Relations between the complex structure of Hurwitz spaces, the cut-and-paste topology, and the complex structure induced by a BPS

Disclaimer: These notes were written in order to not forget a discussion I had with Gabriele Mondello about movings of singular points in $2k\pi$-$\mathbb{CP}^1$-structures, during the preparation of [bps] with Gabriel Calsamiglia and Bertrand Deroin.

Let S be a closed oriented surface of genus at least two. Any branched $\mathbb{CP}^1$-structure on S induces an underlying complex structure on S. We aim to understand the behavior of the complex structure under moving branch-points along twin paths, as defined in Section 3 of our paper. The first natural question is whether or not moving branch-points actually changes the complex structure. If so, we wonder how. The complex structure of the Teichmüller space of S is the one induced on its (co-)tangent bundle by the space of (holomorphic quadratic) beltrami differentials. Therefore, in order to answer the above questions we need to carefully look at the beltrami differentials arising by moving branch-points.

1. Complex structure of Hurwitz spaces in terms of moving branch-points

Hurwitz spaces are defined as follows (directly from [bps]).

Let V be a closed disc, equipped with a Riemann surface structure, and a degree-d covering $\varphi : S^1 \to \partial V$. We consider the set of degree-d smooth branched coverings $\psi : U \to V$, from a closed disc U to V, with no critical value on the boundary, together with an identification $f : S^1 \to \partial U$ such that $\psi \circ f = \varphi$. Two such coverings $\psi_i : U_i \to V$, $i = 1, 2$, are identified if there is a diffeomorphism $\phi : U_1 \to U_2$ such that $\psi_1 = \psi_2 \circ \phi$ and $f_2 = \phi \circ f_1$. The set of equivalence classes under this equivalence relation is denoted by $\mathcal{H}_d(V)$. There is a map $\mathrm{Crit} : \mathcal{H}_d(V) \to Symm^{d-1}(V)$, mapping the equivalence class $[\psi]$ in $\mathcal{H}_d(V)$ to the set of critical values of ψ in V (by Riemann-Hurwitz formula, there are $d-1$ critical values counted with multiplicity). The proof of following Lemma is a standard argument in complex analysis (and can be obtained by "capping" ψ with a covering $z \to z^d$ so that to obtain a branched covering from $\mathbb{CP}^1$ to itself of degree d and of type z^d near ∞).

Lemma 1.1. *Any $\mathcal{H}_d$ is a smooth complex manifold of dimension $d-1$, such that the map* Crit *is a branched holomorphic covering. More precisely, $\mathcal{H}_d$ is biholomorphic tho the space of complex polynomials of degree $d-1$ with roots in the unit disc.*

Thus, the complex structure of an Hurwitz space $\mathcal{H}_d(V)$ is the one induced by the map Crit. The tangent space at a point σ of $\mathcal{H}_d(V)$ is the space of equivalent classes of deformations of σ, that is to say, smooth paths in $\mathcal{H}_d(V)$ through σ under the equivalence relation of first order approximation at σ. These paths are identified with smooth trajectories of the images of branch-points under the map Crit.

An element $\sigma \in \mathcal{H}_d(V)$ is a BPS on the disc, according with our terminology. The movement of a branch-point p of σ along a pair of twin paths γ_1 and γ_2, gives a path γ in $\mathcal{H}_d(V)$, hence a tangent vector $[\gamma]$ at σ. if J denotes the the complex structure of $\mathcal{H}_d(V)$, the vector $J([\gamma])$ is the equivalence class of the path obtained by moving p along twin paths that form an angle of $\pi/2$ with the γ_i's.

The space $\mathcal{M}_{k,\rho}$ of BPS on S with $2k$ branch-points (counted with multiplicity) and holonomy representation ρ, is locally parameterized by a product of Hurwitz space. Therefore, the complex structure of $\mathcal{M}_{k,\rho}$, in terms of moving branch-points, consists in rotating by $\pi/2$ the twin paths used for the movings.

2. Remind: The (co-)tangent space of Teichmüller space

Here we briefly recall some facts about the complex structure of Teichmüller space. The reference is [[ahl] Ahlfors: "Some remarks on Teichmüller's space of Riemann surfaces" Ann. of Math 74 (1) 1961].

