GEODESIC CURRENTS AND LENGTH COMPACTNESS FOR AUTOMORPHISMS OF FREE GROUPS

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Abstract. We prove a compactness theorem for automorphisms of free groups. Namely, we show that the set of automorphisms keeping bounded the length of the uniform current is compact (up to conjugation.) This implies that the spectrum of the length of the images of the uniform current is discrete, answering to a conjecture of I. Kapovich.

1. Introduction

The aim of this paper is to investigate the action of the automorphisms of a free group $F$ on the Cayley graph of $F$. In particular, we are interested to understand how automorphisms can stretch geodesics. One can define the length of an automorphism as the generic stretching factor (see [KKS05]) which is, roughly speaking, the average of the stretching ratios, made over all geodesics (Definition 3.7.) The length function on $\text{Aut}(F)$ is invariant under conjugation, so it descends to a length function on $\text{Out}(F)$. Our main result is the following

**Theorem 1.1** (Length compactness theorem). Let $\Phi_n$ be a sequence of automorphisms of $F$. Then, up to pass to subsequences, there exists a sequence $v_n \in F$ such that the automorphisms $\Psi_n$ defined by $x \mapsto v_n \Phi(x)v_n^{-1}$ satisfy one (and only one) of the following:

- $\Psi_n$ converges to an automorphism $\Phi$.
- $L(\Phi_n)$ goes to $\infty$.

Such a theorem can be formulated as follows:

The set of automorphisms of bounded length is compact up to conjugation. That is, For any $M$, the set $\{[\Phi] \in \text{Out}(F) : L([\Phi]) < M\}$ is finite.

All the work pivots on the fact that the Cayley graph of $F$ is an hyperbolic object. Therefore, a boundary at infinity $\partial F$ of $F$ is well defined, and encodes enough information about the dynamic of the
action of Aut\((F)\). The main idea is that length controls attractors: if \(\Phi_n\) is a sequence of automorphisms of bounded length then, up to conjugation, there are no attractors for the action of \(\Phi_n\) on \(\partial F\). Using this fact we prove that the sequence \(\Phi_n\) keeps bounded the cyclically reduced length of any element of \(F\), and this will be enough to conclude.

As a corollary of Theorem 1.1, we get the following result.

**Corollary 1.2.** The spectrum of the length function is discrete. That is, the set
\[
\{L(\Phi) : \Phi \in Aut(F)\}
\]
is a discrete subset of \(\mathbb{R}\).

Corollary 1.2 was conjectured to be true by I. Kapovich, inspired by computational evidences and partial results. For example, in [KKS05] V. Kaimanovich, I. Kapovich and P. Shupp, among other results, proved that an automorphism of length one must be simple (see below) and estimated the “first gap” of the length function.

A consequence of Corollary 1.2 is the following, that can be viewed as a **Ideal Whitehead Algorithm** (see [Kap05a, Conjecture 5.3].) Recall that, given a free basis \(\Sigma\) of \(F\), an automorphism \(\tau\) is simple if it is either a permutation of \(\Sigma\) or an inner automorphism, while it is a Whitehead automorphism of second kind if there is \(a \in \Sigma\) such that \(\tau(x) \in \{x, xa, a^{-1}x, a^{-1}xa\}\) for all \(x \in \Sigma\).

**Theorem 1.3.** Let \(\Phi \in Aut(F)\) be a non-simple automorphism. Then there exists a factorization
\[
\Phi = \tau_n \tau_{n-1} \cdots \tau_1 \sigma
\]
where \(n \geq 1\), the automorphism \(\sigma\) is simple, each \(\tau_i\) is a Whitehead automorphism of the second kind, and
\[
L(\tau_{i-1} \cdots \sigma) < L(\tau_i \tau_{i-1} \cdots \sigma) \quad i = 1, \cdots, n - 1.
\]

Let us spend some words on Theorem 1.3. The **automorphism problem** for a free group \(F\) asks, given two arbitrary elements \(u, v \in F\), whether there exists \(\Phi \in Aut(F)\) such that \(\Phi(u) = v\). In [Whi36] Whitehead gave an algorithm solving that problem. The first part of the algorithm is to reduce the lengths of \(u\) and \(v\) as much as possible via Whitehead automorphisms. Then, given two minimal elements one proves that they are in the same Aut\((F)\)-orbit if and only if between them there is a sequence of minimal elements, each one obtained from the preceding via a Whitehead automorphism. Roughly speaking, Theorem 1.3 is an **averaged** version of the first part of Whitehead algorithm. We refer the reader to [Kap05a] for a more detailed discussion on the matter. We only notice that, as our proof is “typically hyperbolic,”
one may aspect that it is adaptable to a more general setting, like the one of hyperbolic groups, for which the automorphism problem is still not completely solved.

The main tool we use, is the theory of geodesic currents, that is, $F$-invariant Borel measures on the space of geodesics lines of the Cayley graph of $F$. Geodesic currents where introduced by F. Bonahon in [Bon86] in the setting of hyperbolic manifolds, and turned out to be very useful in group theory (see for example [Mar95, Kap04, Kap05a]).

We also consider measures on the space of geodesic rays (that is, half geodesics) of the Cayley graph of $F$. This is the space of frequency measures (see also [Kap05b].)

Such spaces are in fact homeomorphic, but each one has peculiar characteristics that well-adapt to different situations, and we will jump from one to the other depending on the calculation we’ll need to do. Roughly speaking, the action of Aut($F$) on currents is “more natural”, while frequency measures are “more compact”.

