Abstract. OK guys, I’m going to try to convince you that any holomorphic curve in \( SL(2, \mathbb{C}) \) is parabolic, without saying what exactly this means. OK, I’ll try to show that if one has a holomorphic curve in \( SL(2, \mathbb{C}) \) that projects to a surface in \( \mathbb{H}^3 \), then such a surface has trivial tangent bundle. I’ll do it by showing that the normal bundle of the surface is “intrinsically invariant” (???).

1. Notation for \( \mathbb{H}^3 \)

We fix the disc model of \( \mathbb{H}^3 \), so that \( \partial \mathbb{H}^3 \simeq \mathbb{C}P^1 \) and the action of \( SL(2, \mathbb{C}) \) on \( \mathbb{H}^3 \) is well defined. Each time that we speak about \( \infty \in \partial \mathbb{H}^3 \) we mean that we fixed a point and we use a half-space model of \( \mathbb{H}^3 \) in which such point is \( \infty \). The same holds for \( 0 \in \partial \mathbb{H}^3 \).

2. Notation for \( SL(2, \mathbb{C}) \)

We know that \( SL(2, \mathbb{C}) \) is isomorphic to the trivial bundle over \( \mathbb{H}^3 \) of orthonormal framings on \( \mathbb{H}^3 \), which is isomorphic to \( \mathbb{H}^3 \times O(3) \). The tangent bundle of \( \mathbb{H}^3 \) is isomorphic to \( \mathbb{H}^3 \times \mathbb{R}^3 \). None of such isomorphisms is canonic, so we have to fix something.

We fix once and for all a base-point \( x_0 \) in \( \mathbb{H}^3 \) and an orthonormal basis \((e_1, e_2, e_3)\) of \( T_{x_0} \mathbb{H}^3 \). We also fix a trivialisation \( TH^3 = \mathbb{H}^3 \times \mathbb{R}^3 \).

Now, the identification of \( SL(2, \mathbb{C}) \) with the bundle of orthonormal framing is given by

\[
SL(2, \mathbb{C}) \ni A \mapsto (A(x_0), \text{diff} A[e_1], \text{diff} A[e_2], \text{diff} A[e_3])
\]

Here \( \text{diff} \) means the differential of \( A \) as a diffeomorphism of \( \mathbb{H}^3 \). Why use \( \text{diff} A \) instead of \( dA \)? because later, we’ll need to use a lot of differentials, namely the ones of our holomorphic curves, and it is too much for me, I need to change notation :). Moreover, we’ll more or less never use such a differential (we need it only now, at level of definitions.)

This gives a well defined identification \( SL(2, \mathbb{C}) = \mathbb{H}^3 \times O(3) \).
we’ll use the following notation

\[ \text{SL}(2, \mathbb{C}) \ni A = (s, f) \in \mathbb{H}^3 \times O(3) \]

let me explain what does it means. Suppose you have a holomorphic curve \( C \to \text{SL}(2, \mathbb{C}) \). Then if you look at her at the level of \( \mathbb{H}^3 \), you see a parameterisation of a framed surface: at each point of the surface you have a orthonormal frame. So, in \( A = (s, f) \) the \( s \) stands by surface-coordinate and \( f \) by frame-coordinate.

Suppose now that \( A(z) = (s(z), f(z)) \) is a curve. Then,

\[ s(z) = A(z)(x_0) \quad \text{and} \quad f(z) = \text{diff}(A(z)). \]

Explanation: \( A(z) \) is a curve in \( \text{SL}(2, \mathbb{C}) \), so for each \( z \), \( A(z) \) is a matrix, identified via 1 with

\[ (A(z)(x_0), \text{diff} \, A(z)[e_1], \text{diff} \, A(z)[e_2], \text{diff} \, A(z)[e_3]). \]

Recall that here \( \text{diff} \) is NOT the differential w.r.t. \( z \). It is, for each \( z \), the differential of \( A(z) \) as a diffeomorphism of \( \mathbb{H}^3 \).

3. Right action

\( \text{SL}(2, \mathbb{C}) \) acts on himself by RIGHT action. That is

\[ \text{SL}(2, \mathbb{C}) \ni B : A \mapsto AB \]

How right action affects our identification? Simply by changing the base points and frames

\[ x_0 \mapsto Bx_0 \quad e_i \mapsto \text{diff} \, B[e_i] \]

that is, right multiplications correspond to changes of our identifications.

4. Lines in \( \text{SL}(2, \mathbb{C}) \)

Let \( G < \text{SL}(2, \mathbb{C}) \) be a one-parameter subgroup of \( \text{SL}(2, \mathbb{C}) \). One-parameter means one complex parameter. So \( G = \exp_{t_0}(Cv) \) where \( v \in T_{t_0}\text{SL}(2, \mathbb{C}) \) and \( \exp \) is the usual exponential map. (don’t ask me to speak about Lie algebras!) If we want it parametrised, we can write

\[ G(t) = \exp_{t_0}(tv) \]

What do we see at level of boundary? Well, since \( G \) is Abelian, it fixes a point, say \( \infty \). So we have two cases:

Parabolic. \( G(t) \) is a “translation”, that is \( G(t) : z \mapsto z + \lambda \), with \( \lambda = tv \) or something similar.

