A TRANSFER PRINCIPLE: FROM PERIODS TO ISOPERIODIC FOLIATIONS

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Abstract. In this paper we study the dynamics and topology of the isoperiodic foliation defined on the Hodge bundle over the moduli space of genus $g \geq 2$ curves.

1. Introduction

Let $\Omega\mathcal{M}_g$ be the moduli space of abelian differentials on compact genus $g \geq 2$ smooth curves. The period of an element $(C, \omega) \in \Omega\mathcal{M}_g$ is the element of $H^1(C, \mathbb{C}) \simeq \mathrm{Hom}(H_1(C, \mathbb{Z}), \mathbb{C})$ that is defined by

$$\mathrm{Per}(C, \omega) : \gamma \in H_1(C, \mathbb{Z}) \mapsto \int_\gamma \omega \in \mathbb{C}. \tag{1}$$

The periods of an abelian differential do not allow to recover the abelian differential itself, even infinitesimally. Actually, it is always possible to find non trivial isoperiodic deformations of a given abelian differential, namely an immersed complex submanifold $L \subset \Omega\mathcal{M}_g$ such that the period of a form $(C, \omega) \in L$ is a locally constant function, when we use the local identifications of the $H^*(C, \mathbb{C})$'s given by the Gauss-Manin connection.

The case $g = 2$ is instructive: every genus two curve is a double cover of $\mathbb{P}^1$ ramified over six distinct points, say $0, 1, \infty, x_1, x_2, x_3$. An abelian differential on such a curve can be written as the hyperelliptic integrand

$$\frac{(ax+b)dx}{\sqrt{x(x-1)(x-x_1)(x-x_2)(x-x_3)}}.$$

Picard-Fuchs theory tells us that isoperiodic deformations on $\Omega\mathcal{M}_2$ are integral curves of the following vector field

$$\sum_j \frac{x_j(1-x_j)}{ax_j+b}\frac{\partial}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial a} - \frac{1}{2}\Big(1 + \sum_j \frac{b(x_j-1)}{ax_j+b}\Big)\frac{\partial}{\partial b}. \tag{2}$$

Apart from the invariant closed subsets characterized by topological properties of the image of the periods, there are some interesting known proper closed real analytic invariant submanifolds of the set of abelian differentials of fixed positive volume in $\Omega\mathcal{M}_2$. They were introduced by Calta in [3] and McMullen in [16]. Their image under the Torelli map

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correspond to principally polarized abelian surfaces with complex multiplication by an order of a real quadratic field and the abelian differential is an eigenform with respect to the action of the order. Such a set is diffeomorphic to a circle bundle over the corresponding Hilbert modular surface; we will call them Hilbert modular invariant submanifolds.

In any genus, the collection of all maximal isoperiodic deformations defines a holomorphic foliation $\mathcal{F}_g$ of $\Omega\mathcal{M}_g$, called the isoperiodic foliation[1] (see [18] for further examples). It has dimension $2g-3$, and it is also algebraic: its leaves are solutions of a system of algebraic equations analogous to (2) with respect to the Deligne-Mumford algebraic structure on moduli space. Our main result is the following

Theorem 1.1 (Dynamics of isoperiodic foliations)**.** *Let $g>2$ and $(C,\omega)\in\Omega\mathcal{M}_g$, $V=\frac{i}{2}\int\omega\wedge\overline{\omega}$ its volume, and Λ the closure of the image of its periods. Then the closure of the leaf $L(C,\omega)$ passing through (C,ω) is, up to the action of $GL(2,\mathbb{R})$*

- *(Λ is discrete) a connected component of the Hurwitz space of ramified coverings of the elliptic differential $(\mathbb{C}/\Lambda, dz)$ of volume V,*
- *(Λ is affinely equivalent to $\mathbb{R}+i\mathbb{Z}$) the set of abelian differentials with periods contained in Λ, with primitive imaginary part, and with volume V,*
- *($\Lambda=\mathbb{C}$) the subset of $\Omega\mathcal{M}_g$ consisting of abelian differentials of volume V,*

If $g=2$ the same statement holds, with an extra possibility occurring when ω is an eigenform for real multiplication by an order in a real quadratic field. In this case the closure is a Hilbert modular invariant submanifold.

Moreover, the restriction of $\mathcal{F}_g$ to any of these analytic subsets of $\Omega\mathcal{M}_g$ is ergodic with respect to the Lebesgue class.

The ergodicity part of this result was obtained independently by Hamenstädt in [7]. An example of application of Theorem 1.1 is the existence of an infinite number of abelian differentials with fixed non-discrete periods and only one zero. In the case of the periods of a form associated to a regular polygon these are called fake polygons. This statement was our original motivation. We thank P. Hubert and E. Lanneau for having discussed this problem with us.

Theorem 1.1 is proven by applying Ratner's theory (resp. Moore ergodicity theorem) to the linear action of the integer symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ on the subset of $\mathbb{C}^{2g}$ corresponding to periods of abelian differentials. This latter set has been characterized by Haupt [8]. It consists of periods $p\in\mathbb{C}^{2g}$ such that $\Re p\cdot\Im p>0$ with respect to the usual symplectic product on $\mathbb{R}^{2g}$, that do not correspond to a collapse of $g-1$ handles (see Section 2 for more details). We denote it by $\mathcal{H}_g$. To apply these results in our setting, one needs to study topological properties of the period map

$$\mathrm{Per}:\Omega\mathcal{S}_g\to\mathcal{H}_g\subset\mathbb{C}^{2g}, \tag{3}$$

defined at the level of the Torelli covering $\Omega\mathcal{S}_g\xrightarrow{\pi}\Omega\mathcal{M}_g$. We prove

[1] In the literature, this foliation is also called the kernel foliation, or the absolute period foliation.

Theorem 1.2 (Transfer principle)**.** *The fibers of the period map* (3) *are connected.*

This statement allows to transfer dynamical properties of the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\mathcal{H}_g$ to properties satisfied by the isoperiodic foliation on moduli space. Indeed, Theorem 1.2 is equivalent to the fact that $\mathrm{Sp}(2g,\mathbb{Z})$-invariant subsets of $\mathcal{H}_g$ are in bijective correspondance with $\mathcal{F}_g$-saturated subsets of $\Omega\mathcal{M}_g$ under the map

$$A \subset \mathcal{H}_g \leftrightarrow B = \pi(\mathrm{Per}^{-1}(A)) \subset \Omega\mathcal{M}_g. \tag{4}$$

As a consequence of Theorem 1.2, we deduce Theorem 1.1 by applying Ratner's theory to the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\mathcal{H}_g$. The analysis of Ratner's results in this particular case was carried by M. Kapovich (see [9]) who had the original idea of using Ratner's theory in this setting. After classifying the closed $\mathrm{Sp}(2g,\mathbb{Z})$-invariant subsets of $\mathcal{H}_g$ he gave an alternative proof of Haupt's theorem for genus $g \geq 3$. We review Kapovich's argument in the Appendix 8, and in particular prove Theorem 1.1 assuming the transfer principle following Theorem 1.2.

Theorem 1.2 answers a problem posed by McMullen in [14, p. 2282] for genus $g \geq 4$. For genus $g = 2,3$, he gave a proof by showing that a fiber of the period map on $\Omega\mathcal{S}_g$ is a slice of the Schottky locus in the Siegel space $\mathfrak{h}_g$ of symmetric $g \times g$ matrices with positive definite imaginary part, by a linear copy of the Siegel space $\mathfrak{h}_{g-1}$ inside $\mathfrak{h}_g$ (modulo partially compactifying the fibers by adding nodal abelian differentials of compact type). In genus $g = 2,3$ the Schottky locus is the whole of Siegel space and for higher genera it is a proper analytic set. Since the slice is not contained in the divisors corresponding to curves with nodes, this proves Theorem 1.2 for genus $g = 2$ or 3. However, Schottky's problem - the description of the Schottky locus- is still open for $g > 5$ and the argument seems difficult to generalize for higher genera. Even if we knew the solution of the Schottky problem, determining the components of the intersection with the linear slices does not seem an easy task.

Our road to Theorem 1.2 is different. It is based on an induction argument on the genus (assuming the genus two and three cases) involving degeneration of abelian differentials, i.e. the boundary of $\Omega\mathcal{M}_g$ in its Deligne-Mumford compactification. We partially compactify $\mathrm{Per}^{-1}(p)$ by adding marked stable forms with a simple node and periods p. The added forms can be constructed by using stable forms of lower genus. The inductive hypothesis allows to show that the partial compactification is connected. On the other hand we will prove that the added boundary points do not disconnect the partial compactification. This follows mainly from the fact that at each added point in the boundary of $\Omega\mathcal{M}_g$ there is a unique irreducible *separatrix* of the isoperiodic foliation, that is, a germ of analytic invariant set containing only points of the boundary and points of one leaf. A more detailed account of the difficulties and results used along the proof can be found in subsection 2.10. The full proof takes up most of sections 3 to 7.

Theorem 1.2 is also valid for the restriction of Per to the generic stratum. A question that arises naturally is the description of the connected components of the intersection of the fibers of Per with the other strata. Kontsevich and Zorich gave a description of the connected components of strata of abelian differentials without any condition on the periods in [10]. They found cases with up to three components, so it is plausible to that when we impose further conditions on the abelian differentials, there are also several components. An example is given in genus $g = 2$: the intersection of the non-generic stratum of $\Omega\mathcal{M}_2$ with a leaf of the isoperiodic foliation that does not correspond to discrete periods forms an infinite discrete set (see [14]). A first problem that arises is determining the image of the period map on each such component. Already in the minimal strata there are further restrictions on the periods: if they are discrete, the degree of the associated covering is at least $2g-1$ where g is the genus of the underlying curve. Therefore the periods associated to a collapse of any number of handles onto a connected sum of two copies of an elliptic curve do not occur as periods of such an abelian differential. We shall not pursue this subject further here.

The connectedness of the fibers of the lift of Per to the universal cover of $\Omega\mathcal{M}_g$ fails in general, so the transfer principle does not work at that level. For example, in genus $g = 3$ there are fibers of Per that do not cross the boundary components formed by curves of compact type (see example 5.12). Such a fiber is biholomorphic to a Siegel space, which is simply connected. Thus at the level of the fundamental groups the inclusion $\mathrm{Per}^{-1}(p) \to \Omega\mathcal{M}_g$ is trivial. Therefore there are infinitely many components of the lift of this fiber to the universal cover of $\Omega\mathcal{M}_g$. In fact in genus $g = 2$, Mess showed in [17] that a set of free generators for $\pi_1(\mathcal{M}_2)$ can be constructed with the use of the Torelli map: in Siegel space $\mathfrak{h}_2$ they appear as loops around the components of divisors formed by curves of compact type *with* nodes. We will see that for every $p \in \mathcal{H}_g$ we can find an infinite number of such components that are not crossed by the image of the natural map $\mathrm{Per}^{-1}(p) \to \mathfrak{h}_g$. For $g = 2$ this allows to prove that at the level of the fundamental group the map is not surjective. We thus deduce that there are several connected components in the lift of $\mathrm{Per}^{-1}(p)$ to the universal cover of $\Omega\mathcal{M}_g$. We expect the same to be true for all genera $g \geq 3$.

Another application of Theorem 1.2 is a correspondence between certain classes of integer valued periods and certain classes of representations of fundamental groups of punctured torus to the group of permutations $\mathbb{S}_d$ of d letters. This correspondence comes from the analysis of the transfer principle at the level of the closed leaves of the isoperiodic foliation. Namely, the Hurwitz spaces, i.e. moduli spaces of holomorphic coverings of degree d over an elliptic curve ramifying over $2g-2$ points counted with multiplicity. We do not know whether this correspondence is already known, but it gives an alternative point of view on the irreducible components of Hurwitz spaces.

Theorem 1.3. *For any integers $g \geq 2$ and $d \geq 2$, there is a bijective correspondence between, on one side, the sets of integer periods $\alpha+i\beta \in \mathbb{Z}^{2g}+i\mathbb{Z}^{2g}$ such that $\alpha\cdot\beta = d$ modulo precomposition by $Sp(2g,\mathbb{Z})$ and, on the other side, the set of representations $\rho : \pi \to \mathbb{S}_d$ of a $2g-2$-punctured torus group π sending peripherals to transpositions, modulo precomposition by the braid group of the torus on $2g-2$ braids, and post-conjugation by $\mathbb{S}_d$.*

The paper is organized as follows: in Section 2 we introduce the partial compactification and give the outline the proof of Theorem 1.2. The complete proof takes up Sections 3 to 7; in the Appendix 8 we review Kapovich's results on the dynamical properties of the integer symplectic group on the set of Haupt periods.

