ will be a transversely orientable foliation of codimension 1 on V, at least C^2; if $\partial V \neq \emptyset$ and S is a component of ∂V, then we'll suppose that $\mathcal{F}$ is transverse to S or S is a leaf of $\mathcal{F}$. Moreover we'll suppose a Riemannian metric fixed on V such that:

i) If $x \in \partial V$ and $\mathcal{F}$ is transverse to ∂V at x, then here $\mathcal{F}$ is normal to ∂V.

ii) If $x \in \mathrm{int}(V)$, the trajectory normal to $\mathcal{F}$ and passing through x attains ∂V in a finite time or it is not adherent to ∂V.

Proposition 2.3.3 *Let A be an orientable compact connected surface with non empty boundary. Let $F : [0,1[\times A \mapsto \mathrm{int}(V)$ be a smooth function, of maximum rank at every point and such that:*

1) *$\forall t \in [0,1[\quad F_t = F_{\big|_{\{t\}\times A}}$ is an incompressible embedding, i.e. $F_{t*}: \pi_1(A) \mapsto \pi_1(L_t)$ injective, of A into a leaf L_t.*

2) *$\forall x \in A$ the trajectories $t \mapsto F_t(x)$ are normal to $\mathcal{F}$ and $f = F_{\big|_{[0,1[\times\partial A}}$ is an embedding.*

3) *f has an extension for t=1 to an embedding $f_1 : \partial A \mapsto V$ such that each connected component of $f_1(\partial A)$ is contained in a leaf of $\mathcal{F}$. The set of the leaves which contain $f_1(\partial A)$ will be called L_1.*

4) *$F_0(\mathrm{int}(A)) \cap f([0,1[\times\partial A) = \emptyset$*

Moreover let us suppose that there exists $\bar{x} \in \mathrm{int}(A)$ such that $F_t(\bar{x})$ does not admit limit as $t \to 1$.

Then there exists a leaf $\bar{L}$ without closed transversals passing through it. More particularly, the image of F is contained in a GRC and the image of f_1 is contained on the boundary of such a component.

Proof. Let $\bar{x} \in \mathrm{int}(A)$ be a point without limit as $t \to 1$. Since V is compact, there exist a sequence $t_i \to 1$ and a point $x \in V$ such that $t_i < t_{i+1}$ and $x_i = F_{t_i}(\bar{x}) \to x$. It follows by the choices on the metric that $x \in \mathrm{int}(V)$.

Let L_x be the leaf containing x. In V x has a foliated neighborhood of type $U = D^2 \times (-\varepsilon, \varepsilon)$, with $D^2 \times \{0\} \subset L_x$ and $\mathcal{F}_{\big|_U} = \{D^2 \times \{t\}\}$. So, if $y \in U$ then the normal trajectory passing through y also meets L_x. Since x_i converges to x, definitively $x_i \subset U$ and then, if necessary by changing the sequence of t_i, we can suppose that $x_i \in D^2 \times \{0\} \subset L_x \; \forall i$ (see figure 8).

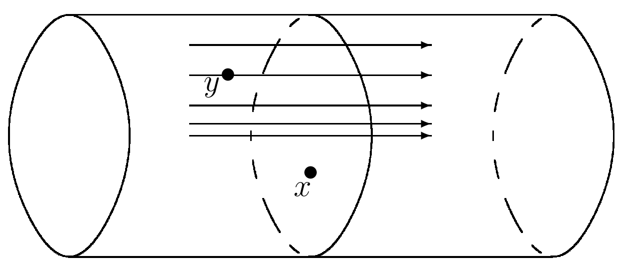


Figure 8: The neighborhood U.

In order to complete the proof we need the following lemma:

Lemma 2.3.4 *Setting $A_i = F_{t_i}(A) \subset L_x$, we have*

a) *If $t < s$ and $F_t(A) \cap F_s(A) \neq \emptyset$, then $F_t(A) \subset F_s(A)$.*

aa) Definitively on i, $A_i \subset A_j$ if $i < j$.

Proof. First of all note that, since $F_t(A) \cap F_s(A) \neq \emptyset$, $F_t(A)$ and $F_s(A)$ lie on the same leaf. If $t < s$ we have $\mathrm{int}(A_t) \cap \partial A_s = \emptyset$. In fact, if a point $p \in \mathrm{int}(A_t) \cap \partial A_s$ exists, by going back along the normal trajectory of p, one contradicts the hypothesis 4) of Proposition 2.3.3. Moreover, by hypothesis 2), we have $\partial A_t \cap \partial A_s = \emptyset$ and therefore $A_t \subset A_s$. Point a) of the lemma is thus proved.

Regarding point aa) it suffices now to prove that definitively on i j, $\mathrm{int}(A_i) \cap A_j \neq \emptyset$. In order to see it, suppose the contrary. Then for the metric induced by V on L_x we have $dist_{L_x}(\partial A_i, \partial A_j) \leq dist_{L_x}(x_i, x_j)$ that converges to 0 (in U we have $dist_V(x_i, x_j) \sim dist_{L_x}(x_i, x_j)$). Let us consider in V a neighborhood O of $F_1(\partial A)$ (on ∂A F admits limit) of type $N \times [0, \delta]$ where N is a neighborhood of $F_1(\partial A)$ in L_1 of type $F_1(\partial A) \times (-\varepsilon, \varepsilon)$ with $F_1(\partial A) \leftrightarrow F_1(\partial A) \times \{0\} \times \{0\}$.

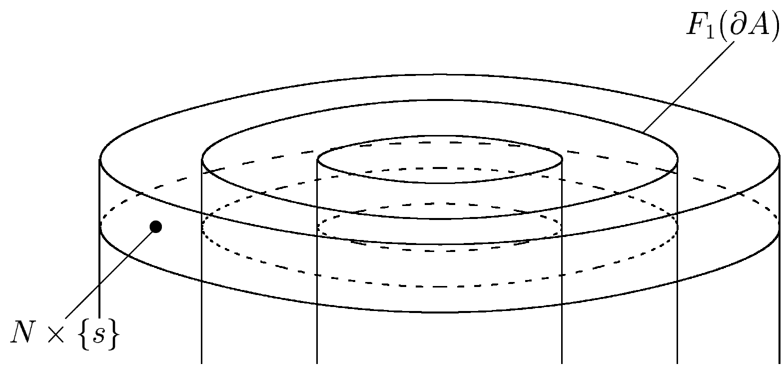


Figure 9: The neighborhood O.

If we take δ and ε small enough, we can make the foliation induced on O be $\{N \times \{x\}\}$. Since the trajectories $F_t(x)$ are normal to $\mathcal{F}$, we have that for t_i "near" 1, $dist_{L_x}(\partial A_i, \partial A_j) \geq 2\varepsilon$. Since $t_i \to 1$, only a finite number of t_i is "far" from 1 and so $\inf_{i<j} dist_{L_x}(\partial A_i, \partial A_j) > 0$; i.e. the distances $dist_{L_x}(\partial A_i, \partial A_j)$ are lower bounded and therefore cannot converge to 0. It follows that $\exists n \mid \forall i, j > n$ $\mathrm{int}(A_i) \cap A_j \neq \emptyset$.$\square$

By the hypotheses on F, $\forall\, t \in [0, 1)$ $\exists \varepsilon > 0$ such that $F_{\big|_{A\times[t-\varepsilon,t+\varepsilon]}}$ is an homeomorphism with its image; in particular F is an open map. Moreover F sends $A \times \{*\}$ on the leaves of $\mathcal{F}$ which so induces on $F(A \times [t-\varepsilon, t+\varepsilon])$ the product foliation. In particular, it follows that the set $\{t \in (t_0, 1) \mid F_t(A) \cap A_0 \neq \emptyset$ (and then $F_t(A) \supset A_0)\}$ cannot have cluster points different from 1 and then, if necessary by changing the sequence of $\{t_i\}$, we can suppose that

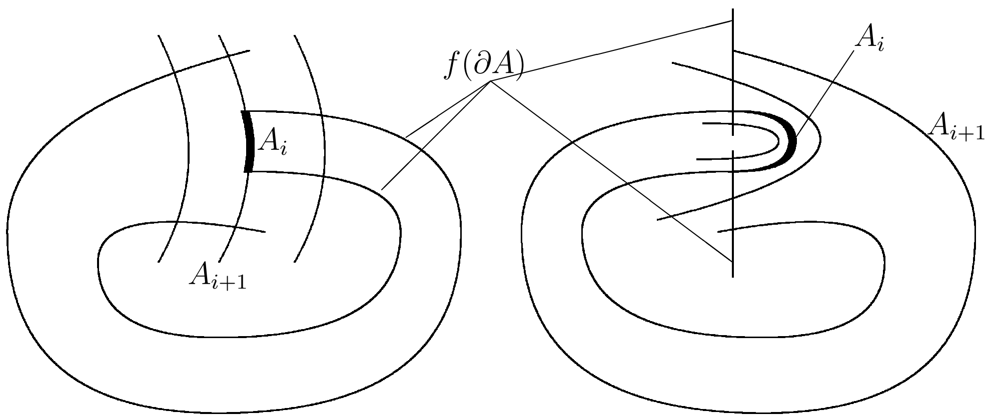


Figure 10: The same position "wrapped up" in two different ways.