Let Σ be a closed surface. A marked Riemann surface is a pair (S, φ) where S is a surface homeomorphic to Σ endowed with a complex structure, and $\varphi : \Sigma \to S$ is an homeomorphism. Two marked Riemann surfaces (S_i, φ_i) $i = 1, 2$ are equivalent if there is a biholomorphism $f : S_1 \to S_2$ in the homotopy class of $\varphi_2 \circ \varphi_1^{-1}$.

A map $f : S \to S$ is holomorphic if $f_{\bar{z}} = 0$. In general we can write

$$f_{\bar{z}} = \mu f_z.$$

For such a formula making sense, the map f must be weakly differentiable with $f_z \neq 0$ a.e., and μ must be a **beltrami differential**, something that in local coordinates is written as

$$\mu \frac{d\bar{z}}{dz}.$$

In other words a anti-holomorphic 1-form valued in the space of holomorphic vector fields.

Locally, if we define the μ-metric as

$$ds_\mu = |dz + \mu d\bar{z}|$$

then a function satisfying $f_{\bar{z}} = \mu f_z$ maps **μ-balls to round balls**.

Since μ is not a function, its value is not intrinsic, i.e. it is not invariant under changes of coordinates (and in particular is well-defined being zero or not). However, its modulo $|\mu|$ is invariant. Therefore, it makes sense to speak of $||\mu||_\infty$.

Theorem 2.1 (Morrey, Ahlfors-Bers). *Let μ be a beltrami differential on S with $||\mu||_\infty < 1$. Then there exists a unique homeomorphism $f^\mu : S \to S$ so that*

- *f^μ has weak L^2 derivatives,*
- *$f_{\bar{z}} = \mu f_z$,*
- *$f^\mu_z \neq 0$ almost everywhere,*
- *f^μ maps nullsets on nullsets.*
- *f^μ is in the homotopy class of the identity.*

Moreover, if $\mu(t)$ is a analytic family of beltrami differentials, then $f^{\mu(t)}(z)$ depends analytically on t for all t and the partial derivatives on $t, z, \bar{z}$ commute.

It follows that every point of Teichmüller space has a neighborhood parameterized by the unit ball of the space of beltrami differentials, which is a complex spaces, whence the complex structure on Teichmüller.

Let μ_t be an analytic family of beltrami differentials with $||\mu_t||_\infty < 1$, and $\mu_0 = 0$. We define

$$f^t = f^{\mu_t}.$$

We denote by $\dot{f}^t(z)$ the derivative on t of $f(z)$, and f^t_z the derivative on z **in local coordinates**. Similarly, we denote by $\dot{\mu}_t$ the derivative on t of μ_t, that is to say

$$\dot{\mu}_t = \lim_{s \to 0} \frac{\mu_{t+s} - \mu_s}{s}.$$

We use the notation

$$\dot{f} = \dot{f}^0$$

Lemma 2.2 (Ahlfors). $\dot{f}_{\bar{z}} = \dot{\mu}_0$

Therefore, we can compute the tangent vector of a deformation in terms of variation of quasi-conformal homeomorphisms. As we said, a beltrami differential has no well-defined point-set value.

However, as we are interested in complex structure, we need only to know, given two beltrami differential μ and ν, whether or not $\mu = J(\nu)$, where J is the complex structure of the space of beltrami differentials.

For that we use the duality with **holomorphic quadratic differentials**. They are section of the double tensor product of the space of holomorphic 1-forms, and in coordinates are written as

$$\phi dz^2.$$

The pairing between beltrami differentials and quadratic holomorphic differentials is given by

$$\langle \mu, \phi \rangle = \int_S \phi \mu.$$

note that $\phi dz^2 \mu \frac{d\bar{z}}{dz} = \phi \mu dz d\bar{z}$ which is a genuine 2-form on S and therefore the integral makes sense.

3. Beltrami differential associated to the movement of a branch-point

Let S be a closed oriented surface endowed with a BPS σ. Let J be the complex structure underlying σ.