Just a final remark on the terminology used through the paper. The space of automorphisms of a free group is discrete. Thus, compactness is equivalent to finiteness, and to say that a sequence converges is equivalent to say that it is finite (and whence eventually constant.) Nevertheless, we preferred to speak about compactness and convergence because we think that this is more appropriate with the spirit of the paper, in which we used “more hyperbolicity than discreteness” (Even if discreteness is necessary, for example in Corollary 1.2.)

The paper is structured as follows: Section 2 contains the notation used in the paper. In Section 3 we give the definition of geodesic current and of frequency measure, we prove some preliminary fact and we introduce the notion of length of automorphisms. In Section 4 we prove Theorem 1.1 and its corollaries. Finally, we collected in Appendix A some basic result about measures that can be helpful through the paper.

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2. Notation

For the remainder of the paper, we fix the following notation.

- $F$ is a free group of rank $k$, with a fixed free basis $\Sigma$. We set $A = \Sigma \cup \Sigma^{-1}$. Any element of $F$ corresponds to a unique freely reduced word in the alphabet $A$, that is, a word not containing subwords of the form $aa^{-1}$ with $a \in A$. We identify $F$ with the set of freely reduced words. We denote by 1 the neutral element of $F$ (the empty word.) A word $w$ is cyclically reduced if all the cyclic permutations of $w$ are freely reduced. For a freely reduced word, $|v|$ denotes its length, and $||v||$ denotes its cyclically reduced length that is, the length of the cyclically reduced word obtained by cyclically reducing $v$.
- We simply write Cayley graph by meaning the Cayley graph of $F$ corresponding to $A$. We denote by 1 the base point of the Cayley graph corresponding to 1 in $F$.
- $\partial F$ is the boundary at infinity of $F$, identified with the set of geodesic rays of the Cayley graph, that is, freely reduced, right-infinite words in the alphabet $A$. The boundary $\partial F$ is endowed with the Cantor-set topology. Namely, for each $v \in F$ we denote by Cyl($v$) the set of rays having $v$ as initial segment. We set Cyl(1) = $\partial F$. Then, a basis for the topology of $\partial F$ is given by $\{\text{Cyl}(v) : v \in F\}$.
- $\partial^2 F$ is the set $\{(x, y) \in (\partial F)^2 : x \neq y\}$. We identify $\partial^2 F$ with the set of oriented bi-infinite geodesics in the Cayley graph. We define the base-ball $B$ of $\partial^2 F$ as the set of geodesics passing through 1.
- For any $x \neq y \in F$, the cylinder Cyl([x, y]) is defined as the subset of $\partial^2 F$ of geodesics passing through the oriented segment joining $x$ and $y$ in the Cayley graph (with the correct orientation.) We set Cyl([x, x]) = $B$.
- We denote by $T:\partial F \to \partial F$ the shift operator deleting the first letter of a ray. It turns out that $T$ is a continuous map.
- Given a topological space $M$, we identify the space of Borel measures on $M$ with the dual of $C_0(M)$ (the space of continuous functions on $M$ with compact support) endowed with the weak-* topology. Namely, measures $\mu_i$ converge to $\mu$ if and only if $\int \varphi \, d\mu_i \to \int \varphi \, d\mu$ for all $\varphi \in C_0(M)$. If $\mu$ is a Borel measure on $M$, $N$ is a topological space, and $f : M \to N$ is a measurable, proper map, we denote by $f_*\mu$ the push-forward of $\mu$, that is the measure on $N$ such that $\int_N \varphi \, d(f_*\mu) = \int_M \varphi \circ f \, d\mu$ for all $\varphi \in C_0(M)$.
3. Definitions and preliminary facts

In this section, we define the spaces of geodesic currents and of frequency measures, and we show that such spaces are homeomorphic. We introduce the notion of length of a current, which is the analogous of the length of a cyclically reduced word. We define the uniform frequency measure and the uniform current, which we use to define the length of automorphisms.

First of all, in order to describe the action of Aut(F) on currents, we need the following classical result, whose proof can be found in [Coo87].

**Theorem 3.1.** Let $\Phi$ be an automorphism of $F$. Then, it induces a map, still named $\Phi$, on the Cayley graph of $F$. Moreover, $\Phi$ extends to a homeomorphism, still named $\Phi$, of $\partial F$.

Since $\Phi$ is a homeomorphisms of $\partial F$, the map $\Phi \times \Phi$ is continuous and proper on $\partial^2 F$. It follows that any automorphism $\Phi$ acts on the space of Borel measures on $\partial^2 F$ via $(\Phi \times \Phi)_\ast$. The inclusion of $F$ in Aut($F$) given by inner automorphisms induces an action of $F$ on the space of Borel measures on $\partial^2 F$. By abusing notation, if $\eta$ is a Borel measure, we will denote by $\Phi \eta$ the measure $(\Phi \times \Phi)_\ast \eta$. We can now give the definition of currents and frequency measures. Our definitions are a little different from those introduced in [Kap04, Kap05b], as we do not require measure to be normalized to probability measures. This is because the quantities we are interested in (lengths of automorphisms) depend on the total mass of the measures we work with.

**Definition 3.2** (Geodesic currents). The space of geodesic currents is the space of $F$-invariant positive Borel measures on $\partial^2 F$. The length $L(\eta)$ of a current $\eta$ is the measure $\eta(B)$ of the base-ball $B$ of $\partial^2 F$.

**Definition 3.3** (Frequency measures). The space of frequency measures is the set of $T$-invariant positive Borel measures on $\partial F$ ($T$-invariant means that $T_*\mu = \mu$.) The total mass of a measure $\mu$ is denoted by $||\mu||$. The unitary ball of the frequency measures, that is the set of probability $T$-invariant measures on $\partial F$, is also called in literature the frequency space of $F$ (see [Kap05b].)