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2. Marked stable forms of genus g and their periods

2.1. Stable curves and forms.

Definition 2.1. A connected complex curve C is said to be stable if its singularities are nodes, and the closure C_i of each component of $C^* := C \setminus \mathrm{Sing}(C)$ is a smooth curve whose group of automorphisms that fix the punctures is finite. The normalization of C is the smooth curve $\hat{C} = \sqcup C_i$. A stable curve C is said of compact type if every node separates C in two components. Otherwise C is said to be of non-compact type.

Definition 2.2. Let C be a stable nodal curve. A stable one-form on C is a holomorphic 1-form on C^* that has at worst simple poles at the nodes and satisfies that the sums of the residues of the branches meeting at each node is zero. $\Omega(C)$ denotes the space of stable forms on C. A stable one form will be sometimes referred to as an abelian differential.

If C has genus g then the dimension of $\Omega(C)$ is g. By the residue theorem if C is of compact type, then all residues of all branches at the nodes have to be zero, and thus the forms have no residues at the nodes.

Of course a stable form is holomorphic and closed outside the nodes and can thus be integrated along paths in C^*. For closed paths the value of the integral does only depend on the homology class in $H_1(C^*, \mathbb{Z})$ and it is called the period of the class. If the residues at nodes are all zero, we can also integrate along paths passing *through* the nodes, and the integral along a closed path depends only on its class in $H_1(C, \mathbb{Z})$.

We are interested in isoperiodic sets of stable forms, that is, sets for which the set of periods around closed curves do not vary, and that coincide with the periods of some abelian differential on a smooth curve. Since for the latter $C^* = C$, it is natural to restrict ourselves to the space of stable forms without residues at the nodes. Thus, the restriction of such a form to any component of the normalization is also an abelian differential. If none of the restrictions is the zero form, we say that the abelian differential has no zero components.

By Riemann-Roch's theorem applied to each component C_i of the normalization of C we have that for an abelian differential $\omega \in \Omega(C)$ with no zero components the degree of its associated divisor (ω) is $\sum 2g_i - 2$ where g_i is the genus of C_i. In particular, if the restriction of ω to a connected component has no zeroes the component is an elliptic curve.

Remark that if the points of C_i where the components are glued coincide with zeroes of the restrictions, then the original form ω has all its zeroes "in the nodes".

Definition 2.3. The order of a stable form (C, ω) at a node q is defined to be

$$\mathrm{ord}_q(\omega) = 2 + \mathrm{ord}_q(\omega_1) + \mathrm{ord}_q(\omega_2)$$

where ω_i denotes the restriction of ω to a branch of C through q. We always have $\mathrm{ord}_q(\omega) \geq 0$. A node $q \in C$ is simple for ω if $\mathrm{ord}_q(\omega) = 2$.

Note that the order of the node cannot be 1. With this definition, given a stable form $\omega \in \Omega(C)$ of genus g without zero components we have

$$\deg(\omega) = \sum_{q \in C} \mathrm{ord}_q(\omega) = 2g - 2$$

2.2. **Flat singular metric associated to a stable form.** A stable form (C, ω) induces a flat metric $\omega \otimes \overline{\omega}$ on $C \setminus \{q \in C : \mathrm{ord}_q(\omega) > 0\}$. At a zero of ω that is not a node of C, the metric has a a singularity of angle $2\pi(\mathrm{ord}_q(\omega) + 1)$. We will sometimes call such a singularity a saddle, because of the structure of the geodesics of a given direction at the point. At a simple node the metric extends regularly to each branch of the node and a small neighbourhood of the node is isometric to two flat discs identified at a point.

The volume of a stable form $\omega \in \Omega(C)$ is defined as the volume of this singular metric

$$\mathrm{vol}(\omega) = \frac{i}{2} \int_C \omega \wedge \overline{\omega}.$$

In particular, $0 \leq \mathrm{vol}(\omega) \leq \infty$ and it is finite if and only if all the residues of ω at the nodes of C are zero.

2.3. **Marked stable forms.** In order to compare periods of abelian differentials of the same genus but not necessarily defined on the same curve, we use a model to compare them, so denote by Σ_g a fixed model of a topological closed oriented surface of genus g. The intersection of two elements $a, b \in H_1(\Sigma_g, \mathbb{Z})$ is denoted by $a \cdot b$ and the associated intersection form defines an integral unimodular symplectic structure on $H_1(\Sigma_g) := H_1(\Sigma_g, \mathbb{Z})$.

Definition 2.4. A marking of a genus $g \geq 1$ nodal curve C (maybe with empty set of nodes) is a surjective map $m : H_1(\Sigma_g) \to H_1(C)$ induced by the isotopy class of a pinching map $\Sigma_g \to C$ that pinches a simple closed curve of Σ_g for each node and is a homeomorphism on the complement.

If a node is separating the class of the associated simple closed curve in $H_1(\Sigma_g)$ is zero. Otherwise it is necessarily a primitive element.

Each separating node in a marked nodal curve (C, m) of genus g induces, via m, a splitting of $H_1(\Sigma_g)$ into an orthogonal direct sum of two proper submodules

$$H_1(\Sigma_g) = V \oplus V^{\perp}$$

satisfying that the restriction of the intersection product to each factor is still a symplectic unimodular form.

Definition 2.5. A submodule of a symplectic (unimodular) module is said to be symplectic if the restriction of the symplectic product is still unimodular.

On the other hand, each non-separating node of (C, m) defines a cyclic module generated by a primitive element in $H_1(\Sigma_g)$, namely $\mathbb{Z}a$ where a is the (primitive) class that is effectively pinched on the node.

Two marked nodal curves (C_i, m_i) for $i = 1, 2$ of genus g are said to share a separating node if for each $i = 1, 2$ there exists a separating node $q_i \in C_i$ that induces the same splitting $V \oplus V^\perp$ of $H_1(\Sigma_g)$. Equivalently, we say that they share a non-separating node if there exist non-separating nodes $q_i \in C_i$ whose associated cyclic submodules of $H_1(\Sigma_g)$ coincide.

Remark that if (C, m) is a marked nodal curve of compact type then m is an isomorphism and $H_1(C) = \oplus_i H_1(C_i)$ where C_i denote the components of the normalization. This decomposition induces, via m, a decomposition of $H_1(\Sigma_g) = \oplus_i m^{-1}(H_1(C_i))$. By construction it is a splitting of $H_1(\Sigma_g)$ into orthogonal submodules with the property that the restriction of the symplectic product to each factor is still unimodular. Up to a choice of a marking m_i of each C_i we get a decomposition

$$H_1(\Sigma_g) = \oplus_i H_1(\Sigma_{g_i}) \text{ where } m = \oplus_i m_i.$$

In the case of a marked curve (C, m) of non-compact type the marking is not an isomorphism anymore, and the homology groups taken into consideration, namely $H_1(\Sigma_g), H_1(C)$ and $H_1(\hat{C})$ are pairwise non-isomorphic. In fact we have that $\mathrm{Ker}(m)$ is isotropic in $H_1(\Sigma_g)$ and m induces an isomorphism

$$\mathrm{Ker}(m)^\perp/\mathrm{Ker}(m) \to H_1(\hat{C}).$$

Definition 2.6. A marked stable form of genus $g \geq 1$ is a triple (C, m, ω) where ω is a stable one-form on a marked nodal curve (C, m) of genus g. We say that ω pinches $a \in H_1(\Sigma_g) \setminus 0$ if a is primitive and $m(a) = 0$.

2.4. **Periods of marked stable forms and Haupt's conditions.** For any marked abelian differential (C, m, ω) of genus g without residues at the nodes we have a well defined notion of **period homomorphism** $\mathrm{Per}(\omega) : H_1(\Sigma_g) \to \mathbb{C}$ defined by

$$\mathrm{Per}(\omega)(\gamma) = \int_{m(\gamma)} \omega \quad \text{for } \gamma \in H_1(\Sigma_g). \tag{5}$$

Any homologically non-trivial curve a in Σ_g pinched by the marking, belongs to $\mathrm{Ker}(m)$ and thus also to $\mathrm{Ker}(\mathrm{Per}(\omega))$. Remark that if $\mathrm{Per}(\omega)$ is injective then the marking is an isomorphism, and thus the curve C must be of compact type.

If ω is an abelian differential without residues or zero components on a marked stable curve (C, m) of genus $g \geq 1$, then the character $p = \mathrm{Per}(\omega) \in H^1(\Sigma_g, \mathbb{C})$ has the following two properties (see Haupt's paper [8]):

(H_1) $\mathrm{vol}(p) := \Re p \cdot \Im p > 0$ and
(H_2) If $g \geq 2$ and $L = p(H_1(\Sigma_g)) \subset \mathbb{C}$ is a lattice, then $p = \pi_* \circ m$ where $\pi : C \to \mathbb{C}/L$ is a map of degree > 1, namely $\pi(z) = \int_{z_0}^{z} \omega$.

The positivity of the volume (condition (H_1)) comes from the fact that, by Riemann's relations, for any symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of $H_1(\Sigma_g)$ we have

$$\mathrm{vol}(\omega) = \frac{i}{2}\int_C \omega \wedge \overline{\omega} = \frac{-1}{2i}\sum_{j=1}^{g} \int_{a_j} \omega \int_{b_j} \overline{\omega} - \int_{a_j} \overline{\omega} \int_{b_j} \omega = \sum_j \Im(\overline{p(a_j)} p(b_j)) = \mathrm{vol}(p) \tag{6}$$

The left hand side of the equation is positive. By the right hand side of the equation, the volume of the metric depends only on the absolute periods of ω.

The second condition is obvious. It has other equivalent statements that we proceed to introduce.

Proposition 2.7. *If $g \geq 2$ and $p \in H^1(\Sigma_g, \mathbb{C})$ satisfies $\mathrm{vol}(p) > 0$ and $L = p(H_1(\Sigma_g)) \subset \mathbb{C}$ is a lattice, then the following are equivalent*

(1) p does not *satisfy condition (H_2)*
(2) Ker(p) is a symplectic submodule of rank $2g-2$.
(3) $\mathrm{vol}(p) = \mathrm{vol}(\mathbb{C}/L)$
(4) p factors by a collapse $\Sigma_g \to \mathbb{C}/L$ of $g-1$ handles.

Proof. The rank of $\mathrm{Ker}(p)$ is already $2g-2$. To see that (2) implies (1), suppose $\mathrm{vol}(p) > 0$ and $\mathrm{Ker}(p)$ is a rank $2g-2$ symplectic submodule. By definition $\mathrm{vol}(p) = \mathrm{vol}(E)$ where $E = \mathbb{C}/L$. On the other hand, the integration of ω produces a branched cover $C^* \to E$ of positive degree. The pull back of the euclidean metric on E to C coincides with that induced by ω on C^* and thus $\mathrm{vol}(p) > \mathrm{vol}(E)$, reaching a contradiction. Therefore $\mathrm{Ker}(p)$ cannot be a symplectic module of rank $2g-2$. The rest of equivalences are easy. □

As we will see in Example 5.12 there abelian differentials without residues or zero components of genus $g \geq 3$ whose period map has kernel of rank $2g-2$, so we cannot drop the condition of being symplectic in item (2). Remark that (H_2) is not necessary if we allow marked abelian differentials to be equal to zero on some component of C^*. In this case, integration of ω would be constant on the components annihilated by ω thus producing collapsing maps, instead of branched covers. In this way we can construct abelian differentials that are zero on some components whose periods factor through a collapsing of $g-1$ handles.

Definition 2.8. A character $p \in H^1(\Sigma_g, \mathbb{C})$ is said to be Haupt if $\mathrm{vol}(p) > 0$ and if $g \geq 2$, $\mathrm{Ker}(p) \subset H_1(\Sigma_g, \mathbb{Z})$ is not a rank $2g-2$ symplectic submodule. The set of Haupt characters in $H^1(\Sigma_g, \mathbb{C})$ will be denoted by $\mathcal{H}_g$.