$F_t(A) \cap (\cup_i A_i) = \emptyset \;\; \forall t \in (t_0, 1) \setminus \{t_i\}$ (between t_i and t_{i+1} the trajectories F_t turn only one time). In order to understand better what is going on, see the figure 10.

Let $\varphi_i = F^{-1}_{t_{i+1}} \circ F_{t_i} : A \mapsto \mathrm{int}(A)$. φ_i is a homeomorphism between A and $\varphi_i(A)$.

For $i < j$ we set $V_{ij} = F(A \times [t_i, t_j])$. We have some simple facts:

- $V_{ij} = V_{j-1\,j} \;\; \forall i < j$ and so we'll speak of V_j rather than of V_{ij}.
- V_j is a manifold, angular at $F_{t_j}(\partial A) \cup F_{t_{j-1}}(\partial A)$, whose boundary is formed by $C_j = (A_j \setminus A_{j-1}) \subset L_x$ and $B_j = f([t_{j-1}, t_j])$.
- F is injective on $A \times (t_{j-1}, t_j)$ in fact, otherwise, by considering two points $(a, s), (b, t) \in A \times (t_{j-1}, t_j)$ such that $F((a, s)) = F((b, t))$ and going back along the normal trajectories passing through these points, one contradicts the fact that the F_t turns only one time between t_i and t_{i+1}. Since F is open, we have that $V_j = F(A \times (t_{j-1}, t_j))$ is homeomorphic to $A \times [0, 1]/\sim$ where we set $(x, 0) \sim (\varphi_j(x), 1)$.
- The positive normals to $\mathcal{F}$ along C_j are all pointing inward V_j or all pointing outward V_j depending on the orientation.

We see now that each leaf of L_1 does not have closed transversals passing through it. Let γ be a transversal passing through a leaf $\bar{L} \subset L_1$. If necessary by changing γ in a neighborhood of $\bar{L}$, we can suppose that γ passes through a point $p \in f_1(\partial A)$ and that here it locally agrees with a normal trajectory.

If j is large enough, then γ enters into V_j passing through B_j (see figure 11).

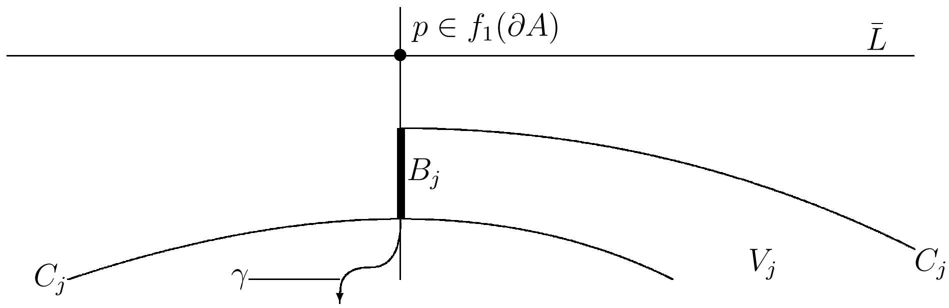


Figure 11: The transversal γ.

In order to be closed, γ must get out from V_j, but it can do this only by passing through B_j and therefore remaining in V_{j+1}. Since this happens $\forall\ i \geq j$, then γ cannot get out from $\bigcup_{i \geq j} V_i = \bigcup_j V_j$ and then it can't be closed. This argument works for every leaf in L_1. It follows that $W = L_1 \cup (\bigcup_j V_j)$ is a non-taut component whose boundary is L_1.

We see now that W is a GRC. First of all we show that each leaf in $\text{int}(W)$ is embedded. Let L be a leaf in $\text{int}(W)$. We have that $L = \cup_j L_j$ where L_j is a leaf in V_j and $L_{j+1} \supset L_j$. Fix now j. Each leaf in V_j is embedded and then there exists a transverse section $\Sigma \subset V_j$ such that $\Sigma \cap L_j = 1$ *point*. Suppose that $\Sigma \cap L_j = p$ and that $\Sigma \cap L_{j+1}$ contains also a point $q \neq p$. In V_j let $F_{t_p}(A)$ and $F_{t_q}(A)$ be the leaves containing p and q, say $t_p < t_q$, and let $F_s(A) = L_{j+1}$; $F_{t_p}(A)$ and $F_{t_q}(A)$ are both contained in $F_s(A)$.

Going back for $(t_p - t_{j-1})$ along the normal trajectories of F, $F_{t_p}(A)$ ends up in A_{j-1}, $F_s(A)$ in $F_{t_{j-1}+s-t_p}$ which so contains A_{j-1} and therefore $F_{t_{j-1}+s-t_p}$ and A_{j-1} must coincide (between t_{i-1} and t_i the trajectories F_t turn only one time). But now the presence of $F_{t_{j-1}+t_q-t_p} \subset A_j$ contradicts the injectivity of $F\big|_{A \times (t_{j-1}, t_j)}$. This argument does not depend on j, it follows that there exists Σ such that $\Sigma \cap L_j = 1$ *point* $\forall j$.

Suppose now that L is not embedded. Then $\Sigma \cap L$ must contain at least two points. Let α be a path in L which connects these points. Since $\text{int}(W) = \cup V_j$, there exists j such that $\alpha \subset V_j$, therefore $\Sigma \cap V_j =$ at least two points and this is a contradiction. It follows that L is embedded.

We see now that W is homeomorphic to $A \times S^1$ and that the leaves in $\text{int}(W)$ are homeomorphic to $\text{int}(A)$. $\varphi_i : A \mapsto \text{int}(A)$ is an embedding of A with its image and by the fact that $\varphi_{i*} : \pi_1(A) \mapsto \pi_1(A)$ is injective follows that $\varphi_i(A)$ is a retract of A (if A is a surface of genus g with k holes, φ_i must send holes around holes) and then $V_j \approx A \times [0,1]/\sim$ is homeomorphic to

$A \times S^1$.

Similarly, by injectivity of φ_{i*} it follows that each $L \subset \text{int}(W)$ is homeomorphic to $\text{int}(A)$ (remember $L = \bigcup_{s\ s.t. A_s \subset L} A_s$).

L_1 is a union of tori and, for i large enough, the normal projection $p : L_1 \setminus f_1(\partial A) \mapsto \partial V_i \setminus B_i = C_i$ is well defined and is an isomorphism (see figure 12).

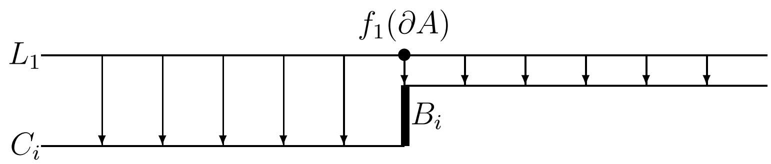


Figure 12: the projection p.

For i large the set M generated by the normal segments between L_1 and ∂V_i is homeomorphic to $L_1 \times [0,1]$ and then $W = M \cup V_i$ is homeomorphic to $A \times S^1$. Therefore W is a GRC and the proof of Proposition 2.3.3 is concluded.$\square$

With the following theorem we end the description of GRCs as an obstruction to displace surfaces. This result plays a fundamental role in the proof of Theorem 2.

Theorem 2.3.5 *Suppose that (V,$\mathcal{F}$) does not contain GRCs* [3]*. Let F be a function satisfying the conditions 1)-4) of Proposition 2.3.3. Then F admits a smooth extension at t=1 to an embedding F_1 of A into a leaf L_1 in such a way that $F_1\big|_{\partial A} \equiv f_1$.*

Proof. By the Proposition 2.3.3, F has a unique extension F_1 in the obvious way. Since A is connected, the image of F_1 is contained in only one leaf L_1. We only have to prove that F_1 is a smooth embedding.