In the following, the letter $\eta$ will be used principally for a current, and the letter $\mu$ for a frequency measure. We refer the reader to Appendix A for some basic facts about currents and measures.
If $\eta$ is a geodesic current, and $x, y \in F$, by $F$-invariance, the value $\eta(Cyl([x, y]))$ depends only on the label $x^{-1}y \in F$. The $F$-invariance of currents plays the role of $T$-invariance for frequency measures. With this in mind, we can construct an isomorphism between the space of geodesic currents and the one of frequency measures given by

$$\eta \leftrightarrow \mu \quad \text{if and only if} \quad \eta(Cyl([x, y])) = \mu(Cyl(x^{-1}y)).$$

More precisely, one can prove (see also [Kap05b, Kap05a])

**Theorem 3.4.** The map $\alpha$ from the space of frequency measures to the space of geodesic currents defined by

$$\alpha(\mu)(Cyl([x, y])) = \mu(Cyl(x^{-1}y))$$

for all $x, y \in F$, is a homeomorphisms with respect to the weak-* topologies. Moreover, under this correspondence, the total mass corresponds to length, that is

$$L(\alpha(\mu)) = ||\mu||.$$

**Proof.** We only sketch the proof. The fact that $\alpha$ is well-defined and bijective can be easily proved using $F$- and $T$-invariance. The weak-* continuity follows from Proposition A.1, while the last claim is a straightforward computation.

The identification between currents and frequency measures induces an action of $\text{Aut}(F)$ on the frequency measures given by

$$\Phi \mu = \alpha^{-1} \circ (\Phi \times \Phi)_* \circ \alpha \mu.$$

Note that the action on frequency measures is not simply the push-forward via $\Phi$ because the push-forward does not commute with $T$.

The fact that length of a current corresponds to the total mass of a frequency measure will be the first ingredient of the proof of the compactness result: bounded length $\rightarrow$ bounded norm $\rightarrow$ weak compactness.

Now, we briefly discuss relations between currents and cyclically reduced words, referring to [Kap04] for more details on the matter.

There is a natural embedding of the space of cyclically reduced words in the space of geodesic currents (or frequency measures). Namely, if $w$ is a cyclically reduced word, we denote by $w^+\infty$ the ray $www \cdots$, by $w^-\infty$ the ray $w^{-1}w^{-1}w^{-1}\cdots$, and by $\gamma_w$ the geodesic joining $w^-\infty$ and $w^+\infty$, that is $\gamma_w = (w^-\infty, w^+\infty) \in \partial^2 F$. Then one can associate at each word $w$ the current

$$\eta_w = \sum_{v \in [w]} \delta_{\gamma_v}.$$
where \([w]\) is the conjugacy class of \(w\) in \(F\) and \(\delta_{\gamma_v}\) denotes the Dirac measure concentrated on \(\gamma_v\). In literature, such currents are often referred to as rational currents. Note that, if \(w\) is not a proper power, then \(||w|| = L(\eta_w) = ||\alpha^{-1}(\eta_w)||\).

**Definition 3.5** (Uniform current and uniform measure). The uniform current \(\eta_A\) and the uniform frequency measure \(\mu_A\) are defined as follows. For all \(v \in F\) we set

\[
\mu_A(Cyl(v)) = \frac{1}{2k(2k - 1)|w| - 1} \quad \text{and} \quad \eta_A = \alpha(\mu_A).
\]

Note that \(L(\eta_A) = 1\) and \(||\mu_A|| = 1\).

**Remark 3.6.** The uniform current is not the product \(\mu_A \times \mu_A\) on \((\partial F)^2\). Indeed, \(\eta_A\) is a measure on \(\partial^2 F \neq (\partial F)^2\), and \(F\)-invariance implies that neighborhoods of the diagonal have infinite measure.

Nevertheless, the current \(\eta_A\) is not so different from \(\mu_A \times \mu_A\). Indeed, we can disintegrate \(\eta_A\) with measures that are in the same density class of \(\mu_A\). This means that if we cut a slice \(S_x\) of \(\partial^2 F\) at the point \(x\), namely \(S_x = \{x\} \times \{\partial F \setminus x\}\), then there exists a continuous function \(\varphi\) on \(\{\partial F \setminus x\}\) such that the measure \(\mu_x\) induced on \(S_x\) by \(\eta_A\) is \(\varphi \cdot \mu_A\). A precise version of this fact is proved in Lemma A.2.

**Definition 3.7** (Length of automorphisms). For any automorphism \(\Phi\) of \(F\) we define the length of \(\Phi\) as the length of the image of the uniform current, that is

\[
L(\Phi) = L(\Phi \eta_A) = \eta_A(\Phi^{-1}(B)).
\]

Because of \(F\)-invariance of currents, \(L(\Phi)\) depends only on the class \([\Phi] \in Out(F)\). We set \(L([\Phi]) = L(\Phi)\).

Some how, the length of \(\Phi\) is the \(\eta_A\)-average of how does \(\Phi\) stretch geodesics.

4. **Proofs of main results**

**Proof of Theorem 1.1.** Let \(\{\Phi_n\}\) be a sequence of automorphisms of bounded length. The strategy of the proof can be summarized as follows.

**Step 1.** The bounded length hypothesis, together with compactness of frequency measures implies that the currents \(\Phi_n \eta_A\) have a limit \(\eta_\infty\) (Lemma 4.1.)