For example, an injective homomorphism $p : H_1(\Sigma_g) \to \mathbb{C}$ defines a Haupt character if and only if $\mathrm{vol}(p) > 0$.

2.5. **Symmetries of the periods.** A last property of the period map p of a marked abelian differential (C, m, ω) without residues at the nodes is that the isotopy class of a Dehn twist around any primitive element $a \in \ker p \setminus 0$, induces a non-trivial symmetry D_a of p, that is, a non-trivial element in the group

$$\mathrm{Aut}(p) = \{h \in \mathrm{Sp}(H_1(\Sigma_g)) : p \circ h = p\}.$$

Hence the marked curves (C, m, ω) and $(C, m \circ D_a, \omega)$ share the same periods. If $a \notin \ker m$ they are distinct as marked stable forms; otherwise $m \circ D_a = m$ and they are equal.

2.6. **The period map.** For $g \geq 1$ denote by $\mathcal{S}_g$ the set of marked compact smooth curves of genus g. It is the covering of $\mathcal{M}_g$ associated to the Torelli group, i.e. the subgroup of the modular group $\mathrm{Mod}(\Sigma_g) \simeq \pi_1(\mathcal{M}_g)$ formed by elements that act trivially on homology. An element in $\Omega\mathcal{S}_g$ can be thought as a marked abelian differential (C, m, ω) on a smooth curve C of genus g.

The natural map $\Omega\mathcal{S}_g \to \Omega\mathcal{M}_g$ that forgets the marking is a covering. The pull-back of the isoperiodic foliation $\mathcal{F}_g$ is a holomorphic regular foliation on $\Omega\mathcal{S}_g$ that admits a global holomorphic first integral

$$\mathrm{Per} : \Omega\mathcal{S}_g \to \mathcal{H}_g \subset H^1(\Sigma_g, \mathbb{C})$$

that is equivariant with respect to the natural actions of the modular group on $\Omega\mathcal{S}_g$ and on $H^1(\Sigma_g, \mathbb{C})$. The map Per is a submersion. The leaves of the foliation underlying Per correspond to connected components of fibers of Per.

2.7. **Topology in the space of marked stable curves.** Let $\overline{\mathcal{M}}_g$ denote the Deligne-Mumford compactification of the moduli space $\mathcal{M}_g$ of genus g smooth curves by adding marked stable curves of genus g. It admits a structure of complex orbifold of dimension $3g - 3$. Let $\overline{\mathcal{S}}_g$ denote the set of marked stable curves of genus g with at most one node. As is shown in [1] and the references therein, there is a topology in the space of marked stable curves of genus g that turns the natural forgetful map $\overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ into a continuous map and extends the covering $\mathcal{S}_g \to \mathcal{M}_g$. The sets $\overline{\mathcal{S}}_g \setminus \mathcal{S}_g$ and $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ are referred to as the boundary of $\overline{\mathcal{S}}_g$ and $\overline{\mathcal{M}}_g$ respectively.

In a small neighbourhood U of a point $(C_0, m_0) \in \overline{\mathcal{S}}_g$ with $\ker m_0 = 0$ (i.e. C_0 is of compact type) the forgetful map is a homeomorphism. If C_0 has a non-separating node (i.e. $\ker m_0 = \mathbb{Z}a \neq 0$), the forgetful map is a branched cover onto its image. The branching occurs on the boundary. Indeed, let D_a denote the map induced by the Dehn twist around the pinched class a in the homology group $H_1(\Sigma_g)$. If $(C, m) \in U$, then the fiber of the forgetful map containing (C, m), i.e.

$$\{(C, m \circ D_a^n) : n \in \mathbb{Z}\}$$

is contained in U. On the other hand we know $\ker m \subset \ker m_0$ is a primitive submodule, so either $\ker m = 0$ and the fiber is infinite or $\ker m = \ker m_0 = \mathbb{Z}a$ and the fiber is a single point.

By Picard-Lefchetz formula applied to the universal curve bundle $\mathcal{C} \to \overline{\mathcal{M}}_g$ around C_0 we deduce that there exists a basis of connected neighbourhoods U of (C_0, m_0) such that $U \cap \mathcal{S}_g$ is connected.

The restriction of the forgetful map to the boundary $\overline{\mathcal{S}}_g \setminus \mathcal{S}_g$ is a local homeomorphism onto its image in $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$. The latter is a normal crossing divisor whose regular points correspond to curves with a single node. The vanishing cycle defined for curves in a neighbourhood in $\overline{\mathcal{M}}_g$ of a regular point of the boundary divisor allows to prove that in the neighbourhood of a boundary point $\overline{\mathcal{S}}_g$ there are only curves that are either non-singular or that share the node with the given point. This allows to parametrize the boundary components of $\overline{\mathcal{S}}_g$ as follows. Consider the set $\mathcal{Z}$ formed by

(1) symplectic submodules $V \subset H_1(\Sigma_g)$ of positive rank and corank and
(2) cyclic primitive submodules $V \subset H_1(\Sigma_g)$.

To a symplectic submodule of type 1) we associate the subset of marked stable curves with one separating node whose associated deocomposition of $H_1(\Sigma_g)$ is $V \oplus V^{\perp}$, i. e. curves of compact type. We will say that $V \in \mathcal{Z}$ is of compact type. To a cyclic module $V \in \mathcal{Z}$ of type 2) we associate the set of marked stable curves with one non-separating node (C, m) such that $m(V) = 0$. It corresponds to curves of non-compact type and we will sometimes say that V is of non-compact type.

In either case the associated boundary set is connected. This can be showed by using attaching maps and the connectedness of the spaces $\mathcal{M}_{g,n}$ for all $(g, n) \in \mathbb{N}^2$. Given two $V, W \in \mathcal{Z}$ we have that the corresponding boundary sets are either equal (if $W = V$ or $V^{\perp}$) or disjoint. Therefore, each such boundary set is a connected component of the boundary. On the other hand every element in the boundary of $\overline{\mathcal{S}}_g$ belongs to at least one of the defined connected components.

By abuse of language we define V to be the boundary component of $\overline{\mathcal{S}}_g$ associated to $V \in \mathcal{Z}$. Depending on the context it will be clear when V is a submodule of $H_1(\Sigma_g)$ or a boundary component of $\overline{\mathcal{S}}_g$.

2.8. **Topology in the space of marked stable forms.** The topologies on $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{S}}_g$ induce topologies on the fiber bundles $\Omega\overline{\mathcal{M}}_g$ and $\Omega\overline{\mathcal{S}}_g$. Their boundaries correspond to abelian differentials over nodal curves with some node, and they are still normal crossing divisors whose regular part corresponds to curves with a single node. Each boundary component of $\Omega\overline{\mathcal{S}}_g$ can be identified with a boundary compnent of $\overline{\mathcal{S}}_g$ and again, by abuse of language we will denote the boundary component and a module $V \in \mathcal{Z}$ that defines it with the same name V.

The forgetful map

$$\Omega\overline{\mathcal{S}}_g \to \Omega\overline{\mathcal{M}}_g$$

is still a branched cover onto its image with branching divisor on the components of the boundary that have a single non-separating node. Around any point in $\Omega\overline{\mathcal{S}}_g$ there is a connected neighbourhood U such that $U \cap \Omega\mathcal{S}_g$ is connected. Any subset of one of those bundles is given the induced topology.

2.9. **Notations.** We introduce the following notational conventions: given a set of marked stable curves $\mathcal{C} \subset \overline{\mathcal{S}}_g$ of genus g we denote by

- $\Omega\mathcal{C}$ the set of marked stable forms over curves in $\mathcal{C}$;
- A point in $\mathcal{C}$ or $\Omega\mathcal{C}$ is said to be in the boundary if the underlying curve has some node;
- $\Omega_0\mathcal{C} \subset \Omega\mathcal{C}$ the subset where the forms have zero residues at the branches of the nodes;
- $\Omega'\mathcal{C} \subset \Omega\mathcal{C}$ formed by marked stable forms in $\Omega\mathcal{C}$ that have at most simple nodes. In particular elements in $\Omega'\mathcal{C}$ have no zero components or residues at the nodes.

If apart from $\mathcal{C} \subset \overline{\mathcal{S}}_g$ we are given a Haupt period $p \in \mathcal{H}_g \subset H^1(\Sigma_g, \mathbb{C})$ we consider

- $\mathcal{C}(p) \subset \Omega_0\mathcal{C}$ formed by stable forms over curves in $\mathcal{C}$ that have no residues at the nodes and periods p;
- $\mathcal{C}'(p) := \mathcal{C}(p) \cap \Omega'\mathcal{C}$.

The genus of the curves is already present in the periods, so we will often not introduce a subindex for it.

Some examples of subsets adapted to these notations are particularly important in the arguments that follow:

- $\mathcal{S}(p) = \mathcal{S}_g(p) = \mathrm{Per}^{-1}(p)$
- $\overline{\mathcal{S}}'(p)$ is the set of abelian differentials with at worst a simple node and periods p
- For $V \in \mathcal{Z}$, we consider its corresponding boundary component V in $\Omega\overline{\mathcal{S}}_g$. Then
$$V'(p) = \overline{\mathcal{S}}'(p) \cap V$$
is the set of abelian differentials on V that have a simple node and periods p.
- $\mathcal{Z}_p = \{V \in \mathcal{Z} : V'(p) \neq \emptyset\}$ corresponds to the set of boundary components having a form of periods p.
- $\mathcal{V}_p = \{V \in \mathcal{Z}_p : \operatorname{rank} V = 2\}$ corresponds to the set of boundary components of compact type whose underlying forms have a genus one component and periods p.

2.10. **Outline of the proof of Theorem 1.2.** We first compactify $\mathcal{S}(p) \subset \Omega\overline{\mathcal{S}}_g$ partially by adding stable abelian differentials of genus g having a simple node and periods p. This union is precisely the previously defined set $\overline{\mathcal{S}}'(p)$. As we will see in Section 4.6, for any sufficiently small neighbourhood U around a point in $\overline{\mathcal{S}}'(p)$, $U \cap \mathcal{S}(p)$ is non-empty and connected. This implies that $\mathcal{S}(p)$ is connected if and only if $\overline{\mathcal{S}}'(p)$ is.

This said, we proceed to prove Theorem 1.2 by induction. The cases $g = 2, 3$ are true by [14]. Suppose $g \geq 4$ and that the fibers of Per are connected for every genus up to $g - 1$. Let $p \in \mathcal{H}_g$ be given. We claim that $\overline{\mathcal{S}}'(p)$ is connected.

In Section 3 we use the inductive hypothesis of Theorem 1.2, attaching maps and Simpson's theorem in [22] to prove that the intersection of $\overline{\mathcal{S}}'(p)$ with a component of the boundary of $\Omega\overline{\mathcal{S}}$ is connected. This implies that the boundary of $\overline{\mathcal{S}}'(p)$ is the disjoint union of connected sets

$$\bigsqcup_{V\in\mathcal{Z}} V'(p).$$

In Section 4 we prove that each component of $\overline{\mathcal{S}}'(p)$ has some point in the boundary. This is true for a component determined by an abelian differential ω on a smooth curve with simple zeroes whose underlying metric $\omega\otimes\overline{\omega}$ has a couple of saddle connections $\gamma_\varepsilon, \gamma'_\varepsilon$ that are parallel geodesics with common endpoints and length $\varepsilon > 0$. Indeed, cutting the surface along them and gluing them in the other possible orientable way, we obtain a (possibly disconnected) stable curve endowed with a flat metric and two disjoint geodesic slits of the same length connecting regular points for the metric.