Since F has maximum rank at every point and the trajectories F_t are normal to $\mathcal{F}$, also F_1 is smooth and it has maximum rank at every point and then the integer degree of F_1 ($= \sharp\{F_1^{-1}(point)\}$) is constant on the connected component of $L_1 \setminus f_1(\partial A)$ and changes by one passing through $f_1(\partial A)$. We complete the proof by showing that $\forall\ p \in f_1(\partial A)\ \ F_1^{-1}(p) = 1\, point$.

Let p be a point in $f_1(\partial A)$. Suppose now that there exists $x \in \text{int}(A)$ such that $F_1(x) = p$, since f_1 is an embedding $\exists!\ y \in \partial A$ such that $f(y) = p$. Consider now a foliated local chart on a neighborhood of p, there are two cases:

[3]For example if $\mathcal{F}$ is taut.

1) The normal trajectories γ_x and γ_y reach L_1 by the same side.

2) γ_x and γ_y reach L_1 by opposite sides.

Going back along the normal trajectories, the case 1) leads to contradicting the hypothesis 4) of Proposition 2.3.3; the case 2) cannot subsist because we have supposed that $\mathcal{F}$ is transversely orientable.□

3 The proof of Theorem 2

3.1 The strategy

Let $(V, \mathcal{F})$ be as before. This section is entirely dedicated to proving the following proposition:

Proposition 3.1.1 *Suppose $(V,\mathcal{F})$ does not contain GRCs* [4]. *Let S be an orientable connected closed surface of genus ≥ 1. Let $\varphi : S \mapsto V$ be an embedding such that:*

- $\varphi_* : \pi_1(S) \mapsto \pi_1(V)$ *is injective.*

- $\exists \ \{R^i\}_{i\in I}$ *where $R^i \neq D^2$ is a regular connected compact surface contained in S, $\partial R^i \neq \emptyset$, such that $j_* : \pi_1(R^i) \mapsto \pi_1(S)$ is injective (j is the inclusion map), and such that $\forall i \ \varphi(R^i)$ is contained in one leaf of $\mathcal{F}$ named L^i (we'll name such a R^i a* contact surface *or a* contact component*).*

- *On $S \setminus \bigcup_i R^i$ the foliation induced by $\mathcal{F}$(the pull-back foliation $\varphi^*(\mathcal{F})$) has only saddle singularities and $\varphi|_{S\setminus \bigcup_i R^i}$ is transverse to $\mathcal{F}$ except in the saddles (there are not any circle tangencies).*

- *φ is transverse to $\mathcal{F}$ on the boundaries ∂R^i.*

- *φ is smooth (like $\mathcal{F}$) except on $\bigcup_i \partial R^i$ where it is angular.*

- *The saddles lie on separate leaves (it is not restrictive to suppose this because we can always obtain it).*

[4]For example if $\mathcal{F}$ is taut.

- *(*)* *For all i there exists a neighborhood W^i of $\varphi(R^i)$ in V of type $U^i \times [-\varepsilon, \varepsilon]$, where U^i is a neighborhood of $\varphi(R^i)$ in L^i, such that, called $W^{i+} = U^i \times (0, \varepsilon]$ and $W^{i-} = U^i \times [-\varepsilon, 0)$, we have $\varphi(S) \cap W^{i+} = \emptyset$ or $\varphi(S) \cap W^{i-} = \emptyset$.*

Then φ is isotopic [5] to a φ' which satisfies the preceding conditions plus one of the following three:

- $\#\{(R^i)'\} \leq 1$
- $\#\{(R^i)'\} < \#\{R^i\}$
- $\#\{(R^i)'\} = \#\{R^i\}$ *and the number of saddles of $\varphi^*(\mathcal{F})$ on the transverse region is strictly smaller than the one of $(\varphi')^*(\mathcal{F})$.*

Remark. $-(\#\text{ saddles}) + \sum_i \chi(R^i) = \chi(S)$ and then I is a finite set.

Remark. If we consider the following order on the set of embeddings which satisfy the conditions of Proposition 3.1.1:

$$\varphi' \lhd \varphi \qquad \Longleftrightarrow \qquad \begin{cases} -\#\{(R^i)'\} < \#\{R^i\} \\ \text{or} \\ -\#\{(R^i)'\} = \#\{R^i\} \text{ and the number of saddles of } \varphi^*(\mathcal{F}) \text{ on the transverse region is strictly less than that of } (\varphi')^*(\mathcal{F}). \end{cases}$$

then the proposition says simply $\varphi' \lhd \varphi$ exists.

Lemma 3.1.2 *The Proposition 3.1.1 implies Theorem 2.*

Proof. Given a reduced embedding φ in the Roussarie sense (1.2.1), if φ does not have circle tangencies we have finished. If φ has some circle tangencies, by flattening out φ around such tangencies, we can find an isotopy from φ to a φ' which has "contact annuli" instead of the circle ones, and which satisfies the hypotheses of Proposition 3.1.1.

By applying the Proposition 3.1.1 and by $\lhd$-induction we find an isotopy from φ' to a $\tilde{\varphi}$ such that $\#\{\tilde{R}^i\} \leq 1$. If $\#\{\tilde{R}^i\} = 1$, then there is only one

[5] Clearly such an isotopy cannot be regular but it will be only C^0 on S and like $\mathcal{F}$on $S \setminus \bigcup_i \partial R^i$.

contact component and then the image of $\tilde{\varphi}$ is contained in only one leaf; if $\#\{\tilde{R}^i\} = 0$, then $\tilde{\varphi}$ is transverse to $\mathcal{F}$ except at $-\chi(S)$ saddles. □

The idea of the proof of Proposition 3.1.1 is to try to make φ adhere to the leaves as much as possible and to examine the obstruction when we must stop.

We start by displacing the contact components along the route fixed by φ; when it is not possible to go on, and Lemma 3.2.1 will say when and why, we define some *standard* local moves by which we'll be able either to go on with the displacement (we'll see that in this case we eliminate one saddle by incorporating it with the contact surface), or to eliminate some contact components by making them transverse to $\mathcal{F}$ (Proposition 3.3.3).

Notation: If there are not ambiguities, we'll name S both S and $\varphi(S)$, R^i both R^i and $\varphi(R^i)$ and, sometimes, we'll omit the index i.

3.2 Displacement of contact surfaces

In order to define and apply the standard moves, we must firstly prove some technical lemmas which provide for the displacement of contact surfaces as far as it is possible. These lemmas also classify the situations in which we cannot go on with the displacement and they fundamentally use the Theorem 2.3.5.

Every saddle q has a foliated neighborhood M_q of type $[-1,1]^3$ (parametrized by (x,y,z)) in which the leaves are the planes $\{z = const.\}$ and $\varphi(S) \cap M_q = \{z = x^2 - y^2\}$. In the following we'll suppose a Riemannian metric is fixed on V such that the manifold $\overline{\varphi(S \setminus \bigcup_i R^i)}$ is normal to $\mathcal{F}$ except in neighborhoods $N_q \supset M_q$ of type $[-1-\varepsilon, 1+\varepsilon]^3$ [6].

Lemma 3.2.1 *In the hypotheses of Proposition 3.1.1, for all i there exist $\tau \geq 0$ and a smooth map $F^i : [0,\tau] \times R^i \mapsto V$ such that:*

i) Setting $\forall\, t\ \ F^i_t(x) = F^i(t,x)$, F^i_0 is an embedding onto R^i and F^i_t is an embedding into a leaf L_t for $0 < t < \tau$.

ii) $f^i = F^i_{|_{[0,\tau[\times\partial R^i}}$ is an embedding into $\overline{\varphi(S \setminus R^i)}$ and, for every $x \in \partial R^i$, the trajectories $t \to f^i(t,x)$ are normal to the traces $\mathcal{F} \cap \varphi(S)$ for the metric induced by V on $\varphi(S)$, if $\tau \neq 0$.

[6] Plus the conditions fixed in section 2 at ∂V.

iii) *The map F^i is of one of the following types:*

α) *F^i_τ is an embedding. $\forall x \in R^i$ the trajectories $t \to F_t(x)$ are normal to $\mathcal{F}$ and F^i has maximum rank at every point (if $\tau \neq 0$). The holonomy below $R^i_\tau = F^i_\tau(R^i) \subset L_\tau$ is not trivial (see figure 13).*

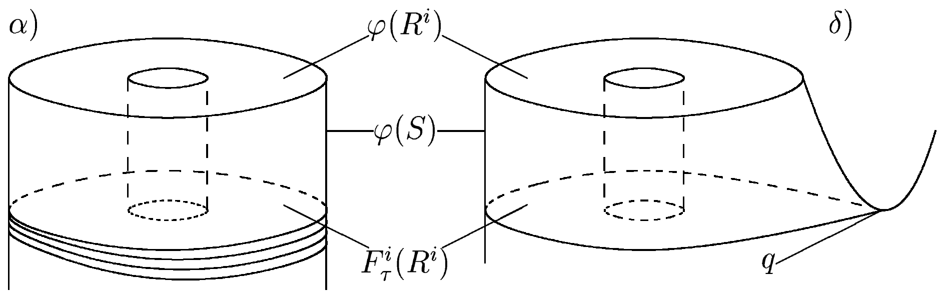


Figure 13: Cases α) and δ).