Hyperbolic. \( G(t) \) fixes also a second point, say 0, and it is a “multiplication”, that is \( G(t) : z \mapsto \lambda z \), with \( \lambda = tv \) or something similar.
In the parabolic case, the image in $\mathbb{H}^3$ of $G$, namely $s(\mathbb{C})$ (if we set $G = (s, f)$ as before) is the horosphere centred at $\infty$ and passing trough $x_0$.

In the hyperbolic case, if $\gamma$ denotes the geodesic from 0 to $\infty$, the image of $G$ in $\mathbb{H}^3$ is the level set of the function distance-from-$\gamma$ passing trough $x_0$.

In both cases we can write

$$s(\mathbb{C}) = \{x \in \mathbb{H}^3 : d(x, \gamma) = d(x_0, \gamma)\}$$

with the convention that $\gamma = \infty$ in the parabolic case and $d(x, \infty)$ is the normalised-at-$O$-Busemann function $B_O(x, \infty)$.

Note that in the hyperbolic case, if $x_0 \in \gamma$ then $s(\mathbb{C})$ is a geodesic.

We call a line passing through $g$ any set $G \subset \text{SL}(2, \mathbb{C})$ of the form

$$\exp_g(\mathbb{C}w)$$

for a $w \in T_g\text{SL}(2, \mathbb{C})$. If we want to emphasise the parameterisation we write $G(t) = \exp_g(tw)$.

If we denote by $\tau_g : \text{SL}(2, \mathbb{C}) \to \text{SL}(2, \mathbb{C})$ the RIGHT translation $A \mapsto Ag$, we have

$$\exp_g(\mathbb{C}w) = \tau_g(\exp(\mathbb{C}v))$$

for a suitable $v \in T_{Id}\text{SL}(2, \mathbb{C})$. Therefore, from the discussion in Section 3, the image of a line passing trough $g$ is computed simply by changing the base point. In particular we have:

For any line $G$, the image in $\mathbb{H}^3$ of $G$ is either a horosphere, a geodesic, or a level set of the function distance-from-a-geodesic.

What’s going on on the framing? Let $G(t)$ be a line of $\text{SL}(2, \mathbb{C})$ passing through $g$, and let $S$ be its image on $\mathbb{H}^3$ (as usual, we set $G(t) = (s(t), f(t))$ and $S$ is the image of $s$; $t$ is a complex parameter.) Writing down explicitly the exponential map, one can convince ourselves that

**Fact 4.1.** For any $v \in T_{s(0)}\mathbb{H}^3$ which is orthogonal to $S$, for any $t \in \mathbb{C}$, the vector $f(t)[v]$ is orthogonal to $S$.

It follows that one can trivialise $T_s\mathbb{H}^3$ as a trivial tangent bundle and a trivial normal (i.e. orthogonal) bundle just by fixing a tangent-normal frame in $s(0)$ and then pushing it to any $s(t)$ using $f(t)$.
5. The Idea

The idea is that, at the first order, a holomorphic curve is a line, so the tangent-normal frames move, at the first order as in a line. This implies that the tangent (normal) frame remains tangent (normal) when pushed via the frame-coordinate of the curve. Therefore, the tangent bundle is trivial.

6. It works?

Let Ω be an open subset of \( \mathbb{C} \), and let

\[ A : \Omega \to \text{SL}(2, \mathbb{C}) \]

be a holomorphic map. As now usual, we set \( A(t) = (s(t), f(t)) \), the surface and frame parameters. Up to parametrisation of \( \Omega \) and Right multiplication in \( \text{SL}(2, \mathbb{C}) \), we can (as we do) suppose that \( 0 \in \Omega \) and \( A(0) = \text{Id} \in \text{SL}(2, \mathbb{C}) \) (so that \( s(0) = x_0 \) and \( f(0) = \text{Id} \in O(3) \)).

Let \( dA(t) \) be the derivative of \( A \) respect to the complex parameter \( t \). We set \( \lambda = dA(0)[1] \), so that, at the first order, \( A \) is approximated by the line

\[ G(t) = \exp_{\text{Id}}(t\lambda) \]

Let \( \langle , \rangle_x \) be the hyperbolic metric of \( \mathbb{H}^3 \) at the point \( x \). Given two vectors \( v, w \) in \( T_{s(0)}\mathbb{H}^3 \), we are interesting to the variation of their scalar product

\[ \langle f(t)[v], f(t)[w] \rangle_{s(t)} \]

as \( t \) varies in \( \Omega \).

\[ \frac{d}{dt}|_{t=0}\langle f(t)[v], f(t)[w] \rangle_{s(t)} = \langle f'(0)[v], w \rangle_{x_0} + \langle v, f'(0)[w] \rangle_{x_0} + \text{NULL} \]

where \( \text{NULL} \) is the term counting the derivative of the metric, that should vanish because Riemannian metric are used to be parallel (it works?)

But now, at the first order, \( A \) is the same as \( G \). Therefore, if \( v \) is tangent to \( S \) and \( w \) is orthonormal to \( S \) (at the point \( x_0 \)) then, by Fact 4.1, the first variation of (2) is zero. Since we can do that for any \( t \), Fact 4.1 holds not only for lines but for a generic holomorphic curve.

It works?