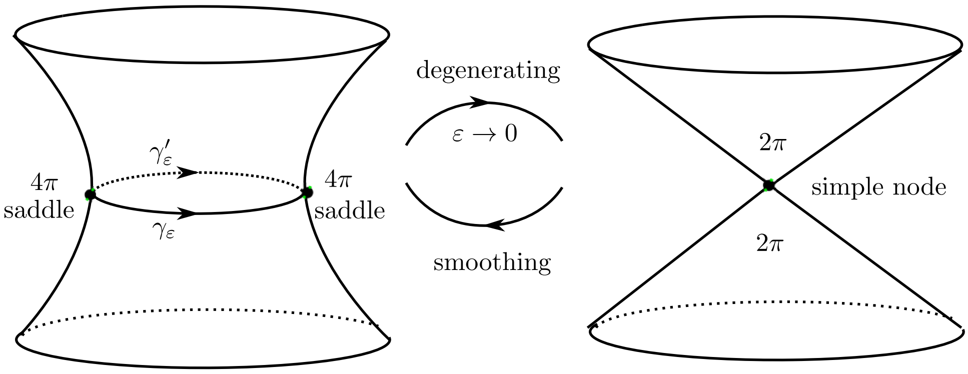

Figure 1. Collapse of two simple zeroes to a simple node

By continuously changing the length of the slits and gluing them back together, we construct a continuous deformation of flat metrics on a genus g surface. By using the existing dictionary between flat surfaces and nonzero abelian differentials (see section 4) we get a path of abelian differentials. When the slit degenerates to a point we obtain a flat metric with isolated singularities on a genus g nodal curve with a single simple node. Along these local surgeries the periods remain constant, so the path is contained in $\overline{\mathcal{S}}'(p)$.

Remark that the last construction can be reversed to show that any stable one form with a simple node and periods $p \in \mathcal{H}_g$ lies in the boundary of $\mathcal{S}(p)$. This reversed construction will be referred to as a **smoothing of a simple node**. In fact we will go further and show that *every* abelian differential in $\mathcal{S}(p)$ sufficiently close to a nodal abelian differential in $\overline{\mathcal{S}}'(p)$ occurs in this way (see 4.5).

It remains to prove that any component of $\overline{\mathcal{S}}'(p)$ contains a form ω as before. By an inductive argument starting with saddle connections of minimal length in any given element in a component of $\mathcal{S}(p)$ and applying Schiffer variations we find an ω in the same component in $\mathcal{S}(p)$ that either has the desired couple of saddle connections, or that has a single zero.

In the second case it also has the desired pair of saddle connections. Indeed, Masur's theorem in [13] on the existence of periodic annuli for ω provides the pair of parallel saddle connections described by the boundary components of the annulus(see Section 4 for the details). This proves that the correesponding component of $\overline{\mathcal{S}}(p)$ has a point in the boundary. Another Schiffer variation argument will imply that the corresponding connected component of $\overline{\mathcal{S}}'(p)$ also contains some point in the boundary.

The rest of the proof relies in finding paths in $\overline{\mathcal{S}}'(p)$ with endpoints in any given pair of boundary components of $\overline{\mathcal{S}}'(p)$. The main tool that we will use to construct paths of isoperiodic abelian differentials that join different boundary components is to construct abelian differentials on stable curves with several simple nodes. Indeed, by smoothing all the simple nodes we obtain an abelian differential ω on a smooth curve with the same periods as the initial form. By undoing the surgery but only at one of the nodes, we connect this element with one on a curve with a single simple node. Thus several abelian differentials on nodal curves with a single simple node and period p can be connected to ω (see Section 6 for details).

To construct stable forms with several simple nodes and prescribed periods $p \in \mathcal{H}_g \subset H^1(\Sigma_g, \mathbb{C})$ we will analyze the algebraic and topological properties of Haupt characters. A special role will be played by splittings $V_1 \oplus \cdots \oplus V_n$ of $H_1(\Sigma_g)$ into pairwise orthogonal symplectic submodules for which $p_{|V_i}$ is either a Haupt homomorphism or a non-injective morphism defined on a rank two submodule. They correspond to periods of certain abelian differentials with simple nodes. The existence of such splittings for any Haupt period can be proven by purely algebraic methods when p is non-injective or $g \geq 3$. This argument will allow to give an algebraic characterization of the modules $V \in \mathcal{Z}$ for which the corresponding boundary component $V'(p)$ is nonempty (in Section 5). Along the proof of this result we give an inductive proof in the spirit of the original proof (see [8]) of

Theorem 2.9 (Haupt)**.** *A character $p \in H^1(\Sigma_g, \mathbb{C})$ is the period of some marked abelian differential on a compact genus g smooth curve if and only if it is a Haupt character.*

The case of genus $g = 1$ is trivial and the case of genus $g = 2$ is proven by the use of the Torelli map and the solution of Schottky's problem in [14]. Given an injective $p \in \mathcal{H}_2$, the existence of elements $V \in \mathcal{Z}$ such that $p_{|V}$ and $p_{|V^\perp}$ are Haupt characters can be proven by taking a form on the boundary of a connected component in $\overline{\mathcal{S}}'(p) \neq \emptyset$.

We will sometimes call a Haupt character a Haupt period. For a modern proof of this result for genus $g \geq 3$ using ergodic theory see [9].

3. Connectedness of the isoperiodic set in each boundary component

Proposition 3.1 (Isoperiodic forms on a boundary component)**.** *Let $g \geq 4$ and $p \in \mathcal{H}_g$. Suppose that the fibers of* Per *are connected for any genus up to $g - 1$. Then, for any $V \in \mathcal{Z}$, $V'(p)$ is connected.*

Proof. If $V'(p) = \emptyset$ the claim is obvious. Otherwise, take $(C, m, \omega) \in V'(p)$. There are two cases, depending on wether ω has a separating or non-separating node.

Case 1: The node of ω is separating. Then $\omega = \omega_1 \vee \omega_2$ and its associated splitting is $V \oplus V^\perp$. By construction $p_1 = p_{|V}$ and $p_2 = p_{|V^\perp}$ correspond to the periods of some abelian differential of genus $< g$, namely ω_1 and ω_2 respectively.

For each $g \geq 1$ consider the universal curve bundle $\mathcal{C}_g \to \mathcal{S}_g$ whose fibre over a marked curve $(C, m) \in \mathcal{S}_g$ is the curve C. This bundle allows to define another curve bundle $\Pi : \Omega\mathcal{C}_g \to \Omega\mathcal{S}_g$ whose fibre over a marked abelian differential (C, m, ω) is still the curve C. Given $p \in \mathcal{H}_g$, we define

$$\mathcal{C}^*(p) = \{(C, m, \omega, q) \in \Omega\mathcal{C}_g : \mathrm{Per}(\omega) = p \text{ and } \omega(q) \neq 0\}.$$

When it is non-empty, the restriction of Π to $\mathcal{C}^*(p)$ defines a topological fibre bundle

$$\mathcal{C}^*(p) \to \mathrm{Per}^{-1}(p)$$

with connected fibers. Hence if $\mathrm{Per}^{-1}(p)$ is connected, then so is $\mathcal{C}^*(p)$.

When $(C, m, \omega) \in V'(p)$, we can write it as $(C, m, \omega) = (C_1 \vee_{q_1 = q_2} C_2, m_1 \oplus m_2, \omega = \omega_1 \vee \omega_2)$ where ω_i is an abelian differential on a curve C_i of genus $0 < g_i < g$ that does not vanish at $q_i \in C_i$, and m_i is a marking of C_i. Write $p_i = \mathrm{Per}(\omega_i)$. By hypothesis $\mathrm{Per}^{-1}(p_i)$ is connected and non-empty, and therefore so is $\mathcal{C}^*(p_i)$.

The natural attaching map

$$\mathcal{C}^*(p_1) \times \mathcal{C}^*(p_2) \to V'(p)$$

that sends a pair $\big((C_1, m_1, \eta_1, q_1), (C_2, m_2, \eta_2, q_2)\big)$ to the nodal abelian differential

$$(C_1 \vee_{q_1 = q_2} C_2, m_1 \oplus m_2, \eta_1 \vee \eta_2)$$

is continuous and therefore its image $V'(p)$ is connected.

Case 2: The node of ω is non-separating. In this case $V \subset H_1(\Sigma_g)$ is a cyclic submodule.

Let $\mathcal{S}_{g,2}$ denote the space of (homologically) marked smooth curves of genus g with two marked points and $\Gamma_g \to \mathcal{S}_{g,2}$ the fiber bundle whose fibre over a point $(C, m, q_1, q_2) \in \mathcal{S}_{g,2}$ is the set of homotopy classes relative to $\{q_1, q_2\}$ of paths with endpoints at q_1, q_2. Consider the composition $\Pi : \Omega\Gamma_g \to \Omega\mathcal{S}_{g,2} \to \Omega\mathcal{S}_g$ where the second arrow is the map that forgets the marked points. Fix $p \in \mathcal{H}_g$ and a complex number $c \in \mathbb{C}$. Define

$$\Gamma^*(p, c) = \{(C, q_1, q_2, m, \omega, [\gamma]) \in \Omega\Gamma_g : \mathrm{Per}(\omega) = p, \omega(q_i) \neq 0, \int_\gamma \omega = c\}.$$

When it is non-empty, the restriction of Π to $\Gamma^*(p, c)$ gives a continuous surjective map

$$\Pi_{(p,c)} : \Gamma^*(p, c) \to \mathrm{Per}^{-1}(p)$$

that has the path lifting property.

Lemma 3.2. *The fibers of* $\Pi_{(p,c)}$ *are connected. In other words, let* ω *be an abelian differential on a smooth curve* C *and* b_0 *and* b_1 *be arcs in* C *such that* $\int_{b_0} \omega = \int_{b_1} \omega$. *Then, there is an homotopy* b_t *between* b_0 *and* b_1 *such that* $\int_{b_t} \omega = \int_{b_0} \omega$ *for any* $t \in [0, 1]$.

Proof. Let Y be the covering of $X = C \times C$ defined as the set of relative homotopy classes of maps $\gamma : [0, 1] \to S$ (relative to $\{0, 1\}$). The covering map is $\pi(\gamma) = (\gamma(0), \gamma(1))$. There is a well-defined map $g : Y \to \mathbb{C}$

$$g(\gamma) = \int_\gamma \omega$$

and the claim is that g has connected fibers. Let π_0 and π_1 be the two projections from $X = C \times C \to C$ and define $\alpha = \pi_1^*\omega - \pi_0^*\omega$. Clearly

$$dg = \pi^*\alpha.$$

By [22, Theorem 1] either the fibers of g are connected or there is an algebraic curve E, a holomorphic 1-form β on E, and a morphism $f : X \to E$ with connected fibers such that $\alpha = f^*\beta$. Therefore, we only need to show that under our hypotheses, also in the latter case, the fibers of g are connected.

First, we prove that E is an elliptic curve. Let $\mathcal{F}$ be the foliation on X defined by $\{\alpha = 0\}$. The singularities of $\mathcal{F}$ are at those pairs of points (x, y) such that ω vanishes at both x and y. In particular, the singularities of $\mathcal{F}$ are isolated. We redefine our notion of leaf. Two (usual) leaves of the foliation that accumulate on the same singular point are considered to be part of the same leaf of $\mathcal{F}$, and we also include the singular point in the leaf. Since $\mathcal{F}$ admits local first integrals, every singular point is contained in a leaf. On the other hand the fibers of f are connected, and clearly $\alpha = 0$ on the fibers, we have $E = X/\mathcal{F}$. Since every leaf has a transversal where α does not vanish, β has no zeroes, whence E is elliptic. Thus $E = \mathbb{C}/\Lambda$ for a lattice Λ and we have the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\quad g=\int_\gamma \omega \quad} & \mathbb{C} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X = C \times C & \xrightarrow{\quad f=\int_x^y \omega \quad} & X/\mathcal{F} = E = \mathbb{C}/\Lambda \end{array}$$

Consider the foliation $\pi^*\mathcal{F}$ on Y, defined by $\{\pi^*\alpha = 0\}$, and let $\hat{C} = Y/\pi^*\mathcal{F}$ with $\varphi : Y \to \hat{E}$ the quotient map. Let $\pi' = f \circ \pi \circ \varphi^{-1} : \hat{E} \to E$. We will see below that $\hat{E}$ is Hausdorff, which implies that $\hat{E}$ is in fact a Riemann surface, and that π' is a covering map.

Since $dg = \pi^*\alpha$, g is constant on the leaves of $\pi^*\mathcal{F}$. Therefore there is a map $h : \hat{E} \to \mathbb{C}$ such that $g = h \circ \varphi$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} & & \overset{g}{\frown\!\frown\!\frown\!\frown\!\rightarrow} & & \\ Y & \xrightarrow{\varphi} & \hat{E} & \xrightarrow{h} & \mathbb{C} \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi'}\downarrow & \swarrow & \\ X & \xrightarrow{f} & E & & \end{array}$$

In particular, $h : \hat{C} \to \mathbb{C}$ is a lift of π'. This implies that $\hat{C}$ is simply connected. If follows that $\hat{E} = \mathbb{C}$ is the universal covering of E, and h is a biholomorphism. Since h is injective and φ has connected fibers by definition, then $g = h \circ \varphi$ has connected fibers.