δ) *$\tau > 0$. $\forall x \in \mathrm{int}(R)$ the trajectories $t \to F^i_t(x)$ are transverse to $\mathcal{F}$. F^i_τ is an immersion, it is injective on $\mathrm{int}(R^i)$ and there exists a saddle q such that $q \in F^i_\tau(\partial R^i)$. F^i has maximum rank except in q, $\forall x \in R$ the trajectories $t \to F^i_t(x)$ are normal to $\mathcal{F}$ except in the neighborhood N_q (see figure 13).*

η) *$\tau > 0$. F^i_τ is an embedding and $j \neq i$ exists such that $\partial R^j \cap F^i_\tau(\partial R^i) \neq \emptyset$. F^i has maximum rank at every point and the trajectories $t \to F^i_t(x)$ are normal to $\mathcal{F}$ (see figure 14).*

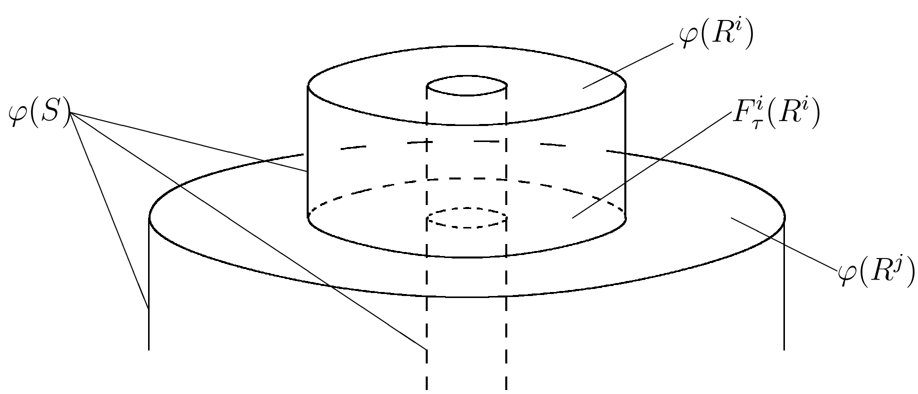


Figure 14: Case η)

Proof. For every i, let us consider the set $\mathcal{K}$ of maps $\Phi : [0,\nu] \times R \mapsto V$ [7] with $\nu \geq 0$, satisfying the conditions i) and ii) of the proposition plus one of the following three:

[7] Read: $\Phi^i : [0,\nu^i] \times R^i \mapsto V$

α') The trajectories $t \to \Phi_t(x)$ are transverse to $\mathcal{F}$ and normal to $\mathcal{F}$ except, if by chance, in the neighborhoods N_q; Φ has maximum rank at every point, if $\nu > 0$. The holonomy below $\Phi_\nu(R)$ is not required to be non trivial.

δ') $\nu > 0$. Φ_ν is an immersion, it is injective on $\text{int}(R)$ and there exists a saddle q such that $q \in \Phi_\nu(\partial R)$. Φ has maximum rank except in q. The trajectories $t \to \Phi_t(x)$ are normal to $\mathcal{F}$ $\forall x \in R$ except in the neighborhood N_q. In N_q the trajectories $t \to \Phi_t(x)$ are transverse to $\mathcal{F}$ for all $x \in R$ and $t \neq \nu$.

η) Like in the proposition.

$\mathcal{K}$ is not empty because it always contains the embedding $R \mapsto R$ ($\nu = 0$). We order $\mathcal{K}$ by the relation

$$\Phi < \Psi \iff \text{Imm}(\Phi) \subset \text{Imm}(\Psi).$$

We'll use the Zorn lemma to obtain a maximal map Φ. Such a Φ will be clearly of type α', δ' or η. If it is of type η, we have finished. If it is of type α', then the holonomy below Φ_ν must be trivial otherwise we can extend the definition interval and obtain so a $\tilde{\Phi} > \Phi$, that contradicts the maximality of Φ which then is of type α. Finally if Φ is of type δ', we can modify it by an isotopy with support in N_q in such a way as to obtain a $\bar{\Phi}$ of type δ.

A maximal Φ of type α or η provides the request F^i, otherwise the F^i is given by $\bar{\Phi}$.

Remark. In the case in which $\nu = 0$ and the maximal Φ is of type α', the hypothesis (*) of Proposition 3.1.1 is essential to conclude that Φ is of type α.

We'll see now that $\mathcal{K}$ is inductive. Let $\{\Phi_i\}_{i\in\mathbb{N}}$ be a sequence of elements of $\mathcal{K}$ with $\Phi_i < \Phi_{i+1}$; since the maps of type δ' and η are clearly maximal for $<$, Φ_i is of type α' for all i.

If necessary by reparameterizing the maps Φ_i, we can suppose that for all i we have

- $\nu_i < \nu_{i+1}$ ($[0, \nu_i]$ is the definition interval of Φ_i)
- $\Phi_{i+1}\big|_{[0,\nu_i]\times R} \equiv \Phi_i$.

We can now define
$$\varphi_i^k = \Phi_i\big|_{\text{k−th component of } \partial R}$$

The images of the φ_i^k are annuli, called S_i^k, embedded in $\overline{\varphi(S \setminus R)}$; $\mathcal{F} \cap S_i^k$ is made up by circles parameterized, via φ_i^k, by $\{t\} \times$(k-th component of ∂R).

Moreover we have

$$S_i^k \subset S_{i+1}^k, \qquad\qquad S_i^k \cap S_i^l = \emptyset \quad \text{if } l \neq k$$

Let $S^k = \overline{\bigcup_i S_i^k}$. If necessary by reparameterizing the maps Φ_i, we can suppose that $\nu_i \to \nu > 0$ and therefore we can define:

- A map $\tilde{\varphi} : [0, \nu] \times \partial R \mapsto V$ such that $\tilde{\varphi}\big|_{[0,\nu_i]\times\partial R} = \Phi_i\big|_{[0,\nu_i]\times\partial R}$ and $\tilde{\varphi}([0, \nu] \times \partial R) = \bigcup_k S^k$
- A map $\Phi : [0, \nu[\times R \mapsto V$ such that $\Phi\big|_{[0,\nu_i]} = \Phi_i$.

There are two cases:

1) $S^k \cap S^l = \emptyset \ \ \forall\, l \neq k$

2) $\exists\, k \neq l$ s.t. $S^k \cap S^l \neq \emptyset$

CASE 1)

Case 1) has two subcases:

1.1) The trajectories $t \to \tilde{\varphi}(t, x)$ are all transverse to $\mathcal{F}$ up to ν included.

1.2) A point $x \in \partial R$ exists whose trajectory $\tilde{\varphi}(t, x)$ becomes tangental to $\mathcal{F}$ for $t = \nu$.

In case 1.1) we directly [8] apply Theorem 2.3.5 to the map Φ and extend it at $t = \nu$, obtaining in this way an upper bound of type α' or η for the maps Φ_i.

Remark. If C^i is a connected component of ∂R^i, $\tilde{\varphi}(C^i)$ is a leaf of $\mathcal{F} \cap \varphi(S)$. In this case $\tilde{\varphi}$ does not touch the singularities of $\mathcal{F} \cap \varphi(S)$ and so, if we are in the case η and $\tilde{\varphi}(C^i) \cap \partial R^j \neq \emptyset$, then $\tilde{\varphi}(C^i) \subset \partial R^j$.

[8]It may be a saddle q exists for which $Imm(\Phi) \cap M_q \neq \emptyset$ and so the trajectories $t \to \Phi_t$ are transverse but not normal to $\mathcal{F}$; if so, we can modify the metric in M_q to make them normal to $\mathcal{F}$ and then apply the Theorem 2.3.5. Observe that in this case the extension of Φ cannot touch more contact components, since $\forall q\, j\ M_q \cap R^j = \emptyset$, and then it must be of type α'.