Associated to a movement of a branch point there is a beltrami differential with compact support. For simplicity, let us consider just the case of moving one branch-point of order two. Let $p \in S$ be the branch-point we are going to move. Let U be a closed disc embedded in S, containint p in its interior and containing no other branch-points. Let W be an open disc containing U. We set Ⓢ $= S \setminus U$. See Figure 1.

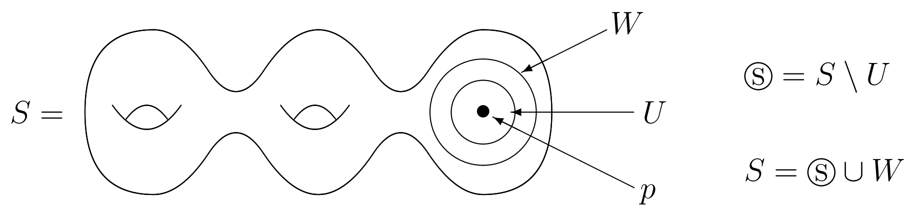

Figure 1. $S, U, W,$ Ⓢ

The developing map gives a branched covering from U to a disc V. We uniformise both U and V with the unit disc so that the branched covering in local coordinates is $z \mapsto z^2$. Thus $p = 0 \in U$.

Let $U_\varepsilon = B(0, \varepsilon) \subset U$ and $V_\varepsilon = B(0, \varepsilon) \subset V$. We choose a path $\gamma : [0, 1] \to V_\varepsilon$ so that $\gamma(0) = 0$ and $\gamma'(0) \neq 0$. Wlog, we can suppose that γ is an embedding and that $\gamma'(0) = 1$. See Figure 2. We now move p along the twin paths pre-image of γ. The structure on Ⓢ remains unchanged, but in W we need to choose accurately a new atlas.

For $t \in [0, 1]$, let U^t_ε denote the preimage of V_ε under the ramified covering

$$z \mapsto z^2 + \gamma(t)$$

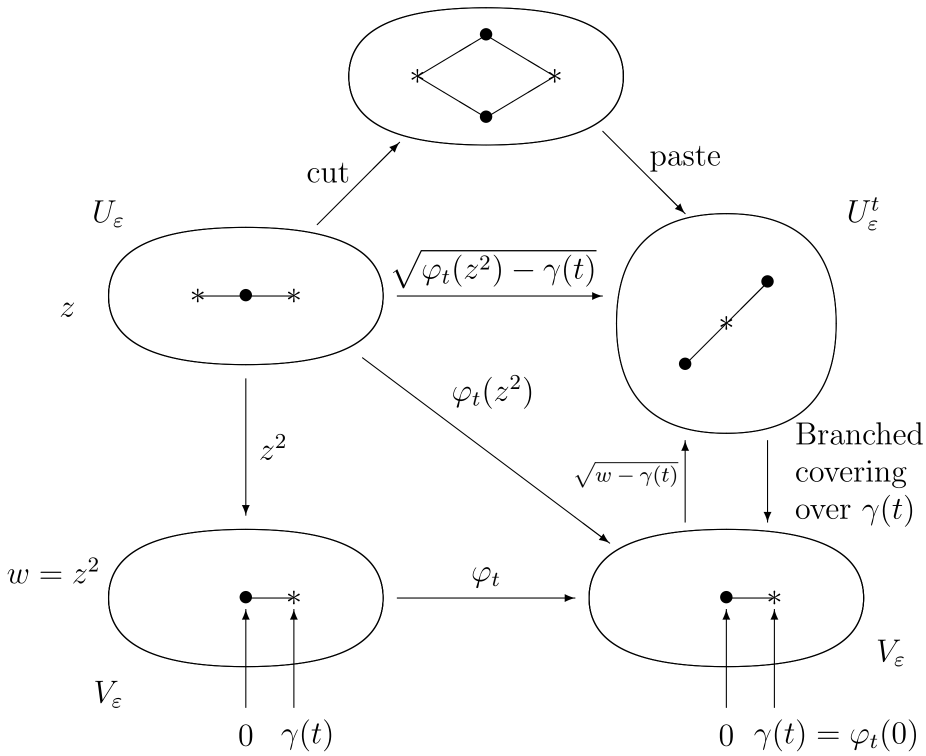

FIGURE 2. Changes of coordinates in U when moving the branch-point

which is ramified on $\gamma(t)$. In Figure 2, zero is the $\bullet$-point and $\gamma(t)$ is the $*$-point. (Note that $U_\varepsilon^0 = U_\varepsilon$.) For ε small enough, U_ε^t is properly contained in U for all $t \in [0, 1]$. See Figure 3.