Let us show that $\hat{E}$ is Hausdorff and π' is a covering. Since E is Hausdorff, two points that cannot be separated must have the same image under π'.

Let F_1 and F_2 be two leaves of $\pi^*\mathcal{F}$ that project on the same leaf of $\mathcal{F}$. We make use of an auxiliary Riemannian metric on X. For each leaf L of $\mathcal{F}$ and for any $\varepsilon > 0$ let $N_\varepsilon(L)$ be the ε-neighborhood of L. Since $\mathcal{F}$ has isolated singularities and compact leaves any two distinct lifts of L to $\pi^*\mathcal{F}$ are disjoint in Y. Moreover, by the same reason and since the leaves of $\mathcal{F}$ are compact, if $x_n \to x \in L$ and L_n is the leaf through x_n, then L_n converges to a sub-set of L in the Gromov-Hausdorff sense. It follows that any $N_\varepsilon(L)$ contains a saturated neighborhood $U_\varepsilon(L)$. Thus that F_1 and F_2 can be separated by saturated neighborhoods. In particular, this implies that $\hat{E}$ is Hausdorff and that π' is a covering map. □

Corollary 3.3. *If* $\mathrm{Per}^{-1}(p)$ *is connected, so is* $\Gamma^*(p, c)$ *for any* $c \in \mathbb{C}$.

Let us proceed to the proof of the connectedness of $V'(p)$. Write $V = \mathbb{Z}a_1$ for a primitive $a_1 \in V$. Complete a_1 to a symplectic basis $a_1, b_1, \ldots, a_g, b_g$ of $H_1(\Sigma_g)$. Denote $W = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1$. Then up to identifying $W^\perp$ with $H_1(\hat{C})$, $p_1 = p_{|V^\perp}$ is the period of an abelian differential, thus a Haupt character in $H^1(\Sigma_{g-1}, \mathbb{C})$ and $\Gamma^*(p_1, p(b_1))$ is non-empty. Consider the continuous attaching map

$$\Gamma^*(p_1, p(b_1)) \to V'(p)$$

that associates to an element $(C, q_1, q_2, m_1, \omega, [\gamma])$ (of genus $g-1$) the abelian differential on a nodal curve of genus g obtained by gluing q_1 and q_2, marked by m_1 on $W^\perp$, pinching a_1 to the node and associating b_1 to the homology class of the cycle γ in the nodal curve. It is a surjective map: it suffices to construct a representative of the class corresponding to b_1 in the nodal curve that touches the node to find a preimage by normalizing the node. By hypothesis and Corollary 3.3 the source is connected, and therefore so is its image $V'(p)$. □

4. Surgeries on abelian differentials and models of degeneracy

In this section we introduce a surgery that allows to move points locally in $\Omega\overline{\mathcal{S}}_g$ without changing the periods on cycles of the homology. As a consequence we will prove that every component of $\mathcal{S}(p)$ has boundary points that belong to the boundary of $\Omega'\overline{\mathcal{S}}$. We also study the local topological properties of the boundary components; this technical part will be crucial in our argument.

4.1. **Stable one forms and singular translation structures.** On each component C_i where a stable form (C, ω) is not identically zero it defines naturally a singular translation structure. Indeed, since ω is a closed form, around a point $p \in C_i^*$ we can locally define a holomorphic function $\phi_p(z) = \int_p^z \omega$ that is a branched cover of degree $\mathrm{ord}_p(\omega) + 1$ ramified over 0 if the degree is at least two. At the intersection of domains two such maps ϕ_p and ϕ_q satisfy

$$\phi_p = \phi_q + \text{const}.$$

Thus any object invariant by translations in $\mathbb{C}$ can be pulled back to $C_i^* \setminus |(\omega)|$ with singularities at the points of $|(\omega)|$ and the nodes. In particular ω induces a singular flat metric on C_i of finite volume, which is defined by $\omega \otimes \overline{\omega}$. At a branch C_i of C around a point $p \in C$ the metric has total angle $2\pi(\text{ord}_p(\omega_{C_i})+1)$. This is also true if ω has non-zero residue at p. In particular, if ω has no zero components we obtain a branched atlas whose transition functions are translations.

The oriented geodesic directional foliation of $\mathbb{C}$ given by an angle $\theta \in \mathbb{S}^1$ is also invariant by translations, so we can also lift it to a singular directional foliation $\mathcal{G}_\theta$ on C_i. At a zero p of ω the foliation has a saddle with $2(\text{ord}_p(\omega)+1)$ separatrices, that alternatively enter and leave the singularity by forming an angle of π. At any other point the foliation is regular.

Reciprocally if we are given a cover U_α of a compact (possibly disconnected) topological surface Σ, and finite branched covers $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}$ satisfying $\phi_\alpha = \phi_\beta + \text{const}$ at the intersections $U_\alpha \cap U_\beta$, we can define a complex structure on Σ by declaring that the ϕ_α's are holomorphic. The abelian differential ω defined locally by $d\phi_\alpha$ is well defined on the obtained Riemann surface $\hat{C}$. By identifying pairs of points in $\hat{C}$, we obtain all nodal curves C that are normalized by $\hat{C}$.

4.2. **Schiffer variations.** A Schiffer variation is a sugery that can be defined for arbitrary $\mathbb{C}P^1$-structures with some branch point on surfaces. It changes the $\mathbb{C}P^1$ structure without varying the underlying topological surface nor the holonomy representation. They were first considered by Schiffer in [21]. A detailed discussion can be found in [2]. We will introduce it only for the case of branched translation structures.

Let ω be a marked abelian differential on a nodal curve C and q be point where $\text{ord}_q(\omega) \geq 1$. Remark that the chart $\phi = \phi_q$ defined by ω around q can be analytically continued along any path in C. Let γ_1 and γ_2 be two embedded paths in C^* starting at q parametrized by $[0,1]$ that are disjoint (we allow q to be a node). We say that γ_1 and γ_2 are **twin paths** if the continuation ϕ_i of ϕ along γ_i satisfies that $\phi_1 \circ \gamma_1(t) = \phi_2 \circ \gamma_2(t) \in \mathbb{C}$ for all t and $t \mapsto \phi_i \circ \gamma_i(t)$ is an embedded path in $\mathbb{C}$.

Given a pair of twin paths γ_1, γ_2 for (C, m, ω) we can consider a path $t \mapsto (C_t, m_t, \omega_t)$ of marked abelian differentials associated to it called the **Schiffer variation** of ω along γ_1, γ_2. To describe the abelian differential at time t, we use the equivalence between abelian differentials and atlas formed by branched covers over open sets in $\mathbb{C}$ and with transitions in the set of translations $z \mapsto z + \text{const}$. Indeed, cut C along the segments $\gamma_1|_{[0,t]}$ and $\gamma_2|_{[0,t]}$ and glue the boundary on the left (resp. on the right) of γ_1 to the boundary on the right (resp. on the left) of γ_2 by identifying points that have the same image for ϕ_i. By construction we get a new orientable (nodal) surface of the same genus equipped with a family of local branched covers. Indeed, on the complement of some topological disc U around $\gamma_1 \cup \gamma_2$ we consider the family of branched coverings given by ω. On the disc U we consider the branched covering defined after the identification. Obviously the critical value has changed: it is at the point $\phi_i \circ \gamma_i(t)$. The associated abelian one form ω_t has the same genus. As we can choose the initial covering of C by open sets not to have intersections on $\gamma_1 \cup \gamma_2$ and generators $a_1, b_1, \dots, a_g, b_g$ for the homology that are contained in $C \setminus \gamma_1 \cup \gamma_2$, the transition functions of the new marked abelian forms are the same and

$$\mathrm{Per}(\omega_t) = \mathrm{Per}(\omega) \text{ for all } t.$$

A Schiffer variation along a short pair of twins based at a simple node describes the smoothing of a simple node as in Figure 1. A Schiffer variation along a short pair of twins at a multiple zero of a stable form splits the saddle into two distinct saddles.

4.3. **Dynamics of directional foliations.** The dynamics of each oriented directional foliation $\mathcal{G}_\theta$ induced by an abelian differential ω without residues or zero components on the underlying curve C is well known. Indeed, by Maier's Theorem (see [11] or [23]) there exist a finite number of saddle connections, that is, leaves $\gamma_1, \dots, \gamma_n$ with the property that both the α and ω limit are singular points. Each component of $C \setminus \cup\gamma_i$ is saturated by $\mathcal{F}$ and is either a **periodic annulus**, i.e. an annulus formed of closed leaves, or **minimal** , i.e. each leaf in the component is dense in the component.

Remark that the length of all leaves in a periodic annulus is the same and coincides with the length of each of its boundary components. On the other hand the saddle structure of the singularities implies that the angle of two leaves at a singular point of the boundary of a periodic annulus is π.

Remark 4.1. If for some directional foliation associated to an abelian differential of genus $g \geq 2$, we can find a periodic annulus whose boundary is formed precisely by a singular point $*$ and two leaves that have $*$ as α and ω limit, then the leaves form a pair of twin geodesics. If we perform a very small Schiffer variation along the same pair of twins, we obtain a form with a pair of twin saddle connections between two distinct zeroes. In

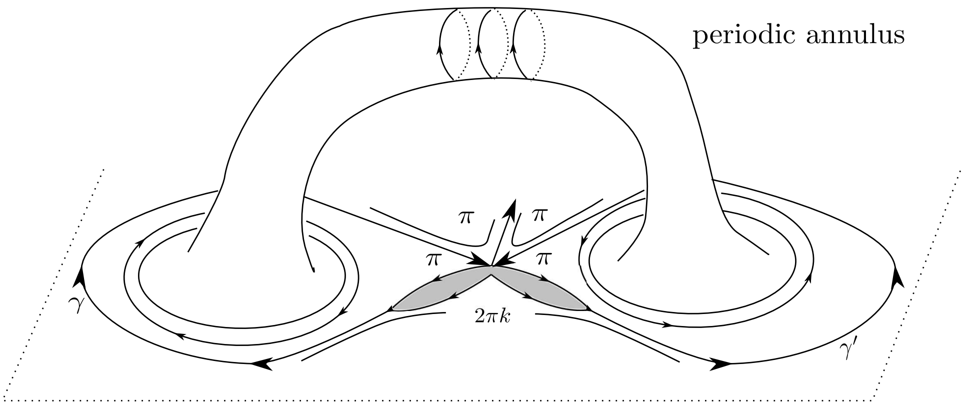

FIGURE 2. Abelian differential slit at a couple of twins

Figure 2 we represent the abelian differential slit at a couple of twins from point of angle

$2\pi(k+2)$. Gluing the boundary components in the other orientable way, the obtained abelian differential has one or two extra saddles (depending on whether $k = 1$ or $k > 1$) and there exists a pair of twins connecting two distinct saddles. Thus, after this surgery we fall in the picture of Figure 1. As can be readily seen, in this situation the Schiffer variation along the pair of saddle conections produces a curve of compact type with a genus one component that is obtained from the periodic annulus. We will use this construction in the next paragraph. The local surgery of splitting a multiple zero into some simpler zeroes will be referred to as a **splitting of a multiple zero**.

4.4. **Twin saddle connections between distinct simple zeroes.** In the next Lemma we show that any abelian differential on a non-singular curve can be connected via Schiffer variations to one having some pair of twins as in Figure 1.

Lemma 4.2. *Given a marked abelian differential (C, m, ω) on a non-singular curve C, we can perform a finite number of Schiffer variations so that the resulting abelian differential (C', m', ω') has no nodes, only simple zeroes and a pair of twin saddle connections between distinct zeroes of ω'.*

Proof. First notice that it suffices to perform a finite number of Schiffer variations so that the resulting form (C'', m'', ω'') has no nodes, and a pair of twin saddle connections between distinct zeroes of ω''. Indeed, if the form contains multiple zeroes, or if the interior of the twins contain zeroes of ω'', then performing small Schiffer variations at these zeroes or splitting the zeroes furnishes our desired form (C', m', ω').