Case 1.2).
Since the foliation $\mathcal{F}$ is tangental to $\varphi(S)$ only in saddle points, $\tilde{\varphi}(t,x)$ converges, for $t \to \nu$, to a saddle q. Moreover, since the trajectories $t \to \tilde{\varphi}(t,y)$ are normal to the traces $\mathcal{F} \cap \varphi(S)$, there is at most one point $y \in \partial R$, $y \neq x$ such that the trajectory $\tilde{\varphi}(t,y)$ converges to q. Finally, since the saddles lie on different leaves (by the hypothesis of Proposition 3.1.1), there are no other points of ∂R whose trajectories become tangental to $\mathcal{F}$ for $t \to \nu$.

Observe that Φ satisfies the hypotheses of Theorem 2.3.5 except in the neighborhood N_q in which the trajectories $t \to \Phi_t(x)$ are not normal to $\mathcal{F}$. Suppose that Φ comes into M_q from the top and that $y \neq x$ exists with $\tilde{\varphi}(t,y)$ converging to q; i.e. that $Imm(\tilde{\varphi}) \cap M_q = \{(x,y,z) \in [-1,1]^3 \ s.t. \ z = x^2 - y^2 \ and \ z \geq 0\}$ [9].

We can modify Φ near q, by an isotopy with support contained in a neighborhood $N_q \supset M_q$, and obtain a Φ' such that the trajectories $t \to \Phi'_t(x)$ are normal to $\mathcal{F}$ and in such a way that Φ' satisfies all the hypotheses of Theorem 2.3.5.

It follows that Φ' admits an extension at $t = \nu$. We can now modify, near q, the extension of Φ' to obtain an extension of Φ which is of type δ'. The extended Φ is an upper bound for the maps Φ_i.

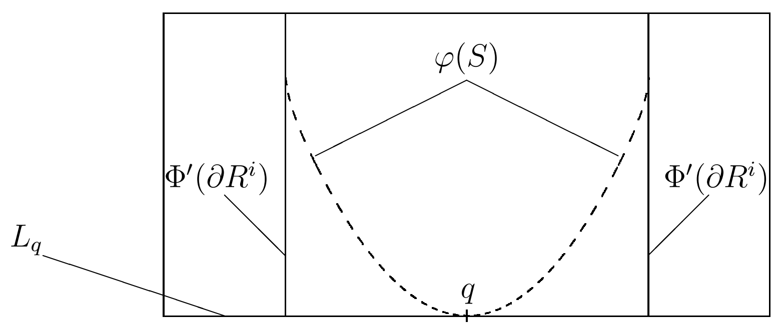


Figure 15: A possible section of the neighborhood N_q ($\{y = 0$ and $x \geq 0\}$).

CASE 2)

Observe that since $\mathcal{F}$ is transversely orientable, in this case the trajectories $t \to \tilde{\varphi}(t,x)$ cannot be all transverse to $\mathcal{F}$ up to ν included.

Therefore, in case 2), there exists $x \in \partial R$ whose trajectory $\tilde{\varphi}(t,x)$ becomes tangental to $\mathcal{F}$ at $t = \nu$.

[9]The hypothesis that Φ comes from the top is not restrictive.
If $y \neq x$ does not exist with $\tilde{\varphi}(t,y)$ converging to q, then $Imm(\tilde{\varphi}) \cap M_q = \{(x,y,z) \in [-1,1]^3 \ s.t. \ z = x^2 - y^2$, $z \geq 0$ and $x \geq 0\}$ and the proof works mutatis mutandis.

Then we proceed exactly as in case 1.2) and we find an upper bound of type δ' for the maps Φ_i.

Therefore $\mathcal{K}$ is inductive and the lemma is so proved. $\square$

Remark. In the case in which the map F^i is of type δ, we can suppose that the holonomy below $F^i_\tau(R)$ is trivial. In fact, above it is trivial since we come from the top with F^i and , if the holonomy below $F^i_\tau(R)$ is not trivial, it suffices to push up a little the saddle q and we can make this by an isotopy with support contained in M_q and in such a way as not to modify the previous conditions.

In the following we'll suppose that, in the case δ, the holonomy of $F^i_\tau(R)$ is trivial.

After applying the Lemma 3.2.1 to a contact component R, we reparametrize the map F in such a way that $\tau = 1$ (if $\tau \neq 0$).

In the cases α and η we have the maps $F : R \times [0,1] \mapsto V$ which are immersions and embeddings locally in t, i.e. each $t \in [0,1]$ has a neighborhood U such that $F|_{R\times U}$ is an embedding.

In the case δ we have two cases:

- f_1 is injective on ∂R
- f_1 is not injective on ∂R.

It follows from the proof of Lemma 3.2.1 that if f_1 is injective, then the map F is an immersion and an embedding locally in t, and one $x \in \partial R$ exists such that $f_t(x)$ converges to a saddle q (i.e. $f_1(X) = q$). If f_1 is not injective, there exists x and $y \neq x \in \partial R$ such that $f_1(x) = f_1(y) = q$. We'll consider now what happens in this case.

Lemma 3.2.2 *In the case δ $\nexists\ z \in \mathrm{int}(R)$ such that $F_1(z) = q$.*

Proof. Suppose on the contrary that $F_1(z) = q$ with $z \in \mathrm{int}(R)$. Remember that for all t $F_t(R)$ is contained in the same leaf L_t and that the trajectories $t \to F_t(x)$ are transverse to $\mathcal{F}$ for all $x \in \mathrm{int}(R)$. Let us suppose that F comes from the top in M_q.

If the trajectory $F_t(z) \cap M_q$ is contained in $\{(x,y,z) \in M_q$ such that $z < x^2 - y^2\}$, then it cannot be transverse to $\mathcal{F}$ (in effect it becomes tangental to $\mathcal{F}$ in q); then $F_t(z) \cap M_q \subset \{(x,y,z) \in M_q\ s.t.\ z > x^2 - y^2\}$.

It follows that f_1 is not injective and that, if x and y are the points of ∂R for which $f_1(x) = f_1(y)$, for t near to 1 an embedded path $\xi : [0,1] \mapsto F_t(R)$ exists from $F_t(x)$ to $F_t(y)$ contained in $L_t \cap M_q$ (see figure 16).

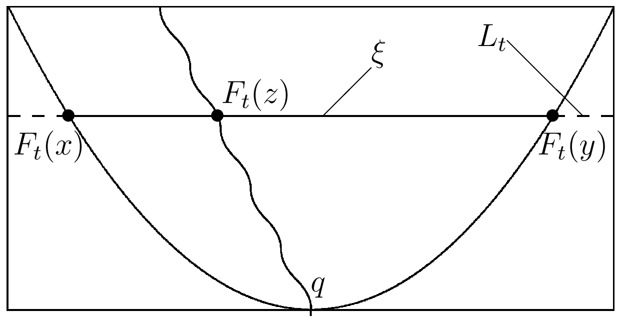


Figure 16: How the path ξ may be in the section $\{y = 0 \text{ and } x \geq 0\}$ of a neighborhood N_q.

The loop $\xi \cup (\bigcup_{s>t} F(s,x) \cup F(s,y))$ is contractible in V; by going back along the trajectories F_s passing through ξ, we can find a loop β embedded in $\varphi(S)$ which contains the trajectories passing through x and y, and which is contractible in V.

Since $\varphi_* : \pi_1(S) \mapsto \pi_1(V)$ is injective, β is contractible also in $\varphi(S)$ and then it is the boundary of a D^2 embedded in $\varphi(S)$.

In the induced foliation $\varphi^*\mathcal{F}$, the orbit through q is an "eight" (contained in $F_1(\partial R)$) and, since the trajectories $f_t(x)$ are normal to the traces $\mathcal{F} \cap \varphi(S)$ and since $\mathcal{F}$ is transversely orientable, one of the two lobes of the eight is contained in D^2 (see figure 17).

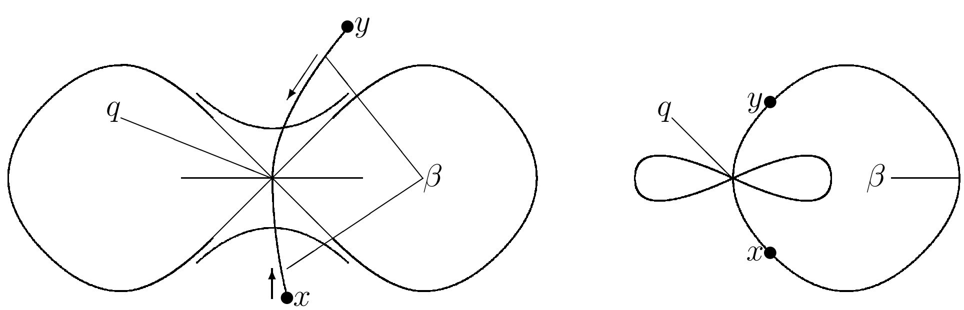


Figure 17: In S the leaf of q is an "eight".