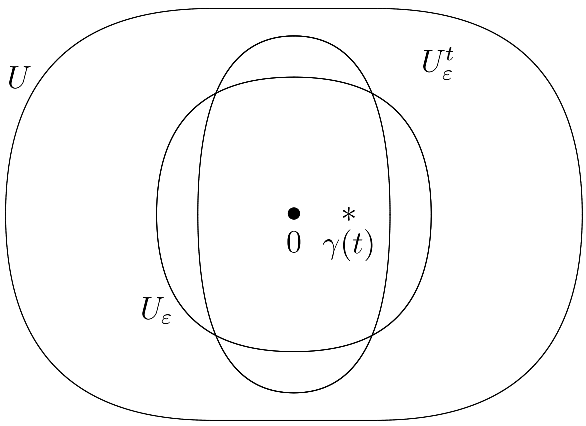

FIGURE 3. The sets U, U_ε, and U_ε^t.

Let $\varphi_t : V_\varepsilon \to V_\varepsilon$ be a diffeomorphism so that:

- $\varphi_0 = Id$;
- φ_t has compact support for every t (is the identity outside a compact);
- $\varphi_t(0) = \gamma(t)$;
- there is $r < \varepsilon$ so that $\dot\varphi_0(z) = 1$ for $z \in U_r$ (where $\dot\varphi_t(z) = \lim_{s\to 0} \frac{\varphi_{t+s}(z)-\varphi_t(z)}{s}$)
- $\varphi_t(z)$ depends analytically on t.

Now, let $f^t : U_\varepsilon \to U^t_\varepsilon$ be defined by

$$f^t(z) = \sqrt{\varphi_t(z^2) - \gamma(t)}$$

being the square-root well defined because φ_t sends 0 to $\gamma(t)$. Extend now f^t to a diffeomorphism

$$f^t : U \to U$$

so that f^t is the identity on a neighborhood of ∂U.

A developing map for the structure σ_t obtained by moving p along $\gamma|_{[0,t]}$ is obtained by replacing the local chart in U with

$$f^t(z)^2 + \gamma(t),$$

which reduces to $\varphi_t(z^2)$ on U_ε.

If J is the complex structure underlying σ and J_t is the one underlying σ_t, then the map $(f^t)^{-1}$ sends J-round balls to J_t-round balls. Therefore, the map f^t — which maps J_t-round balls to J rounds balls — is the one we are looking for (if we want to use Theorem 2.1 and Lemma 2.2).

Note that since the map f^t is the identity near ∂U, it extends to a map

$$f^t : S \to S$$

which is the identity outside U. So both f^t and $(f^t)^{-1}$ make sense globally. But we remark that Lemma 2.2 holds in local coordinates only, because globally $\dot f$ is not well-defined.

If μ_t is the sequence of beltrami differentials so that f^t satisfies $f^t_{\bar z} = \mu_t f^t_z$ (in the terminology of previous section this translates to $f^t = f^{\mu_t}$), then Lemma 2.2 says that $\dot\mu_0 = \dot f_{\bar z}$.

Near ∂U we have $\dot f^t = 0$, and in U_ε we have

$$\dot f^t(z) = \frac{d}{dt}\left(f^t(z)\right) = \frac{d}{dt}\left(\sqrt{\varphi_t(z^2) - \gamma(t)}\right) = \frac{\dot\varphi_t(z^2) - \dot\gamma(t)}{2\sqrt{\varphi_t(z^2) - \gamma(t)}}$$

for $t = 0$ we have

$$\dot f(z) = \dot f^0(z) = \frac{\dot\varphi_0(z^2) - \dot\gamma(0)}{2\sqrt{\varphi_0(z^2) - \gamma(0)}} = \frac{\dot\varphi_0(z^2) - 1}{2z}$$

and then

$$\dot\mu_0 = \dot f_{\bar z} = \left(\frac{\dot\varphi_0(z^2)-1}{2z}\right)_{\bar z} = \left(\frac{\dot\varphi_0(z^2)}{2z}\right)_{\bar z} - \left(\frac{1}{2z}\right)_{\bar z} = \left(\frac{\dot\varphi_0(z^2)}{2z}\right)_{\bar z}$$