**Step 2.** The core of the proof. We study of the action of \(\Phi_n\) on \(\partial^2 F\) and on \(\partial F\). The main idea is that unbounded lengths of \(\Phi_n\) correspond
to the fact that the maps $\Phi_n$ accumulate all the boundary on some points (the attractors.) The key point is now that, on one hand, the bounded length hypothesis excludes the presence of attractors, while on the other hand, the existence of an element of $f \in F$ such that $||\Phi_n(f)||$ is unbounded implies the presence of attractors (Lemma 4.4 and Lemma 4.6.) Therefore, the maps $\Phi_n$ keep bounded the cyclically reduced length of all elements of $F$.

**Step 3.** If $\Phi_n$ keep bounded the cyclically reduced length of all elements of $F$, then $\Phi_n$ has a subsequence that converges (Lemma 4.7.)

We now work out all the details.

**Lemma 4.1.** Up to pass to subsequences, the currents $\Phi_n \eta_A$ have a limit $\eta_\infty$ which is a geodesic current.

**Proof.** Since the lengths $L(\Phi_n \eta_A)$ are bounded, the total mass of the corresponding frequency measure $\Phi_n \mu_A$ is bounded. The set of positive measures with bounded mass on a compact space is weak-* compact. Since $\partial F$ is compact, up to pass to subsequences, $\Phi_n \mu_A$ has a limit $\mu_\infty$. Such a limit is $T$-invariant because the push-forward via a continuous map is weak-* continuous. Thus, the limit is a frequency measure, which therefore corresponds to a geodesic current $\eta_\infty$. By continuity of the correspondence $\alpha$ between frequency measures and geodesic currents, it follows that $\Phi_n \eta_A \to \eta_\infty$. $\Box$

Now, we start the study of attractors. As the next lemma shows, attractors correspond to singularities on the limit current $\eta_\infty$. More precisely, we say that a current $\eta$ is absolutely continuous w.r.t. $\eta_A$ if for any Borel set $C$, $\eta_A(C) = 0$ implies $\eta(C) = 0$. We say that $\eta$ has a part concentrated on a set $C$ if $\eta_A(C) = 0$ and $\eta(C) > 0$. Then we have

**Lemma 4.2.** Let $\Psi_n$ be a sequence of automorphisms such that the currents $\Psi_n \eta_A$ have a limit current $\eta$. Let $p, q \in \partial F$ be two distinct points. Suppose that there exist cylinders $P_n = Cyl(p_n), Q_n = Cyl(q_n) \subset \partial F$ such that $p_n \to p$ and $q_n \to q$, and such that there is a positive constant $c$ for which $\eta_A(\Psi_n^{-1}(P_n \times Q_n)) > c$. Then, the current $\eta$ has a part concentrated on the geodesic $(p, q)$.

**Proof.** Any cylinder $C \subset \partial^2 F$ containing $(p, q)$, eventually on $n$ contains also $P_n \times Q_n$. Therefore, by definition of push-forward and by hypothesis, eventually on $n$ we get

$$\Psi_n \eta_A(C) > \Psi_n \eta_A(P_n \times Q_n) = \eta_A(\Psi_n^{-1}(P_n \times Q_n)) > c$$
By Proposition A.1 it follows that the limit current satisfies \( \eta(C) > c \) for any cylinder \( C \) containing \((p,q)\). This implies that \( \eta((p,q)) > c \) while \( \eta_A((p,q)) = 0 \), that is, \( \eta \) has a part concentrated on \((p,q)\).

The following is a standard fact, which says that the only currents that can have a part concentrated on a geodesic are essentially the rational currents.

**Lemma 4.3.** Any frequency measure, and hence the limit \( \mu_\infty \), has not a part concentrated on a non-periodic ray. Therefore, any current, and hence the limit \( \eta_\infty \), has not a part concentrated on a non-periodic geodesic.

**Proof.** Any frequency measure \( \mu \) has finite mass and it is \( T \)-invariant. Therefore, if it has a part concentrated on a point \( x \), it has a part concentrated on each point of the \( T \)-orbit of \( x \), with the same weight. It follows that such an orbit must be finite and therefore \( x \) is periodic.

Now the aim is to prove that the maps \( \Phi_n \) keep bounded the lengths of all elements of \( F \), so that we can apply Lemma 4.7. We do it in the following two lemmata. Namely, in Lemma 4.4 we show that if this not the case, then there are no attractors on \( \partial F \) and at most a unique attractor on \( \partial F \). Lemma 4.6 will show that, up to conjugation, we can avoid the presence of a unique attractor on \( \partial F \).

**Lemma 4.4.** Suppose that \( \Phi_n \eta_A \) has a limit current \( \eta_\infty \). Suppose that there exists a word \( f \) such that the length of the cyclically reduced word obtained from \( \Phi_n(f) \) goes to \( \infty \). Then, up possibly to pass to subsequences, \( \Phi_n \) has a limit which is almost everywhere constant. That is, there exists \( y \in \partial F \) such that for \( \mu_A \)-almost all \( x \in \partial F \), \( \Phi_n(x) \to y \).

**Proof.** Recall that we consider \( \Phi_n(f) \) and \( \Phi_n(f^{-1}) \) as freely reduced words. Let \( v_n \) be the maximal initial segment shared by \( \Phi_n(f) \) and \( \Phi_n(f^{-1}) \), and let \( \Psi_n \) be the map \( x \mapsto v_n^{-1} \Phi_n(x)v_n \). Note that \( \Psi_n \eta_A = \Phi_n \eta_A \) and that \( \Psi_n(f) \) is cyclically reduced. Up to pass to subsequences, \( \Psi_n(f) \) and \( \Psi_n(f^{-1}) \) have limits which we denote by \( r_+ \) and \( r_- \). Since \( \Psi(f) \) is cyclically reduced, \( r_+ \neq r_- \). Note that this also implies that \( r_+ \) and \( r_- \) have no common initial segment, that is, the geodesic \((r_-,r_+)\) passes through 1, the base-point of the Cayley graph. We want to show that \( \Psi_n \) converges almost everywhere either to \( r_- \) or to \( r_+ \).