It follows that such a lobe is contractible in S and then it is the boundary of a disc D in S (see figure 18).

Observe that, by the hypotheses of Proposition 3.1.1, for all i $j_* : \pi_1(R^i) \mapsto \pi_1(S)$ is injective (j is the inclusion map) and then in D there are no contact components.

The vector field positively normal to $\mathcal{F} \cap \varphi(S)$ for the metric induced by V on $\varphi(S)$ is transverse to ∂D [10] and so, by the Hopf theorem (see for example [M]),

$$1 = \chi(D) = \sum \text{ index of the vector field}$$

and then in D there are singularities of positive index (i.e. of type maximum or minimum) but this is a contradiction because, by hypotheses of Proposition 3.1.1, the only possible singularities of $\mathcal{F} \cap \varphi(S)$ are of saddle type and then they ave index -1. □

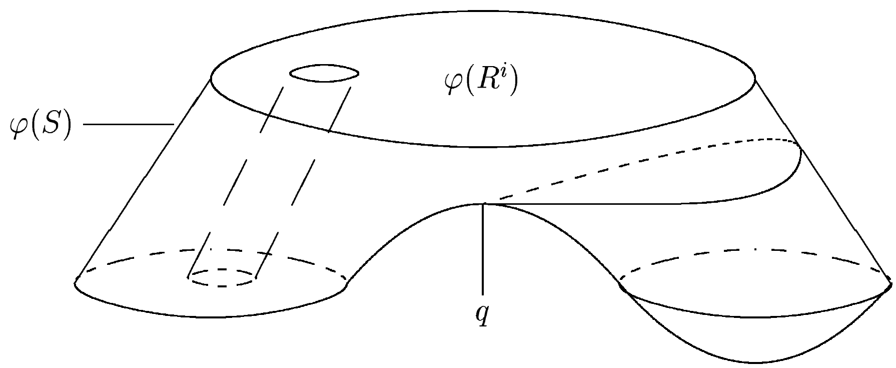


Figure 18: A lobe of the saddle is contractible. This cannot be.

As a corollary of Lemma 3.2.2 we have that, in case δ, if f_1 is not injective F_1 is an embedding of $R/x \sim y$ and F is an immersion, and an embedding locally in t, of $R \times [0,1]/(x,1) \sim (y,1)$.

We'll name the cases δ:

$$\delta_1 \text{ if } f_1 \text{ is injective} \quad , \quad \delta_2 \text{ if } f_1 \text{ is not injective.}$$

Remark. As we had observed in the proof of Lemma 2.3.4, for all i if $t < s$ and $F_t^i(R) \cap F_s^i(R) \neq \emptyset$ then $F_t^i(R) \subset F_s^i(R) \neq \emptyset$.

3.3 The moves

Definition 3.3.1 We say that F^i is a *primitive* immersion, or that R^i is a *primitive* component, iff

$$F^i(R \times [0,1]) \cap \varphi(S) = F^i(\partial R \times [0,1]) \cup F_0^i(R^i)$$

[10] If one observes that at q the vector field is not defined, we modify the disc D a little near q so as to obtain a smooth disc for which the positively normal vector field is defined and transverse to the boundary.

Lemma 3.3.2 *If F is primitive then φ is isotopic to an embedding φ' which coincides with φ out of $\varphi^{-1}(F(R\times[0,1]))$ and which maps $\varphi^{-1}(F(R\times[0,1]))$ on $F_1(R)$.*

Proof. By using F, which is an embedding locally in t, we find a $t>0$ and an isotopy between φ and a φ' in such a way that φ' coincides with φ out of $\varphi^{-1}(F(R\times[0,t[))$ and it maps $\varphi^{-1}(F(R\times[0,t[))$ on $F_t(R)$.

Let now

$$s=\sup\{t\in[0,1] \text{ such that } \ \forall\,\tau<t\ \varphi \text{ is isotopic to such a } \varphi' \text{ as above}\}$$

If $s\neq 1$, by using a neighborhood $U\subset[0,1]$ of s for which $F\big|_{R\times U}$ is an embedding, we contradict the maximality of s. The thesis follows. $\square$

Proposition 3.3.3 *Let φ be an embedding that satisfies the hypotheses of Proposition 3.1.1. If φ has a primitive component R, then the Proposition 3.1.1 is valid for φ.*

Proof. The proof of this proposition consists of the exhaustive analysis of the possible cases. For each of them we'll define a move which eliminates the component R or incorporates a saddle with R.

CASE δ)

Remember we have supposed that the holonomy below $F_1(R)$ is trivial. Since R is a compact orientable surface, it is diffeomorphic to a sphere with k holes and l handles, since $\partial R\neq\emptyset$ then $k>0$.

Let $A\subset\mathbb{R}^2$ be a smooth compact surface obtained from a disc by removing $k-1+2l$ little discs; R is diffeomorphic to $A/\!\sim$ where $\sim$ is the equivalence relation which identifies two by two the boundaries of $2l$ discs in such a way as to obtain the l handles (see figure 19).

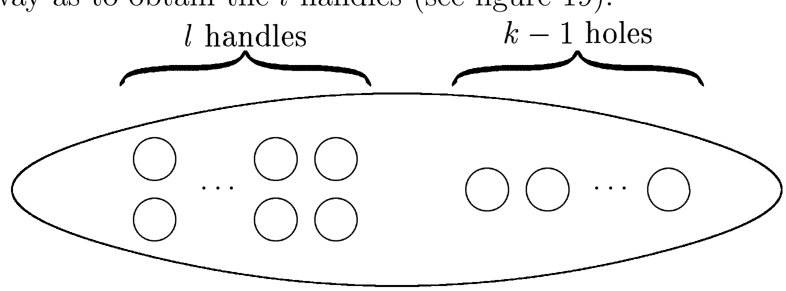


Figure 19: The surface A.

Case δ_1)

Let us now consider a surface, which we name A too, obtained from A by "pinching" it; i.e. by creating a cusp on the boundary of the first disc (the bigger one) in such a way that$F_1(R)$ is diffeomorphic to $A/\sim$ (the new A). Let us fix a diffeomorphism $G : F_1(R) \leftrightarrow A/\sim$. If there are no ambiguities we'll name q both the saddle q and $G(q)$ (the cup).

Named (γ_1^i, γ_0^i) $i = 1, \ldots, l$ the pair of discs identified by $\sim$ to make the handles, for $\xi > 0$ small enough let be:

$B_1 = \{x \in \mathbb{R}^2 \text{ such that } dist(x, A) < \xi\}$,
$B_2 = \{x \in \mathbb{R}^2 \setminus A \text{ such that } dist(x, \gamma_j^i) < \xi \text{ for some } 1 \leq i \leq l,\ j = 0, 1\}$,
$M = B_1 \setminus B_2$,
$H = M \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3$

Let U be in V a foliated neighborhood of $F_1(R)$ diffeomorphic to $H/\sim$ [11], by a diffeomorphism ψ, such that on $F_1(R)$ ψ coincides with G, i.e. $\psi(F_1(R)) = (G(F_1(R)), 0)$. Such a neighborhood exists since the holonomy of $F_1(R)$ is trivial.

The idea is to incline $F_1(R)$, by an isotopy, in such a way as to make it transverse to $\mathcal{F}$ except, clearly, at the right number of saddles (see figure 20).

We'll define the inclining isotopy in H and, by composing with ψ, we'll obtain the isotopy looked for.

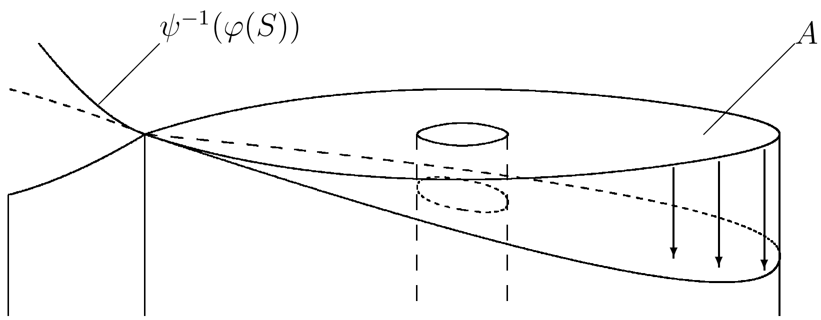


Figure 20: The isotopy which inclines A in H.