4. Contraction of $\dot\mu_0$ with quadratic differentials

Let ϕ be a holomorphic quadratic differential on S. We want to compute

$$\langle \dot\mu_0, \phi\rangle.$$

The first observation is that f^t is the identity outside U and in a neighborhood of ∂U, so $\dot\mu_0$ has support contained in the interior of U.

In the local coordinates of previous section we have

$$\langle \dot\mu_0, \phi\rangle = \int_S \phi\cdot\dot\mu_0 d\bar z dz = \int_U \phi\cdot\dot\mu_0 d\bar z dz = \int_U \phi\cdot\left(\frac{\dot\varphi_0(z^2)}{2z}\right)_{\bar z} d\bar z dz$$

now, since ϕ is holomorphic, for $z\neq 0$ we have

$$\phi\cdot\left(\frac{\dot\varphi_0(z^2)}{2z}\right)_{\bar z} d\bar z dz = d\left(\phi\cdot\frac{\dot\varphi_0(z^2)}{2z}dz\right)$$

and by Stokes we get

$$\langle \dot\mu_0, \phi\rangle = \int_{U_r} \phi\cdot\left(\frac{\dot\varphi_0(z^2)}{2z}\right)_{\bar z} d\bar z dz + \int_{\partial U}\phi\cdot\frac{\dot\varphi_0(z^2)}{2z}dz - \int_{\partial U_r}\phi\cdot\frac{\dot\varphi_0(z^2)}{2z}dz.$$

The term $\int_{\partial U}$ vanishes because $\dot\varphi_t = 0$ on ∂U. The term $\int_{U_r}$ vanishes because $\dot\varphi_0 = 1$ on U_r and $1/2z$ is holomorphic. Thus by Cauchy formula we get

$$\langle \dot\mu_0, \phi\rangle = -\pi i\phi(0)\cdot\dot\varphi_0(0) = -\pi i\phi(0) \tag{1}$$

Remark 4.1. *One could conclude that $\phi(0)$ is therefore well-defined by the above formula. In other words, one may conclude that the pointwise value of a quadratic differential is well-defined. This is not true, as Lemma 2.2 holds only in local coordinates. In other words, if we change coordinates we get a different value, depending on $\dot\varphi_0(0)$. What is intrinsic is the product $\dot\varphi_0(0)\cdot\phi(0)$.*

5. Consequence 1: the map Teich is holomorphic

The first consequence of our calculation is that the map

$$\text{Teich} : \mathcal{M}_{k,\rho} \to \mathcal{T}eich(S)$$

is holomorphic. Indeed, if we change γ with $J(\gamma)$, in our local coordinates this reduces to the condition

$$\dot\varphi_0(z) = i \ \forall z\in U_r$$

hence the contraction with any quadratic holomorphic differential of the new beltrami differential is multiplied by i. So, if we use the terminology $\dot{\mu}_0(\gamma)$ we have

$$\dot{\mu}_0(J(\gamma)) = i\dot{\mu}_0(\gamma).$$

6. Consequence 2: moving 1 branch-point changes the complex structure

Moving a single branch-point actually change the structure. Indeed, to show that $\dot{\mu}_0 \neq 0$ it suffices to prove the existence of a holomorphic quadratic differential ϕ so that $\phi(p) \neq 0$. Such a condition is intrinsic (if $\phi(p) \neq 0$ is some coordinates, so is in any coordinates). And this is easily constructed.