Cut \( \Psi_n(f) \) in two segments of equal length. More precisely, we set

\[
\Psi_n(f) = s_ne_n^{-1}
\]

where the starting segment \( s_n \) and the ending one \( e_n \) have both length \( |\Psi_n(f)|/2 \) (approximated to the nearest integers.) We have \( s_n \to r_+ \).
and \( e_n \to r_- \). In particular, eventually on \( n \), \( \text{Cyl}(e_n) \cap \text{Cyl}(s_n) = \emptyset \), which implies \( \text{Cyl}([e_n, s_n]) = \text{Cyl}(e_n) \times \text{Cyl}(s_n) \). For large \( n \), let \( C_n \) be such a cylinder

\[
C_n = \text{Cyl}([e_n, s_n]) = \text{Cyl}(e_n) \times \text{Cyl}(s_n).
\]

For all \( x \in \partial F \) either \( \Psi_n(x) \in \text{Cyl}(e_n) \) or \( \Psi_n(f x) \in \text{Cyl}(s_n) \), so either \( x \in \Psi_n^{-1}(\text{Cyl}(e_n)) \) or \( f x \in \Psi_n^{-1}(\text{Cyl}(s_n)) \), whence \( f(\partial F \setminus \Psi_n^{-1}(\text{Cyl}(e_n))) \subset \Psi_n^{-1}(\text{Cyl}(s_n)) \). Similarly, either \( \Psi_n(x) \in \text{Cyl}(s_n) \) or \( \Psi_n(f^{-1} x) \in \text{Cyl}(e_n) \). Thus, by Lemma A.4, it follows that

\[
\mu_A(\Psi_n^{-1}(\text{Cyl}(s_n))) \geq \frac{1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(e_n)))}{(2k - 1)\#l}
\]

and

\[
\mu_A(\Psi_n^{-1}(\text{Cyl}(e_n))) \geq \frac{1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(s_n)))}{(2k - 1)\#l}
\]

from which we get

\[
\mu_A(\Psi_n^{-1}(\text{Cyl}(s_n))) \mu_A(\Psi_n^{-1}(\text{Cyl}(e_n))) \\
\geq \frac{1}{(2k - 1)^2\#l} (1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(e_n)))) (1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(s_n))))
\]

By definition of push-forward and Lemma A.3

\[
\Psi_n \eta_A(C_n) = \eta_A(\Psi_n^{-1}(C_n)) = \eta_A(\Psi_n^{-1}(\text{Cyl}(s_n)) \times \Psi_n^{-1}(\text{Cyl}(e_n))) \\
\geq \mu_A(\Psi_n^{-1}(\text{Cyl}(s_n))) \mu_A(\Psi_n^{-1}(\text{Cyl}(e_n)))
\]

Putting together such inequalities, we get

\[
\Psi_n \eta_A(C_n) \geq \frac{(1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(e_n))))(1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(s_n))))}{(2k - 1)^2\#l}
\]

If \( \Psi_n \eta_A(C_n) \to 0 \) then, either \( \mu_A(\Psi_n^{-1}(\text{Cyl}(e_n))) \) or \( \mu_A(\Psi_n^{-1}(\text{Cyl}(s_n))) \) converges to 1 and therefore, up to subsequences, \( \Psi_n \) converges almost everywhere either to \( r_- \) or to \( r_+ \), and we have done.

Otherwise, there exists a constant \( c \) such that \( \Psi_n \eta_A(C_n) > c \), uniformly on \( n \). We now show by contradiction that this is impossible.

If \( \Psi_n \eta_A(C_n) > c \), then by Lemma 4.2, \( \eta_\infty \) has a part concentrated on the geodesic \( (r_-, r_+) \), which is therefore periodic by Lemma 4.3; let \( w \) be its period. We have \( r_- = w^{-\infty} \) and \( r_+ = w^{\infty} \). Note that the words \( \Psi_n(f) \) cannot be a proper power of a finite-length word for infinitely many \( n \). Indeed, otherwise we would get \( f = v^l \), with \( l \to \infty \), and hence \( f \) would have infinite length. Therefore, eventually on \( n \), we can write \( \Psi_n(f) \) as

\[
\Psi_n(f) = w^{(n)} u_n w^{(n)}
\]
with $u_n$ that does not starts, nor ends, by $w$, the exponents $i(n)$ and $j(n)$ are non-negatives and go to infinity as $n$ does. Without loss generality, we can suppose $i(n) \leq j(n)$, so that $s_n$ starts by $w^{i(n)}$.

For any $0 \leq h \leq i(n)$ Let $C_n^h$ be the cylinder

$$C_n^h = \text{Cyl}([w^{-h}e_n, w^{-h}s_n]).$$

Note that the $C_n^h$'s are pairwise disjoint, because $u_n$ does not starts, nor ends by $w$. Moreover, the condition $0 \leq h \leq i(n) \leq j(n)$ implies that the geodesic segment from $w^{-h}e_n$ and $w^{-h}s_n$ passes through 1, thus $C_n^h \subset B$ for all $0 \leq h \leq i(n)$. By $F$-invariance of currents, we have

$$\Psi_n\eta_A(C_n^h) = \Psi_n\eta_A(C_n) > c$$

uniformly on $n$. It follows that

$$L(\Phi_n) = L(\Psi_n) = \Psi_n\eta_A(B) > ci(n)$$

which goes to infinity as $n$ does. But, since $\Psi_n\eta_A \to \eta_\infty$, by Proposition A.1 and the definition of length of automorphisms, the lengths $L(\Psi_n)$ converge to $L(\eta_\infty) < \infty$, a contradiction.