For $j = 0, 1$, $i = 1, \ldots, l$ and $\xi > 0$, we define the annuli

$$C_j^i = \{x \in A \text{ such that } dist(x, \gamma_j^i) \leq \xi\}$$

If ξ is small enough, $C_j^i \cap C_n^m = \emptyset$ when $(i, j) \neq (m, n)$. Moreover, if necessary by changing the local model of A, we can suppose that for all i we have

$$dist(q, C_0^i) > dist(q, C_1^i).$$

[11] We understand that $\mathbb{R}^3$ is foliated by planes parallel to $\mathbb{R}^2$. Clearly such a foliation respects $\sim$.

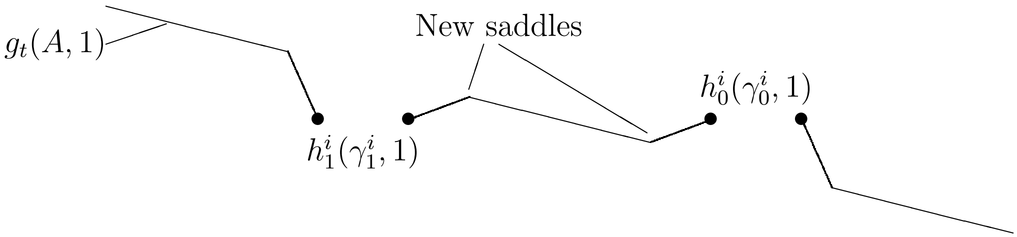


Figure 21: The results of h_j^i. There are two new saddles for each handle.

For all i let τ_i be fixed such that $dist(q, C_0^i) > \tau_i > dist(q, C_1^i)$.

Now we are ready to define the isotopy. For $t \in [0,1]$ let be:

$$g_t : (A \setminus \bigcup_{i,j} C_j^i) \mapsto H$$
$$g_t(x) = (\lambda(x), -dist(q,x) \cdot at)$$

where λ is a scale map of A which keeps ∂A on $\psi^{-1}(\varphi(S))$, and $a > 0$ is such that $a \cdot \sup_{x \in A}\{dist(q,x)\} < \varepsilon$.

For all $j = 0,1 \quad i = 1,\ldots,l$ let be (see figure 21)

$$h_j^i : (C_j^i) \times [0,1] \mapsto H$$
$$h_j^i(x,t) = (x, -at(\tau_i(1 - \frac{dist(x,\gamma_j^i)}{\xi}) + dist(x,q)\frac{dist(x,\gamma_j^i)}{\xi}))$$

We can glue the maps g and h since they coincide along the sets ∂C_j^i and, by the definition of the maps h, the result respects $\sim$ and is a $\mathcal{C}^0$-isotopy which makes $A/\!\sim$ transverse to the foliation on $H/\!\sim$ [12] except at the right number of saddles.

At this point it is easy to find an isotopy r which inclines also $\psi^{-1}(\varphi(S) \setminus F(R \times [0,1]))$ and which links well to the g and the h.

The angular points, which are present after we had applied the isotopy, are easily smoothable and the composition with the diffeomorphism ψ provides, with the Lemma 3.3.2, an isotopy of φ which eliminates the contact component R by making it transverse to $\mathcal{F}$.

In the case δ_1 the Proposition 3.3.3 is thus proved.

Remark. By the elimination of R, we eliminate also q but a saddle appears

[12]The maps h_j^i are $\mathcal{C}^0$ and make some angular circles appear along the handles of A, but it is not hard to find some smooth h and so we can speak of transversality.

for each little disc of A; the number of saddles on the transverse region of φ is so increased by $-\chi(R)$.

Case δ_2)

In this case we prove the Proposition 3.3.3 by finding an isotopy which incorporates the saddle q in the contact component R.

As in the case δ_1, we construct a standard local model in R^3 in which we'll work.

In the following, we'll name x_1 and x_2 the two different points of ∂R for which $f_1(x_1) = f_1(x_2) = q$.

Let P be a pant, embedded in $[1-,1]^3$ foliated by the horizontal parallel planes ($\{(x,y,z) \in [-1,1]^3$ such that $z = const.\}$), with the saddle at level 0, as in the figure 22. Let $A \subset [-1,1]^3 \cap \{z = 0\}$ be a neighborhood of $P \cap \{z = 0\}$ such that $A \times [-1,1]$ is a neighborhood of P and let $\partial A = b_1 \cup b_2$ as in the figure 22.

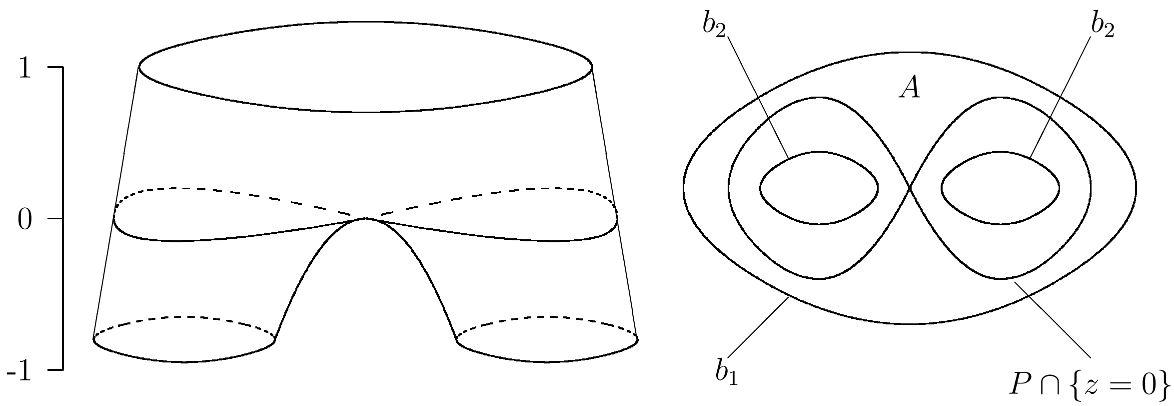


Figure 22: The pant P and the neighborhood A

As we have seen in the proof of Lemma 3.2.2, the leaf l_q of $\mathcal{F} \cap \varphi(S)$ which passes through q is an eight contained in $F_1(\partial R)$.

If we set $A' = \{p \in [-1,1]^3$ such that $z = 0$ and $dist(p, A) < \varepsilon\}$, for ε small enough, and by using the trivial holonomy below $F_1(R)$, we can find a foliated neighborhood U of l_q in V which is diffeomorphic to $A' \times [-1,1]$ via a diffeomorphism ψ, such that $\psi(\varphi(S \setminus R)) = P$.

If x_1 and x_2 lie on the same connected component of ∂R, then we are coming from the top, i.e. $\psi(F(R \times [-1,1])) \subset \{z \geq 0\}$. In this case $F_1(R)$ lies outside the pant, i.e. $\psi(F(R_1) \setminus l_q)$ is connected. In fact, if on the contrary it lies inside P, we have the contradiction that a lobe of l_q is contractible.

If x_1 and x_2 lie on different connected components of ∂R, then we are coming from the bottom, i.e. $\psi(F(R \times [-1,1])) \subset \{z \leq 0\}$ and in this case $F_1(R)$ lies inside the pant, i.e. $\psi(F(R_1) \setminus l_q)$ is not connected.

In $A \times [-1,1]$ it is easy to find an isotopy which sends P into P', where P' is the union of the three following sets (see figure 23):

$$A, \qquad b_1 \times [0,1], \qquad b_2 \times [1-,0].$$

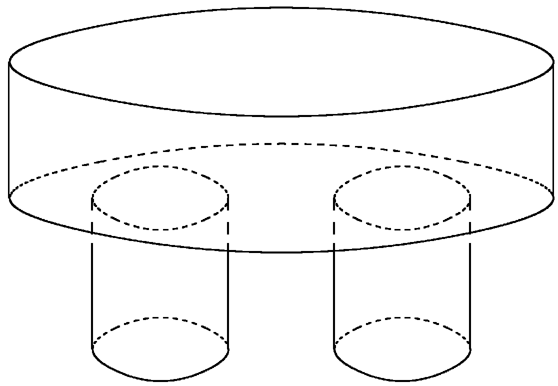


Figure 23: The pant P'

By composing by ψ^{-1}, we find an isotopy of φ in φ' such that $\varphi = \varphi'$ out of U and the new map $F' : R \times [0,1] \mapsto V$, coincident with F out of U, satisfies the conditions $i)$ and $ii)$ of Lemma 3.2.1 plus the following:

> F'_1 is an embedding, F' has maximum rank at every point and the trajectories $t \to F'_t(x)$ are normal to $\mathcal{F}$.