This said, our statement is analogous to [2, Proposition 8.1], and the proof is basically the same. The idea is to inductively try to gather the zeroes of the form ω, by performing Schiffer variations, in order to get a form with one multiple zero. If we succeed then our lemma is an easy application of a theorem of Masur. It may likely happen that along the induction, we already find the desired twin saddle connections before getting to a form with a single zero. In fact the same argument shows that if we reach an abelian differential with a single zero, it means by Remark 4.1 that there is an abelian differential with the same periods on a curve of compact type. However, as was already mentioned, there are examples of periods that occur on abelian differentials on non-singular curves that cannot be attained on stable forms of compact type with at least a node (see Example 5.12)

Up to small Schiffer variations, we can suppose that the integral of ω between any pair of zeroes is not $\mathbb{R}$-colinear to the integral of ω between any other pair of zeroes, nor to any period of ω. Let x, y be a pair of distinct zeroes that minimizes the distance. It follows that any shortest geodesic γ between them is smooth. By our assumption on the position of the zeroes of ω, any twin γ' of γ at x is a geodesic segment starting from x that does not meet any singular point other than y. Moreover, γ' does not meet y in the time interval $[0, d(x, y))$. If $\gamma'(d(x, y)) = y$ then the pair γ, γ' gives the solution to the lemma, up to small Schiffer variations as we explained in the first paragraph. Otherwise we succeed in joining y to x by performing the Schiffer variation along the pair of twins γ, γ' starting from x, hence decreasing the total number of zeroes of ω.

Repeating this procedure inductively, we either find a pair of twin saddle connections on the road with common starting and endpoints as before, or we end up with an abelian differential on a non-singular curve of genus g with a unique zero of order $2g-2$. In the latter case, we appeal to a theorem of Masur [13, Theorem 2], stating that our abelian differential has a maximal cylinder formed by parallel geodesics. Each boundary component of this cylinder is necessarily a geodesic through the (unique) saddle point. A choice of orientation in one of the geodesics provides an orientation of the boundary components that makes them a couple of oriented geodesic twins. Performing a small Schiffer variation along those twins gives the solution to the Lemma (see Remark 4.1).

□

4.5. **Degeneracy.**

Proposition 4.3. *Every connected component of* $\overline{\mathcal{S}}'(p)$ *intersects* $\mathcal{S}(p)$ *and the boundary.*

Proof. Suppose first that the chosen connected component has an element (C,m,ω) on a smooth curve C. Then, up to applying 4.2 we can suppose ω admits a pair of twin saddle connections γ,γ' between two distinct simple zeroes. The closed curve $\sigma=\gamma'\star\gamma^{-1}$ satisfies $\int_\sigma \omega = 0$. The Schiffer variation along this pair of twins produces a marked stable form with a single simple node that pinches $[\sigma]$ and has the same periods $p=\mathrm{Per}(\omega)$. If the homology class $[\sigma]\in H_1(\Sigma_g)$ is trivial, the node is separating. If it is non-trivial, then the node is non-separating. In either case it determines a boundary point in $\overline{\mathcal{S}}'(p)$.

If (C,m,ω) has a node, it is simple, and we can apply a smoothing of simple node as in Figure 1 to obtain an abelian differential on a smooth curve that has the same periods.

4.6. **Local topology around boundary points.** The next Lemma tells us that around any pair of points in $\mathcal{S}(p)$ that are close to a boundary point of $\overline{\mathcal{S}}'(p)$ lie in the same connected component of $\mathcal{S}(p)$.

Lemma 4.4. *Let* (C,m,ω) *be a marked stable form of genus* $g\geq 2$ *with a single simple node and periods* $p\in\mathcal{H}_g$*. Then there exists a basis of connected neighbourhoods* $U\subset\overline{\mathcal{S}}'_g(p)$ *of* (C,m,ω)*, such that* $U\cap\mathcal{S}(p)$ *is nonempty and connected.*

Proof. Let U be a connected neighbourhood of (C,m,ω) in $\Omega'\overline{\mathcal{S}}_g$ and consider its projection $V=\{(C,\omega)\in\overline{\Omega\mathcal{M}_g} : (C,m,\omega)\in U\}$. First we construct a holomorphic map

$$G : W \to \Omega\overline{\mathcal{M}}_g$$

defined on a trivial horizontal foliation of $W=\mathbb{D}^{2g-3}\times\mathbb{D}^{2g}$ such that $G(0)=(C,\omega)$, $V\subset G(W)$, each horizontal leaf in W is mapped onto an isoperiodic set and $\Delta=\{w\in W : G(w) \text{ has a node }\}$ forms a divisor transverse to the leaves.

The construction of the map G depends on the type of node.

Case 1: The node of ω is separating.

Then $\omega=\omega_1\vee\omega_2$ where ω_i is an abelian differential on a smooth curve C_i of genus g_i with a marked point $q_i\in C_i$ that is not a zero of ω_i. Let U_i be a flow box of the isoperiodic foliation $\mathcal{F}_i$ around (C_i, ω_i, q_i) in the space $\Omega\mathcal{M}_{g_i,1}$ of abelian differentials with a marked point. Define $W = U_1 \times U_2 \times (\mathbb{D}, 0)$ and for $(\eta_1, \eta_2, \varepsilon) \in W$ define $G(\eta_1, \eta_2, \varepsilon)$ to be the abelian differential constructed by attaching η_1 and η_2 in the following way: slit each η_i following the number ε from the marked point q_i and glue the slit forms together in the orientable way. Call it $\eta_1 \sqcup_\varepsilon \eta_2$. When $\varepsilon = 0$ we obtain a nodal abelian differential $\eta_1 \vee \eta_2$ with a simple node. Otherwise it is an abelian differential on a non-singular surface of genus g having a pair of twin saddle connections between two distinct zeroes obtained from the glued slits. The map G is holomorphic and by construction, if L_i is a plaque $\mathcal{F}_i$ on U_i, the image of $L_1 \times L_2 \times (\mathbb{D}, 0) \setminus 0$ by G is contained in a leaf of the isoperioic foliation $\mathcal{F}_g$. On the other hand the stable forms in the image of $L_1 \times L_2 \times 0$ have all the same periods. Remark that by construction $G_{|U_1 \times U_2 \times 0}$ is a foliated biholomorphism onto its image.

Case 2: The node of ω is non-separating.

Let (C_1, ω_1) be the normalization of ω marked at two distinct points q_1 and q_2 where ω_1 does not vanish. Choose a flow box U_1 of the isoperiodic foliation $\mathcal{F}_{g-1}$ on $\Omega\mathcal{M}_{g-1,2}$ around $(C_1, \omega_1, q_1, q_2)$. Define $W = U_1 \times (\mathbb{D}, 0)$ and for each $(\eta, \varepsilon) \in W$ define $G(\eta, \varepsilon)$ as the abelian differential defined by slitting η on the segment of length and direction ε at the marked points q_1, q_2 and gluing the two slits together in the orientable way. Again, for any plaque L of $\mathcal{F}_{g-1}$, the image of $L \times \mathbb{D} \setminus 0$ is contained in a leaf of the isoperiodic foliation and $L \times 0$ is sent to a family of isoperiodic nodal abelian differentials with non-separating simple nodes. Again, $G_{|U_1 \times 0}$ is a foliated biholomorphism onto its image.

The points of W corresponding to boundary points via G form a divisor $\Delta \subset W$. The product of the horizontal foliations of the U_i's in $W \setminus \Delta$ is mapped by G to the isoperiodic foliation. By construction it extends to a regular foliation in W that is transverse to Δ. The intersection of a leaf with this divisor is a regular proper analytic set in the divisor and in the leaf. Therefore, it is locally connected and does not locally disconnect. Taking out the nodal forms to a leaf in W does not disconnect the leaf.

Remark that all points in the image of G that have non-singular underlying curves have a pair of twin geodesics that join two simple zeroes. The following lemma shows that the image of G contains all leaves of the isoperiodic foliation on $\Omega\mathcal{M}_g$ that accumulate on (C, ω).

Lemma 4.5. *Let $r > 0$ and $(C, \omega) \in \Omega\overline{\mathcal{M}}_g$ be a stable form with a single simple node. Then there exists a sufficiently small neighbourhood U of (C, ω) such that every form $(C', \omega') \in U$ satisfies one, and only one, of the following properties:*

- $\mathrm{vol}(\omega') = \infty$
- *C' has a simple node on a vanishing cycle*
- *C' is non-singular and there exists a unique couple of geodesics γ, γ' of length $< r$ forming a pair of twin saddle connections between two distinct simple zeroes of ω' and such that $\gamma' \star \gamma^{-1}$ is a vanishing cycle (see Figure 1).*

Proof. See the proof of Proposition 5.5 in [1]. □

By construction, all the points that belong to the image of G have finite volume. On the other hand, the leaves of the isoperiodic foliation are also contained in the part of finite volume. Therefore, only one leaf of the isoperiodic foliation accumulates on (C, ω). It is the one containing the image $L = G(L_0)$ of the leaf $L_0 \subset W \setminus \Delta$ whose closure contains the origin. For any neighbourhood U of (C, ω), $U \cap L$ is the image of a connected set under the map G.

If C has a separating node then there are no elements in the neighbourhood of (C, ω) with infinite volume. Hence G is surjective onto a neighbourhood of (C, ω). Therefore we have a holomorphic extension of the isoperiodic foliation to a regular holomorphic foliation around (C, ω) that is transverse to the boundary component containing (C, ω). The forgetful map $\pi_g : \Omega\overline{\mathcal{S}}_g \to \Omega\overline{\mathcal{M}}_g$ is a local homeomorphism around (C, m, ω), so the topological properties of L_0 describe those of $\overline{\mathcal{S}}'(p)$ at (C, m, ω).

The case of C with a non-separating node is more delicate. There are elements in any neighbourhood of ω that have infinite volume, namely those with non-zero residues at the branches of the nodes. However, the image of G is contained in the set of forms whose integral on the vanishing cycle is zero, and its image covers a neighbourhood of (C, ω) in this set. The isoperiodic foliation cannot be extended topologically to a non-singular foliation in the neighbourhood of the point (C, ω). A local picture of the isoperiodic foliation around (C, ω) can be found in Figure 3. The main point here is that the infinite volume part of the boundary component $\mathbb{Z}a$ that contains (C, ω) can be also foliated by a foliation of dimension $2g - 3$ defined by the equivalence relation of having the same periods along all cycles. Its leaves are formed by isoperiodic sets that can be obtained by applying small Schiffer variations to the $2g - 3$ zeroes of the forms (recall that when there is some residue at the node there are $2g - 2$ zeroes of the forms that we can move to change the abelian differential without changing the periods on cycles). Since in any neighbourhood of (C, ω) there are leaves of this foliation in $\mathbb{Z}a$ and leaves of the isoperiodic foliation that go across the boundary component, we deduce that the isoperiodic foliation cannot be extended as a regular foliation to (C, ω). Nevertheless, the previous argument shows that there is only one leaf of the isoperiodic foliation on $\Omega\mathcal{M}_g$ that accumulates on (C, ω). It is not difficult to see that the union of the local leaf with its accumulation forms a germ of invariant analytic set in $\Omega\overline{\mathcal{M}}_g$. In the language of singular holomorphic foliation theory, such a subset is called a separatrix of the singularity.

To deduce the result of the Proposition in the case where C has a non-separating node, we need to show that $\pi_g^{-1}(L)$ is still connected.

We need just to check that there is a closed path in L_0 that is mapped by G to a loop in $\Omega\mathcal{M}_g$ that, when lifted to $\Omega\mathcal{S}_g$, joins a point (C_0, m_0, ω_0) to $(C_0, m_0 \circ D_a, \omega_0)$, where D_a denotes the action of the Dehn twist around the pinched class a on homology. Given a sufficiently small $r > 0$, the path

$$t \mapsto G(\omega_1, re^{2\pi i t}) \text{ for } t \in [0, 1]$$

does the job. This can be shown directly by marking the surface and following the marking or by using Picard-Lefschetz theory to the tautological curve bundle over the disc $G(\omega_1, \mathbb{D})$.