We have so proved that, after passing to a subsequence, the maps $\Psi_n : x \mapsto v_n^{-1}\Phi_n(x)v_n$ converge to a map which is almost everywhere constant (either to $r_-$ or to $r_+$). Up possibly pass to a subsequence, $v_n$ converges to a limit $v_\infty$, that may be finite or infinite. Since the elements $v_n$ where the maximal initial segments shared by $\Phi_n(f)$ and $\Phi_n(f^{-1})$, the words $v_n\Psi_n(f)$ and $v_n\Psi_n(f^{-1})$ are freely reduced. It follows that also the maps $\Phi_n$ converge, up to pass to the same subsequence and almost everywhere, to a constant – which is either $v_\infty$ (if $v_\infty$ is an infinite ray) or $v_\infty r_-$ or $v_\infty r_+$. \qed

**Remark 4.5.** The proof of Lemma 4.4 can be adapted to prove the following more general fact. If we replace the hypothesis “$\Phi_n\eta_A$ as a limit $\eta_\infty$” with “$\frac{\Phi_n}{L(\Phi_n)}\eta_A$ has a limit $\eta_\infty$” – which is always true up to subsequences – then, we get that $\eta_\infty$ has not a part concentrated on a geodesic. Indeed, if $\eta_\infty$ has a part concentrated on a geodesic $\gamma$, then there exists a positive constant $c$ such that for any cylinder $C$ containing $\gamma$ we have $\eta_A((\Phi_n \times \Phi_n)^{-1}(C)) \sim cL(\Phi_n)$. As above, we must have $\gamma = (w^{-\infty}, w^{+\infty})$ for some $w \in F$, and conjugating $\Phi_n$ by a suitable power of $w$, we reach a contradiction. Indeed, if $X$ denotes the set $(\Phi_n \times \Phi_n)^{-1}(C)$, then $(\Phi_n \times \Phi_n)^{-1}(wC) = \Phi_n^{-1}(w)X$ which is contained in $(\Phi_n \times \Phi_n)^{-1}(B)$, whose $\eta_A$-measure is $L(\Phi_n)$ by definition. If a geodesic belongs to $X \cap \Phi_n^{-1}(w)X$, then it passes through 1, whence $\eta_A(X \cap \Phi_n^{-1}(w)X) \leq 1$. Since $\eta_A(X) \sim cL(\Phi_n)$, we can conjugate by $w$.
approximatively at most for $1/c$ times, while if $\gamma = (w^{-\infty}, w^{+\infty})$, then we can do that infinitely many times.

After Lemma 4.4, it remains to deal with the case in which $\Phi_n$ converges almost everywhere to a constant. What is the behavior of such a sequence? An example can be constructed by taking a fixed $\Phi$ and conjugating with elements $v_n$ whose length goes to infinity. Next lemma shows that more or less this is the only case.

**Lemma 4.6.** Let $\Phi_n$ be a sequence of automorphisms of $F$. Then, there exists $v_n \in F$ such that, up possibly to pass to a subsequence, the maps $x \mapsto v_n^{-1}\Phi_n(x)v_n$ have no subsequence converging to a constant $\mu_A$-almost everywhere.

**Proof.** The rough idea is that, up to conjugation, we can force the "barycenter of $\Phi_n" to stay in a fixed compact.

For any freely reduced word $w$ of length $M$, define $B_n(w)$ as the set of rays $x$ such that $\Phi_n(x)$ starts by $w$, namely

$$B_n(w) = \{ x \in \partial F : \Phi_n(x) \in \text{Cyl}(w) \} = \Phi_n^{-1}(\text{Cyl}(w)).$$

Obviously $B_n(1) = \partial F$. Moreover, for each $n$ we have:

$$\lim_{M \to \infty} \sup_{|w|=M} \mu_A(B_n(w)) = 0.$$ 

Indeed, otherwise for all $M$ there exists $w_M \in F$ of length $M$ such that $\mu_A(B_n(w_M)) > c > 0$. Up to subsequences, $w_M$ converges to a ray $R$, and $\Phi_n(B_n(w_M)) = \text{Cyl}(w_M) \to R$ contradicts the fact that $\Phi_n$ is an homeomorphism of $\partial F$ (in this argument $n$ is fixed).

Now, let $v_n$ be a freely reduced word of maximal length such that $\mu_A(B_n(v_n)) \geq \frac{1}{2}$. Let $\widetilde{\Phi}_n$ be the map $x \mapsto v_n^{-1}\Phi(x)v_n$ and let

$$\widetilde{B}_n(w) = \{ x \in \partial F : \widetilde{\Phi}_n(w) \in \text{Cyl}(w) \}.$$ 

Let $l_n \in A$ be the last letter of $v_n$. Since $\mu_A(B_n(v_n)) \geq \frac{1}{2}$ we get $\mu_A(\widetilde{B}_n(l_n^{-1})) \leq \frac{1}{2}$. On the other hand, for any $a \in A$, different from $l_n^{-1}$, maximality of the length of $v_n$ implies

$$\mu_A(\widetilde{B}_n(a)) \leq \frac{1}{2}.$$ 

Hence, such an inequality holds for all $a \in A$. It follows that the sequence $\widetilde{\Phi}_n$ cannot have any subsequence converging to a constant almost everywhere. \hfill \qed

Since conjugation does not affect the length of cyclically reduced words, Lemma 4.4 and Lemma 4.6 can be summarize as follows (recall
that for \( f \in F \), \(|f|\) denotes its length, while \(||f||\) denotes the length of the cyclically reduced word obtained from \( f \).