This suffices to apply the Lemma 3.3.2. At this point we glue in the obvious way $F'_1(R)$ to $\psi^{-1}(A)$ and we obtain a new contact component $\tilde{R}$. Such a gluing provides the attachment of a pant to R which so incorporates the saddle q.

The inclusion map $i_* : \pi_1(\tilde{R}) \mapsto \pi_1(S)$ remains injective since the lobes l_q cannot be contractible.

Therefore φ' verifies the conditions of Proposition 3.1.1, $\#\{(R^i)'\} = \#\{(R^i)\}$ and the number of saddles on the transverse region of φ' is smaller than the one of φ. Then, also in the case δ_2 the Proposition 3.3.3 is proved.

CASE η)

As we have remarked in the proof of Lemma 3.2.1, if C is a connected component of ∂R^i such that $F_1(C) \cap \partial R^j \neq \emptyset$, then $F_1(C) \subset \partial R^j$. After the Lemma 3.3.2, the gluing of $F_1(R^i)$ to R^j is immediate.

Let $T = F_1(R^i) \cup R^j$. If T is a contact component which satisfies the conditions of Proposition 3.1.1, then the new embedding obtained by Lemma 3.3.2 has one contact component less than φ.

It is easy to see that we only have to check the following conditions:

j) $i_* : \pi_1(T) \mapsto \pi_1(S)$ is injective.

jj) The condition $(*)$ of Proposition 3.1.1

j). The elements of $\pi_1(T)$ which are elements of $\pi_1(R^i)$ or of $\pi_1(R^j)$ are not contractible in S since the inclusion maps $\pi_1(R^i) \mapsto \pi_1(S)$ and $\pi_1(R^j) \mapsto \pi_1(S)$ are both injective; in particular each component of ∂T, as an element of $\pi_1(S)$, is not zero.

Let now $[\gamma] \in \pi_1(T)$ be a regular path homotopically trivial in S; γ disconnects S, and therefore T, in two regions one of which, called D, is contractible to a point. From the above, D cannot contain components of ∂T and so γ is homotopically zero also in T.

jj). If $(*)$ is satisfied, then we have finished. If $(*)$ is not satisfied, then we'll make T transverse to $\mathcal{F}$ as follows.

We share out the components of ∂T in:

- $\mathcal{C}^+ = \{$ components C such that there exists a neighborhood $H \subset S$ of C such that $\varphi(H) \cap W^- = \emptyset\}$
- $\mathcal{C}^- = \{$ components C such that there exists a neighborhood $H \subset S$ of C such that $\varphi(H) \cap W^+ = \emptyset\}$

Note that $\mathcal{C}^+ \subset F_1(\partial R^i)$ and $\mathcal{C}^- \subset \partial R^j$ or vice versa.

As in case δ_1, we can construct a local model of type $A \times [-1, 1]$ with A, diffeomorphic to T, obtained from a disc by removing some little discs; in the model we can find an isotopy, like the maps g and h, which pushes up $\mathcal{C}^+$ and pushes down $\mathcal{C}^-$, and so making T transverse to $\mathcal{F}$ except at $-(\chi(R^i) + \chi(R^j))$ saddles (see figure 24).

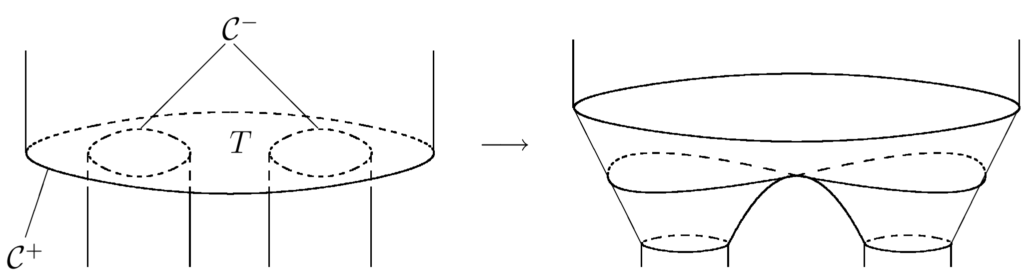


Figure 24: The isotopy in a local model.

The result of the isotopy is an embedding of S which satisfies the conditions of Proposition 3.1.1 and which has two contact components less than φ. Therefore also in case η the Proposition 3.3.3 is proved.

CASE α)

First of all, if $\tau \neq 0$, we can apply the Lemma 3.3.2 to displace R and therefore we'll suppose $\tau = 0$.

Let γ be an S^1 embedded in R such that the holonomy below γ is not trivial and let C be a neighborhood of γ in R homeomorphic to $S^1 \times [-1, 1]$.

Near R, the transversality of $\mathcal{F}$ is the same as a foliation with trivial holonomy; in the sense that, considering a neighborhood of R in V of type $W = U \times (-\varepsilon, \varepsilon)$, where U is a neighborhood of R in L_1, and the foliation $\mathcal{E}$ on W with horizontal leaves ($\{(u, s)$ such that $s = const\}$), if we can make φ transverse to $\mathcal{E}$ in W, then for ε small enough, we can make φ transverse also to $\mathcal{F}$.

In particular, as in the case η, we can find an isotopy of φ which pushes down $R \setminus C$ and which fix C.

It follows that in case α we only have to study the case in which R is an annulus. In the following we suppose R is an annulus.

Let N be the fence normal to γ of length ε. N is a cylinder and then the induced foliation $\mathcal{J} = \mathcal{F} \cap N$ consists of leaves homeomorphic to S^1 and leaves homeomorphic to $\mathbb{R}$. Moreover, since the holonomy below γ is not trivial, for each $\varepsilon \geq 0$ there exist leaves homeomorphic to $\mathbb{R}$ which pass near γ more than ε. Finally, we can parameterize N by $S^1 \times (-\varepsilon, \varepsilon)$ in such a way that the tangent line to the leaves $\sim \mathbb{R}$ is never horizontal.

If necessary by changing the annulus C, we can find a neighborhood H of C of type $C' \times (-\varepsilon, \varepsilon)$, where C' is an annulus in R which contains C, in which the foliation induced by $\mathcal{F}$ is $\mathcal{J} \times (-1, 1)$ (we reparameterize C' by $S^1 \times (-1, 1)$).

Moreover we can find such a neighborhood in such a way that $\varphi(S) \cap H = S^1 \times [-\frac{1}{2}, \frac{1}{2}] \times 0 \cup S^1 \times \{-\frac{1}{2}\} \times [0, \varepsilon) \cup S^1 \times \{\frac{1}{2}\} \times [0, \varepsilon)$

Let us consider an isotopy in H, from φ to a φ', which displaces C in a region foliated by planes; the tangent planes of the foliation $\mathcal{J} \times (-1, 1)$ are never horizontal and it follows that φ' is transverse to $\mathcal{F}$.

So also case α is proved and the proof of Proposition 3.3.3 is completed. $\square$

We complete now the proof of Proposition 3.1.1 by the following lemma

Lemma 3.3.4 *In the hypotheses of Proposition 3.1.1, once we had applied the Lemma 3.2.1, up to isotopy, φ has some primitive components.*

Proof. Let R^i be a contact component. If R^i is not primitive, then $F^i(R^i \times [0, 1]) \cap \varphi(S)$ contains some other contact components.

Let $\bar{R}$ be a component which is minimal by the inclusion in $F^i(R^i \times [0,1])$, i.e. by the relation $R^j < R^k$ if and only if

$$F^j(R^j \times [0,1]) \cap F^i(R^i \times [0,1]) \subset F^k(R^k \times [0,1]) \cap F^i(R^i \times [0,1]).$$

As it is minimal, $\bar{R}$ is relatively primitive, i.e.

$$\bar{F}(\bar{R}\times[0,1])\cap\varphi(S)\cap F^i(R^i\times[0,1]) = (\bar{F}(\partial\bar{R}\times[0,1])\cap F^i(R^i\times[0,1]))\cup\bar{F}_0(\bar{R}).$$

If $\bar{R}$ is primitive, we have finished. If not, then the holonomy below $F^i_1(R^i) \cap \bar{F}(\bar{R} \times [0,1])$ is trivial. We can therefore apply the techniques of the proof of Lemma 3.3.2 to $\bar{F}$ and so we can find an isotopy of φ which displaces $\bar{R}$ out of $F^i(R \times [0,1])$.

So, by induction on the number of contact components in $F^i(R \times [0,1])$, we find a primitive component (if the worst comes to the worst we make R^i primitive). □

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