7. Consequence 3: moving up to $2g-2$ branch-points change the complex structure

Formula (1) applied to multiples movings of points $p_1, \dots, p_n$ would give

$$\langle \dot{\mu}_0, \phi \rangle = -\pi i \sum_i \phi(p_i).$$

Unfortunately, such a formula makes no-sense if we are not working in local coordinates because $\phi(p_i)$ is not intrinsically defined. Nonetheless, using that formula we can prove that

Theorem 7.1. *Any moving of $n < 2g-2$ (strict inequality) points (where g is the genus of S) actually changes the underlying complex structure.*

Proof. Let $n < 2g-2$ and let $p_1, \dots p_n$ be the n branch-points that we are going to move. Let ϕ_t^i be the local diffeomorphism around the point p_i used by computing the beltrami differentials as in previous sections. If we are doing a non-trivial movement, we can arrange things so that it is non-trivial at first order approximation, namely, that some of the $\dot{\phi}_0^i(0) \neq 0$ for some i in local coordinates, say $i = 1$ for simplicity. If we are able to find a holomorphic quadratic differential ϕ so that

$$\phi(p_1) \neq 0, \qquad \phi(p_i) = 0 \, \forall i > 1,$$

then we are done. Indeed, to be 0 or not at a point, is an intrinsic property of ϕ, and in this case Formula (1) gives

$$\langle \dot{\mu}_0, \phi \rangle = -\pi i \phi(p_1) \neq 0$$

showing that the vector $\dot{\mu}_0$ tangent to our variation is not null, so the structure actually changes.

We are thus reduced to prove the following fact.

Lemma 7.2. *Given points $p_1, \dots, p_n \in S$ with $n < 2g-2$, there exists a holomorphic quadratic differential ϕ so that $\phi(p_1) \neq 0$ and $\phi(p_i) = 0$ $\forall i > 0$.*

Proof. We need Riemann-Roch Theorem. Here some remind about that (more for me than for you, the source is the Kapovich note [kap]):

- **Divisors on** S, multiplicative notation: $A = \Pi_{p \in S} p^{\alpha(p)}$, where $\alpha : S \to \mathbb{Z}$ is not zero only at finitely many points.
- **Degree** of a divisor A: $\deg(A) = \sum_p \alpha(p)$.
- **Divisor of a meromorphic function**: $\operatorname{div}(f) = \sum_p p^{\operatorname{ord}_p(f)}$ where $\operatorname{ord}(p)$ is the first non-zero term in the local Laurent expansion of f.
- **Divisor of a meromorphic (q-)differential**: the same definition as as for f. (q means quadratic.)
- **Group omomorphism**: $\operatorname{div}(\omega\eta) = \operatorname{div}(\omega)\operatorname{div}(\eta)$.
- **Principal divisors**: A is principal if $A = \operatorname{div}(f)$ for some meromorphic function f.
- **Degree of a function**: $\deg(\operatorname{div}(f)) = 0$ for any merom. f.
- **Canonical divisors**: A is canonical if $A = \operatorname{div}(\omega)$ for some nontrivial meromorphic differential ω. A is q-canonical if $A = \operatorname{div}(\phi)$ for some non-trivial quadratic differential ϕ.
- **Divisor class group**: is the quotient of the group of divisors on S by the sub-group of principal divisors.
- **Degree of a divisor class**: The degree is well defined because if $A = B\operatorname{div}(f)$ then $\deg(A) = \deg(B) + 0$.
- **Canonical divisor class**: Any canonical divisor defines the same class because for any ω, η meromorphic differentials we have that ω/η is a meromorphic function. Such class is denoted by K. The same argument shows that the q-canonical class is well defined and it is K^2.
- **Polar and zero divisors**: The polar divisor of f is $f^{-1}\infty$ with orders of multiplicity (considered positive), the zero is $f^{-1}(0)$, with orders of multiplicity. With this notation, $\operatorname{div}(f) = f^{-1}(0)/f^{-1}(\infty)$ so the polar and the zero divisor of f are in the same divisor class.
- **Order**: we say $A > B$ if $\alpha(p) > \beta(p)$ for all $p \in S$.
- **Effective divisors**: A is called integral (or effective) if $A \geq 1$, and strictly integral if $A > 1$.
- **Space** L: $L(A) = \{\text{meromorphic functions } f : \operatorname{div}(f) \geq A\}$.
- **Space** Ω: $\Omega(A) = \{\text{meromorphic differentials } \omega : \operatorname{div}(\omega) \geq A\}$.
- **Space** Ω_q: $\Omega_q(A) = \{\text{meromorphic quadratic differentials } \phi : \operatorname{div}(\phi) \geq A\}$.