Let \( \Phi_n \) be a sequence of automorphisms. If there is \( M \) such that \( L(\Phi_n) < M \), then for each \( f \in F \) there is \( M(f) \) such that \(||\Phi_n(f)||\) < \( M(f) \).

As the experts know, this is enough to conclude. We include the proof of the following Lemma 4.7 by completeness.

**Lemma 4.7.** Let \( \{\Phi_n\} \) be a sequence of automorphisms such that for each \( f \in F \) there is \( M(f) \) such that \(||\Phi_n(f)||\) < \( M(f) \). Then, there exist elements \( v_n \in F \) such that a subsequence of \( \{v_n^{-1}\Phi_nv_n\} \) converges to an automorphism.

**Proof.** Up to pass to a subsequence, the maps \( \Phi_n \) converges to a map \( \Phi_\infty \) up to conjugation. More precisely, there exists a map \( \Phi_\infty : A \to F \) and maps \( w_n : A \to F \) such that \( \Phi_\infty(f) \) is cyclically reduced and, up to subsequences, for all \( f \in A \) we have

\[
\Phi_n(f) = w_n(f)\Phi_\infty(f)w_n(f)^{-1}.
\]

Choose an element \( a \in A \). Up to conjugation we can suppose that \( w_n(a) = 1 \), that is, \( \Phi_n \) really converges on the subgroup generated by \( a \). Let \( G \subseteq A \) be a maximal set of generators \( g \) such that \(|\Phi_n(g)|\) stay bounded. If \( G = A \) we have done, because, up to subsequences, \( \Phi_n \) converges on \( A \), whence on \( F \). Otherwise, there exists \( f \in A \) such that the length of \( w_n(f) \) goes to infinity. Since

\[
\Phi_n(af) = \Phi_\infty(a)\Phi_n(f) = \Phi_\infty(a)w_n(f)\Phi_\infty(f)w_n(f)^{-1}
\]

has bounded cyclically reduced length, and since \( \Phi_\infty(a) \) has finite length, we get that, eventually on \( n \), \( w_n(f) \) must start either by \( \Phi_\infty(a) \) or by \( \Phi_\infty(a)^{-1} \). Iterating that argument we get that \( w_n(f) \) is the product of a power of \( \Phi_\infty(a) \) and a word of bounded length. Thus, up to subsequences, we get

\[
w_n(f) = \Phi_\infty(a)^mu
\]

for some \( m \in \mathbb{Z} \) with \(|m| \to \infty \) as \( n \to \infty \), and \( u \) is a finite word (which depends on \( f \)).

It follows that, up to conjugating \( \Phi_n \) by \( \Phi_\infty(a)^m \), we can suppose that \( G \) has at least two elements \( a, b \) and that \( \Phi_n \) is eventually constant on the subgroup generated by \( a \) and \( b \). If \( G \neq A \), let \( f \) be as above. As in (1), we get

\[
w_n(f) = \Phi_\infty(a)^mu
\]

for some \( m \) with \(|m| \to \infty \) as \( n \to \infty \), and \( u \) is a finite word (which depends on \( f \)).
for some exponents \( m, l \) such that \(|m|, |l|\) go to infinity as \( n \) does, and fixed words \( u, v \) (depending on \( f, a, b \)). Therefore, as \( n \) goes to infinity, we get that the unoriented geodesics \( (\Phi_\infty(a)^{-\infty}, \Phi_\infty(a)^{+\infty}) \) and \( (\Phi_n(b)^{-\infty}, \Phi_n(b)^{+\infty}) \) coincide.

This implies that \( \Phi_n(b) \) is cyclically reduced. In particular, we get \( \Phi_n(b) = \Phi_\infty(b) \), and therefore \( \Phi_\infty \) is an automorphism on the group generated by \( a \) and \( b \). Moreover, the above inequalities imply that

\[
\Phi_\infty(b)^{|\Phi_\infty(a)|} = \Phi_\infty(a^{\pm 1})^{|\Phi_\infty(b)|}
\]

whence

\[
b^{|\Phi_\infty(a)|} = a^{\pm 1}|\Phi_\infty(b)|
\]

which is impossible because \( F \) is free. Thus \( G = A \), and hence there exists a subsequence of \( \{\Phi_n\} \) which converges. \( \square \)

By Lemma 4.6, up to conjugation, the sequence \( \Phi_n \) does not subconverges almost everywhere to the same point; by Lemma 4.4 we can apply Lemma 4.7, and the proof of Theorem 1.1 is complete. \( \square \)

**Proof of Corollary 1.2.** We have to prove that the spectrum of the length function is discrete. Suppose the contrary, and take a sequence \( \Phi_n \) of automorphisms such that \( L(\Phi_n) \) has a limit \( \lambda \), with \( L(\Phi_n) \neq \lambda \) for all \( n \). By Theorem 1.1 there exist elements \( v_n \) and a subsequence \( n_i \) such that the maps \( \Psi_{n_i} : x \mapsto v_{n_i} \Phi_{n_i}(x) v_{n_i}^{-1} \) converge to an automorphism \( \Psi \). Thus, the sequence \( \Psi_{n_i} \) is eventually constant, and therefore also the sequence of lengths \( L(\Psi_{n_i}) \) is eventually constant. But \( L(\Psi_{n_i}) = L(\Phi_{n_i}) \) is therefore eventually equal to \( \lambda \), a contradiction. \( \square \)

**Proof of Theorem 1.3.** This immediately follows from Corollary 1.2 and [Kap05a, Proposition 5.2]. Indeed, I. Kapovich proved that for any non-simple automorphism \( \Phi \) there exists a Whitehead automorphism \( \tau \) such that

\[
1 \leq L(\tau \Phi) < L(\Phi)
\]

and the claim follows from Corollary 1.2. \( \square \)

**Appendix A.**

Trough the paper, we used some standard results about currents and measures. This section contains the proofs of such facts.