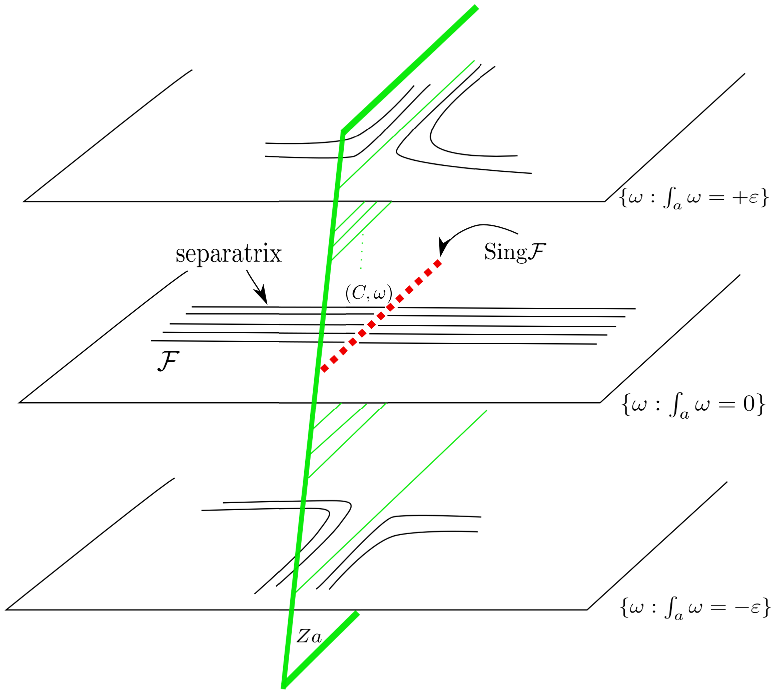

FIGURE 3. Local structure of $\mathcal{F}$ around (C,ω) with a non-separating node

□

□

5. Boundary components containing stable forms of periods $p \in \mathcal{H}_g$

Given $p \in \mathcal{H}_g$ we define

$$\mathcal{Z}_p = \{V \in \mathcal{Z} : V'(p) \neq \emptyset\}.$$

It corresponds to the boundary components that cut $\overline{\mathcal{S}}'(p)$.

By Proposition 4.3 we already know that $\mathcal{S}(p) \neq \emptyset$ implies that $\mathcal{Z}_p \neq \emptyset$.

In this section we will show that $\mathcal{S}(p) \neq \emptyset$ whenever $p \in \mathcal{H}_g$, i.e. Theorem 2.9. This will allow us to give an algebraic characterization of the elements $V \in \mathcal{Z}$ that belong to $\mathcal{Z}_p$.

Given an abelian differential (C,ω) without residues or zero components on a nodal curve, its restriction $\omega_j = \omega_{|C_j}$ to each component C_j of its normalization $\widehat{C}$ defines a nonzero $\omega_j \in \Omega(C_j)$ without residues that satisfy

$$\mathrm{vol}(\mathrm{Per}(\omega)) = \mathrm{vol}(\omega) = \sum \mathrm{vol}(\omega_j) = \sum \mathrm{vol}(\mathrm{Per}(\omega_j)) \tag{7}$$

Suppose $H_1(\Sigma_g) = V_1 \oplus \cdots \oplus V_k$ where V_j are pairwise orthogonal symplectic modules and there exists a marking $m : H_1(\Sigma_g) \to H_1(C) \to H_1(\widehat{C})$ satisfying $m(V_j) = H_1(C_j)$. The periods of ω_j marked by V_j via $m_{|V_j}$ coincide with $p_{|V_j}$ by construction. Hence, up to identifying V_j with $H_1(\Sigma_{g_j})$, the homomorphism $p_{|V_j}$ is a Haupt homomorphism for every j. This motivates the following algebraic definitions:

Definition 5.1. A homomorphism $h : V \to \mathbb{C}$ from a symplectic $\mathbb{Z}$-module V of rank $2g$ to $\mathbb{C}$ is said to be Haupt if under some symplectic identification $V \simeq H_1(\Sigma_g)$ it is a Haupt character in $H^1(\Sigma_g, \mathbb{C})$.

Definition 5.2. Given $p \in H_1(\Sigma_g, \mathbb{C})$ with $\mathrm{vol}(p) > 0$, a decomposition $V_1 \oplus \ldots \oplus V_k$ of a submodule $V \subset H_1(\Sigma_g)$ into orthogonal pairwise symplectic submodules is p-admissible if each $p_{|V_j}$ is a Haupt homomorphism.

Definition 5.3. If V is a symplectic $\mathbb{Z}$-module and $h : V \to \mathbb{C}$ is a homomorphism we define $\mathrm{vol}(h) = \Re h \cdot \Im h$.

To calculate volumes effectively we use the following

Lemma 5.4. *Let V be a symplectic $\mathbb{Z}$-module and $h : V \to \mathbb{C}$ a homomorphism. Then for any symplectic basis $\{a_i, b_i\}$ of V, that is a basis where all pairwise products are zero except for $a_i \cdot b_i = 1$ and $b_i \cdot a_i = -1$, we have*

$$\mathrm{vol}(h) = \sum \Im(\overline{h(a_i)}h(b_i)).$$

Lemma 5.5. *Suppose V_1 is a symplectic module of rank ≥ 4, and V_2 one of rank ≥ 2. Let $p_i : V_i \to \mathbb{C}$ be a homomorphism for $i = 1, 2$. Suppose p_1 is Haupt and $\mathrm{vol}(p_2) \geq 0$. Then $p = p_1 \oplus p_2 : V_1 \oplus V_2 \to \mathbb{C}$ is a Haupt homomorphism.*

Proof. We already have $\mathrm{vol}(p) = \mathrm{vol}(p_1) + \mathrm{vol}(p_2) \geq \mathrm{vol}(p_1) > 0$. If p were not Haupt, then $\mathrm{vol}(p) = \mathrm{vol}(\mathbb{C}/Im(p))$. On the other hand $\mathrm{Im}(p_1) \subset \mathrm{Im}(p)$ are discrete and therefore

$$\mathrm{vol}(\mathbb{C}/Im(p)) \leq \mathrm{vol}(\mathbb{C}/Im(p_1)) < \mathrm{vol}(p_1) \leq \mathrm{vol}(p)$$

where the strict inequality comes from the Haupt condition for p_1.

□

Since the second condition of a Haupt homomorphism is automatically satisfied for injective homomorphisms, we have that for an injective $p \in \mathcal{H}_g$ every symplectic submodule $V \subset H_1(\Sigma_g)$ of intermediate volume $0 < \mathrm{vol}_p(V) < \mathrm{vol}(p)$ induces a p-admissible decomposition $V \oplus V^\perp$.

In $\Omega\overline{\mathcal{S}}$ we have only forms with at most one node we analyze the boundary components that correspond to a single node.

5.1. Boundary components with a separating node.

Given a marked abelian differential ω with a single separating node and periods $p \in \mathcal{H}_g$ we have a well defined p-admissible decomposition $H_1(\Sigma_g) = V \oplus V^\perp$ that corresponds to the boundary component $V'(p)$ containing ω.

In this subsection we will analyze the possibilites of the volumes for rank two submodules from an algebraic point of view.

Lemma 5.6. *Let W be a unimodular symplectic module of rank $2g \geq 4$ and $p : W \to \mathbb{C}$ a non-trivial homomorphism. Let $a \in W \setminus \ker p$. Then if one of the following conditions hold:*

(1) $\operatorname{rank}(a^\perp \cap p^{-1}(\mathbb{R}p(a))) \leq 2g-3$ *or*
(2) if $\ell \neq \mathbb{R}p(a)$ is a real line in $\mathbb{C}$ satisfying $\operatorname{rank}(p(W) \cap \ell) > 2$,

then for every $\varepsilon_1 < \varepsilon_2$ there exists a symplectic submodule $V \subset W$ of rank 2 such that $a \in V$ and $\varepsilon_1 < \operatorname{vol}_p(V) < \varepsilon_2$.

Proof. Let $b \in W$ be such that $a \cdot b = 1$, $e \in a^\perp$ and $b' = b + e$ and denote $\alpha = p(a)$, $\beta = p(b)$. The volume of $V = \mathbb{Z}a + \mathbb{Z}b'$ is given by

$$\operatorname{vol}_p(V) = \Im(\beta\overline{\alpha}) + \Im(p(e)\overline{\alpha}). \tag{8}$$

If 1) holds, the form $e \in a^\perp \mapsto \Im(p(e)\overline{\alpha}) \in \mathbb{R}$ has image a submodule of $\mathbb{R}$ of rank

$$\operatorname{rank} \quad a^\perp - \operatorname{rank}(a^\perp \cap p^{-1}(\mathbb{R}\alpha)) \geq 2g-1-(2g-3)) \geq 2.$$

Therefore its image is dense in $\mathbb{R}$ and we can find the desired e for any given ε's. On the other hand, 2) implies 1) so the same conclusion holds. □

Proposition 5.7. *Let W be a unimodular symplectic module of rank $2g \geq 4$, and $p : W \to \mathbb{C}$ be a homomorphism whose image is not contained in a real line. Suppose that either p is injective or* $\operatorname{rank}(p) \geq 5$. *Then at least one of the following possibilities occur*

(1) there exists an element $a \in W \setminus \ker p$ such that for any pair of real numbers $\varepsilon_1 < \varepsilon_2$, there exists a rank two symplectic submodule $V \subset W$ containing a such that

$$\varepsilon_1 < \operatorname{vol}_p(V) < \varepsilon_2$$

.

(2) $g = 2$, and for every real line $l \subset \mathbb{C}$, the preimage $p^{-1}(l)$ is either $\{0\}$ or a Lagrangian submodule of W.

Moreover, if $g \geq 3$ there exists a proper submodule $\ker p \subset I \subset W$ such that the conclusion is true for every primitive $a \in W \setminus I$. If $I = \ker p$ does not have the property, then there exists a unique real line $\ell \subset \mathbb{C}$ such that $\operatorname{rank}(p(W) \cap \ell) > 2$. *In this case, the module $I = p^{-1}(\ell)$ does the job.*

Proof. We first treat the case $g \geq 3$. Assume that for every real line $l \subset \mathbb{C}$, $p(W) \cap l$ has rank ≤ 2. Take $a \in W \setminus \ker p$. Then

$$\operatorname{rank}(a^\perp \cap p^{-1}(\mathbb{R}p(a)) \leq \operatorname{rank} p^1(\mathbb{R}p(a)) \leq \operatorname{rank}\ker p + \operatorname{rank}(p(W) \cap \mathbb{R}p(a)) \leq 2g-5+2 = 2g-3$$

and we conclude by Lemma 5.6. Therefore in this case the conclusion with $I = \ker p$ is valid. If there exists a real line $l \subset \mathbb{C}$ such that $p(W) \cap l$ has rank > 2 the conclusion follows by Lemma 5.6. In this case the submodule $I = p^{-1}(l)$ does the job. For the uniqueness of the module I as defined: suppose that there exists $a \in W \setminus \ker p$ and an interval $(\varepsilon_1, \varepsilon_2)$ in $\mathbb{R}$ such that no symplectic submodule $V \subset W$ containing a satisfies $\varepsilon_1 < \operatorname{vol}_p(V) < \varepsilon_2$. Then there exists a real line $l \subset \mathbb{C}$, $p(W) \cap l$ has rank > 2. On the other hand, no real line

$l \neq \mathbb{R}p(a)$ can have this property, since otherwise a would belong to rank two submodules of W of arbitrary volume. Hence the only possibility is that $l = \mathbb{R}p(a)$. The submodule $I = p^{-1}(l)$ is the only of this type that has the desired property.

Next suppose $g = 2$. Then p is injective by assumption. If there exists a real line $l = \mathbb{R}p(w) \subset \mathbb{C}$ with $\mathrm{rank}(a^\perp \cap p^{-1}(l)) = 1$ or $\mathrm{rank}(p^{-1}(l)) > 2$, we can use Lemma 5.6 to find the desired element $a \in W \setminus 0$. Otherwise we have that for every $a \in W \setminus 0$

$$\mathrm{rank}(a^\perp \cap p^{-1}(\mathbb{R}p(a))) = 2 \text{ and } \mathrm{rank}\, p^{-1}(\mathbb{R}p(a)) \leq 2.$$

By injectivity of p this means that $p^{-1}(\mathbb{R}p(a)) \subset a^\perp$ for every a, so $p^{-1}(\mathbb{R}p(a))$ is a Lagrangian.