- **Basic important fact 1**: $L(1) = \mathbb{C}$.
- **Basic important fact 2**: If $\deg(A) > 0$ then $L(A) = \{0\}$.
- **Basic important fact 3**: For any non-trivial meromorphic differential ω,
$$\dim(\Omega(A)) = \dim(L(A/\operatorname{div}(\omega))).$$
- **Basic important fact 4**: For any non-trivial meromorphic differential ω,
$$\dim(\Omega_q(A)) = \dim(\Omega(A/\operatorname{div}(\omega))) = \dim(L(A/\operatorname{div}(\omega)^2)).$$
- **Prove of facts 3 and 4**: The map $\eta \mapsto \eta/\omega$ gives the isomorphisms of the required spaces.
- **Non-trival** ω: Non-trivial meromorphic differentials exist.
- **Spaces L, Ω, Ω_q for classes**: although different spaces, if $A = B\operatorname{div}(f)$ then the spaces $L(A), \Omega(A), \Omega_q(A)$ are isomorphic to $L(B), \Omega(B), \Omega_q(B)$, so the dimensions are the same. Indeed, if $A = B\operatorname{div}(f)$ then the multiplication by f gives isomorphism between $L(A)$ and $L(B)$. Given ω a non-trivial meromorphic differential we have $\dim(\Omega(A)) = \dim(L(A/\operatorname{div}(\omega))) = \dim(L(B/\operatorname{div}(\omega))) = \dim(\Omega(B))$. Similarly for Ω_q.
- **Basic facts 3 and 4 reloaded for divisor classes**:
$$\dim(\Omega(A)) = \dim(L(A/K))$$
$$\dim(\Omega_q(A)) = \dim(\Omega(A/K)) = \dim(L(A/K^2)).$$
- **Quadratic differentials with prescribed zeroes**: A holomorphic quadratic differential ϕ so that $\phi(p_1) = \cdots = \phi(p_n) = 0$ is an element of $\Omega_q(P)$, where $P = \Pi_{i=1}^n p_i^1$.
- **The Riemann-Roch Theorem**: If g is the genus of S and A any divisor, then:
$$\dim(L(A^{-1})) - \dim(\Omega(A)) = \deg(A) - g + 1$$
equivalently
$$\dim(L(A^{-1})) - \dim(L(A/K)) = \deg(A) - g + 1$$
or in additive notation
$$\dim(L(-A)) - \dim(L(A - K)) = \deg(A) - g + 1$$

Now we can prove Lemma 7.2. Let $P = \Pi_{i=1}^n p_i$. We have
$$\dim(\Omega_q(P)) = \dim(\Omega(P/K))$$
and by Riemann-Roch
$$\dim(L(K/P)) - \dim(\Omega(P/K)) = \deg(P/K) - g + 1$$

since $n < 2g - 2 = \deg(K)$, we have $\deg(K/P) > 0$ and by the Basic important fact 2 we get $\dim(L(K/P)) = 0$. Thus

$$\dim(\Omega_q(P)) = \dim(\Omega(P/K)) = g - 1 - \deg(P/K) = 3g - 3 - n > 0$$

In particular, if $Q = \Pi_{1=2}^{n} p_i^1$, then

$$\dim(\Omega_q(Q)) - \dim(\Omega(P)) = 1$$

That is to say, the space of holomorphic quadratic differentials that are null at $p_1, \ldots, p_n$ is a sub space of codimension one of the space of those that are null at $p_2, \ldots, p_n$. In particular there are plenty of ϕ so that $\phi(p_1) \neq 0$ but $\phi(p_2) = \cdots = \phi(p_n) = 0$. □

□

We remark that if $n = 2g - 2$, the lemma is no longer true. Indeed, consider $P = K$. In this case, with above notation, we find $\dim(\Omega_q(Q)) = \dim(\Omega_q(P))$. In particular, if the branch-points we want to move are the divisor of a meromorphic differential, then there are non-trivial movements that leave unchanged the complex structure underlying our BPS.

References

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