**Proposition A.1.** Let \( m \) be a Borel measure on \( \partial F \) or \( \partial^2 F \). Then, \( m \) is determined by its value on cylinders. Moreover, if \( \{m_i\} \) is a sequence of Borel measures, then \( m_i \) converges to \( m \) if and only if for all cylinders \( C \), \( m_i(C) \rightarrow m(C) \).
Proof. The proof for $\partial F$ and $\partial^2 F$ is the same. Assume $m$ is a Borel measure on $\partial F$. The characteristic function of any cylinder belongs to $C_0(\partial F)$, and the space $V$ generated by the characteristic functions of cylinders is dense in $C_0(\partial F)$ (the topology of $C_0(\partial F)$ is the one of uniform convergence.) The first claim follows.

Also, this implies that if $m_i$ converges to $m$, then for any cylinder $C$, $m_i(C) \to m(C)$. On the other hand, suppose that $m_i(C) \to m(C)$ for all cylinders $C$. Then for any $\chi \in V$, $\int \chi \, dm_i \to \int \chi \, dm$. Therefore, for any $\varphi \in C_0(\partial F)$, if $\{\chi_k\} \subset V$ is a sequence converging to $\varphi$,

$$\left| \int \varphi \, d(m_i-m) \right| \leq \left| \int |\varphi-\chi_k| \, dm_i \right| + \left| \int \chi_k \, d(m_i-m) \right| + \left| \int |\chi_k-\varphi| \, dm \right|$$

where the first plus the last term are bounded by $||\varphi-\chi_k||(||m_i|| + ||m||)$, which goes to zero as $k \to \infty$, uniformly on $i$. The second term goes to zero for any $k$. \hfill \Box

**Lemma A.2.** For any $(x,y) \in \partial^2 F$ let $L(x,y)$ be the length of the maximal initial segment shared by $x$ and $y$. Then we have

$$\eta_A = 2k(2k-1)^{2L(x,y)-1} \mu_A \times \mu_A.$$

Proof. Let $D, E \subset \partial F$ be two disjoint cylinders. Since $D$ and $E$ are disjoint, there exist $v, w \in F$ such that $D = Cyl(v)$, $E = Cyl(w)$, and such that $v$ is not the initial segment of $w$ nor vice versa. Let $L$ be the length of the maximal initial segment shared by $v$ and $w$ (possibly $L = 0$.) Now, let $D' = Cyl(v') \subset D$ and $E' = Cyl(w') \subset E$ be two cylinders. We set $|v'| = L + a$ and $|w'| = L + b$. We have $D' \times E' = Cyl([v', w'])$ and, by definition (Definition 3.5)

$$\eta_A(Cyl([v', w'])) = \frac{1}{2k(2k-1)^{|v'|^{-1}w'|^{-1}}} = \frac{1}{2k(2k-1)^{a+b-1}}$$

which can be written as

$$\frac{2k(2k-1)^{2L-1}}{(2k(2k-1)^{L+a-1})(2k(2k-1)^{L+b-1})} = 2k(2k-1)^{2L-1} \mu_A(D') \mu_A(E')$$

So, by Proposition A.1, the restriction of $\eta_A$ to $D \times E$ is given by $2k(2k-1)^{2L-1} \mu_A \times \mu_A$. Since for each $(x,y) \in D \times E$ we have $L(x,y) = L$, we get that the restriction of $\eta_A$ to $D \times E$ is given by

$$2k(2k-1)^{2L(x,y)-1} \mu_A \times \mu_A.$$

Since this holds for any $D, E$, the claim follows by Proposition A.1. \hfill \Box

An immediate corollary of Lemma A.2 is the following.

**Lemma A.3.** Let $E, D \subset \partial F$ be two Borel subsets of $\mu_A$-positive measure. Then $\eta_A(E \times D) \geq \mu_A(E) \mu_A(D)$. 


Proof. Just apply Fubini-Tonelli theorem, using Lemma A.2, the fact that $L(x, y) \geq 0$ and that $\frac{2k}{2k-1} > 1$.

**Lemma A.4.** Let $E$ be a Borel subset of $\partial F$. Then for all $f \in F$

$$\mu_A(fE) \geq \frac{\mu_A(E)}{(2k-1)|f|}$$

In particular, if $E$ has $\mu_A$-positive measure, then $fE$ has $\mu_A$-positive measure.

Proof. It suffices to prove the claim for $f \in A$. Let $E_0 = E \cap \text{Cyl}(f^{-1})$ and $E_1 = E \setminus E_0$. By definition of $\mu_A$ and Proposition A.1 we have

$$\mu_A(fE_0) = (2k-1)\mu_A(E_0) \quad \mu_A(fE_1) = \frac{\mu_A(E_1)}{2k-1}$$

and the claim follows. \qed

**References**