□

Example 5.8. In $\mathcal{H}_2$ there are examples of injective homomorphisms for wich the volume of symplectic submodules can only take discrete values. They necessarily correspond to case (2) in Proposition 5.7. These examples correspond precisely to the periods of forms belonging to Hilbert modular invariant submanifolds. Assume W is of rank 4, and that the homomorphism $p : W \to \mathbb{C}$ is given on a symplectic basis a_1, b_1, a_2, b_2 by

$$\alpha_1 = 1,\ \beta_1 = i\sqrt{D},\ \alpha_2 = \sqrt{D},\ \beta_2 = i,$$

where $D \geq 2$ is an integer. Then, we claim that for any symplectic submodule V of W we have

$$\mathrm{vol}_p(V) \in \sqrt{D} + \mathbb{Z}. \tag{9}$$

Indeed, taking a symplectic pair a, b of W, and writing

$$a = \sum m_i a_i + m_i b_i, \quad b = \sum_i m'_i a_i + n'_i b_i,$$

we have

$$a \cdot b = m_1 n'_1 - n_1 m'_1 + m_2 n'_2 - n_2 m'_2 = 1.$$

A straightforward computation shows that the volume of $V = \mathbb{Z}a + \mathbb{Z}b$ is

$$\mathrm{vol}_p(V) = n'_2 m_1 - m'_1 n_2 + D(n'_1 m_2 - m'_2 n_1) + \sqrt{D},$$

which ends the proof of equation (9). Even if the possible volumes of symplectic submodules forms a discrete set, there are an infinite number of elements in $\mathcal{V}_p$, all having volumes in a finite set of values. In the given example with $D = 2$ the only possibilities for the volume of a module in $\mathcal{V}_p$ are $\sqrt{2}$ or $\sqrt{2} + 1$.

The previous results will be used to show the existence of p-admissible decompositions with a factor of rank two. They will play an important role in the sequel.

Definition 5.9. Given $p \in H^1(\Sigma_g, \mathbb{C})$ with $\mathrm{vol}(p) > 0$ we define $\mathcal{V}_p \subset \mathcal{Z}$ to be the set of rank two symplectic submodules $V \subset H_1(\Sigma_g)$ satisfying that $p_{|V}$ and $p_{|V^\perp}$ are Haupt homomorphisms. In other words, $V \oplus V^\perp$ is a p-admissible decomposition. A primitive element $a \in H_1(\Sigma_g) \setminus 0$ is said to be p-admissible if it belongs to some $V \in \mathcal{V}_p$.

The role played by the rank of p on lines in Lemma 5.6 makes it useful to introduce

Definition 5.10. Given a homomorphism $p : W \to \mathbb{C}$ from a $\mathbb{Z}$-module W we define its **line rank** as

$$r(p) = \max_{a \in \mathbb{S}^1} \mathrm{rank}_{\mathbb{Z}}(p^{-1}(a\mathbb{R})).$$

Remark that if W is symplectic and $\mathrm{vol}(p) > 0$ then $r(p) < 2g$. Also, if p is injective and $r(p) > g$, then the maximum is attained by a unique real line $\ell_{max} \subset \mathbb{C}$ containing 0.

Corollary 5.11. *Let $g \geq 2$. If $p \in \mathcal{H}_g$ is injective, then the sets $\mathcal{V}_p$ and $\mathcal{Z} \setminus \mathcal{V}_p$ are infinite. If there exists a primitive element $a \in H_1(\Sigma_g)\backslash 0$ that is not p-admissible, then $r(p) \geq 2g-2$. If moreover $g \geq 3$, every real line $l \neq \ell_{max} = \mathbb{R}p(a)$ satisfies* $\mathrm{rank}\, p^{-1}(l) \leq 2 < r(p)$.

Proof. If p falls into case (2) of Proposition 5.7 we appeal [14, Theorem 1.2, p. 2274]. Otherwise it suffices to make distinct choices of intervals $(\varepsilon_1, \varepsilon_2)$ in $(0, \mathrm{vol}(p))$ or $\mathbb{R} \setminus [0, \mathrm{vol}(p)]$ to construct examples of modules in $\mathcal{V}_p$ or $\mathcal{Z} \setminus \mathcal{V}_p$ close to any given volume. By choosing different volumes all the constructed submodules are different. The rest of the statements are direct applications of Lemma 5.6. ☐

In genus $g \geq 3$ there exist examples of $p \in \mathcal{H}_g$ with big kernel that do not admit any p-admissible decompositions. Hence $\mathcal{V}_p = \emptyset$.

Example 5.12. Take W a symplectic module of rank $2g \geq 6$ and a homomorphism p defined on a symplectic basis $\{a_i, b_i\}$ by $p(a_1) = p(a_2) = 1$, $p(b_1) = p(b_2) = i$ and zero elsewhere. In this case $\mathrm{vol}(p) = 2$ and $\mathrm{rank}(\mathrm{Ker}(p)) = 2g - 2$, but it is not a symplectic submodule, so p is a Haupt homomorphism. The volume of any symplectic submodule of W is an integer. Suppose there exists a p-admissible decomposition $V_1 \oplus V_2$ of $H_1(\Sigma_g)$. The volume of each component is a positive integer. Hence, the only possibility is that each component has volume one and thus $\mathrm{vol}(V_i) = \mathrm{vol}(\mathbb{C}/\mathrm{Im}(p))$. One of both factors, say V_1, has even rank ≥ 4 so $p|_{V_1}$ is not a Haupt homomorphism.

In the next subsection we will see that in the case of non-injective $p \in \mathcal{H}_g$ we can always find cyclic submodules in $\mathcal{Z}$.

5.2. **Boundary components with a non-separating node.** Let (C, m, ω) be an abelian differential having a single non-separating node that has zero residues. Let $p \in \mathcal{H}_g$ be its periods. Then the normalization $\hat{C}$ of C is a genus $g - 1$ smooth curve. The module $\mathrm{Ker}(m) = \mathbb{Z}a \neq 0$ belongs to $\mathcal{Z}$ and the unimodular symplectic form on $H_1(\Sigma_g)$ induces a unimodular symplectic form on $a^\perp/\mathbb{Z}a$. The latter is naturally isomorphic to $H_1(\hat{C})$ via the marking. The homomorphism

$$p_a : a^\perp/\mathbb{Z}a \to \mathbb{C}$$

induced by p corresponds to the periods of the form induced by ω on $\hat{C}$, hence p_a is a Haupt homomorphism.

Remark 5.13. Given any symplectic submodule V of rank two containing a primitive element $a \in \ker p$, there is a natural symplectic isomorphism between $V^\perp$ and $a^\perp/\mathbb{Z}a$. Under this identification $p_{|V^\perp}$ is equal to p_a.

The next lemma describes the boundary components of $\Omega\overline{\mathcal{S}}_g$ with a non-separating node that are candidates to contain an abelian differential of periods $p \in \mathcal{H}_g$.

Lemma 5.14. *Let $g \geq 2$ and $p \in \mathcal{H}_g$ be a Haupt character with $\ker p \neq 0$. Then either all primitive elements $a \in \ker p$ satisfy that p_a is a Haupt homomorphism or $g \geq 3$, $\ker p$ has rank $\geq 2g-3$ and contains a symplectic submodule W of rank $2g-4$. All primitive $a \in W$ satisfy that p_a is a Haupt homomorphism.*

Proof. Suppose that there exists a primitive element $a \in \ker p$ such that p_a is not Haupt and take a rank two sympelctic submodule V containing a. Hence $p_{V^\perp}$ is of positive volume. If V is contained in $\ker p$, then $p_{|V^\perp}$ must be Haupt, since otherwise $\ker p$ would be a symplectic submodule of rank $2g-2$. So, the only possibility is $p(V) \neq 0$.

If $g = 2$ then $V^\perp$ has rank two and must have positive volume, so $p_{V^\perp}$ is Haupt. A contradiction. Thus in genus $g = 2$ every $a \in \ker p$ satisfies that p_a is Haupt.

If $g \geq 3$, then $p_{|V^\perp}$ is of positive volume and is not Haupt. Thus $\ker(p)$ contains a symplectic submodule W of rank $2g-4$ and every element in W is contained in a symplectic submodule of $\ker p$ of rank 2. □

5.3. **Haupt's Theorem, Realization of p-admissible decompositions and pinchable classes.** In this subsection we will give a proof of Theorem 2.9 in the spirit of Haupt's original proof by using p-admissible decompositions and pinchable elements. The proof goes by induction.

For genus $g = 1$ the condition $\mathrm{vol}(p) > 0$ implies that p is injective and the image of p is a lattice $\Lambda \subset \mathbb{C}$. Therefore the abelian differential dz on $\mathbb{C}/\Lambda$ has periods p.

For $g = 2$ the argument in Proposition 2.3 of [14] shows that any character $p \in H^1(\Sigma_2, \mathbb{C})$ of positive volume is the period map of an abelian differential ω on a nodal curve of compact type of genus two [2]. If ω does not have a zero component we can smooth out the node to obtain an abelian differential on a smooth curve. Otherwise p is a pinching of a handle, contrary to assumption.

Suppose $g \geq 3$ and that every Haupt period of genus up to $g-1$ is the period of some abelian differential on a marked smooth curve. Let $p \in \mathcal{H}_g$.

Case 1: If p is injective. By Lemma 5.7 there exists a p-admissible decomposition $V_1 \oplus V_2$ of $H_1(\Sigma_g)$ where V_1 has rank two. The homomorphism $p_i = p_{V_i}$ satisfies Haupt'conditions. By inductive hypothesis, for $i = 1, 2$ take an abelian differential ω_i of periods p_i, choose points q_i where $\omega_i(q_i) \neq 0$ and consider the stable form $\omega_1 \vee \omega_2$. By smoothing the node we obtain an abelian differential on a smooth curve with periods $p = p_1 \oplus p_2$.

[2]This argument uses that any marked principally polarized abelian variety of dimension two is in the Schottky locus, i.e. the image of the Torelli map. Haupt uses a different argument for the case $g = 2$

Case 2: Suppose $\ker p \neq 0$. By Lemma 5.14 there exists a class a for which p_a is a Haupt homomorphism.

To prove that the candidate classes to be pinched are effectively pinched by some abelian differential we will use the following

Lemma 5.15. *Let (C, ω) be an abelian differential on a smooth curve of genus $g \geq 1$. For any $z \in \mathbb{C}$ there is an immersed arc β in C so that $\int_\beta \omega = z$. Moreover, if $g \geq 2$, the arc β can be chosen to be embedded with distinct endpoints.*

Proof. Let $\tilde{C}$ be the universal cover of C and $\tilde{\omega}$ the lift of ω to $\tilde{C}$. The map $I : \tilde{C} \to \mathbb{C}$ defined by $x \mapsto \int_{x_0}^x \tilde{\omega}$ is onto. Therefore there exists an immersed arc β in C, starting from x_0, such that $\int_\beta \omega = z$. If $g \geq 2$ we can require x_0 to be a zero of ω. In this case the map I is a branched covering of degree > 1 near x_0. Thus, if the endpoints of both β coincide with x_0, we can move them a little so that $\int_\beta \omega$ does not change and they become distinct. Now, since β is an arc, via an homotopy relative to endpoints we can eliminate all self-intersections so that β becomes embedded. □

Lemma 5.16. *Suppose Theorem 2.9 is true up to genus $g-1$ and let $p \in \mathcal{H}_g$. Then a primitive element $a \in \ker p$ is pinched by some abelian differential of periods p if and only if p_a is Haupt.*

Proof. Let $a \in \ker p$ be a primitive element such that p_a is Haupt. Choose b such that $a \cdot b = 1$, define $V_1 = \mathbb{Z}a \oplus \mathbb{Z}b$, $V_2 = V_1^\perp$, and $p_i = p_{|V_i}$. By construction $V_2 \cong a^\perp/\mathbb{Z}a$ as symplectic modules and $p_2 = p_a$ under this equivalence, meaning that p_2 is a Haupt homomorphism.

Suppose first that $g = 2$. Then p_2 is of positive volume and V_2 being of rank two, this implies that $p_2(V_2)$ is a lattice. On the other hand $p(b_1)$ does not belong to the lattice Λ, since otherwise $\mathrm{vol}(p) = \mathrm{vol}(\mathbb{C}/\Lambda)$ and p would not be a Haupt homomorphism. Therefore $p(b_1) \in \mathbb{C}$ describes a path with distinct endpoints in $(\mathbb{C}/\Lambda, dz)$ that we can glue to obtain a nodal curve of genus 2. We leave it to the reader to describe the appropriate marking to guarantee that its periods are given by p and a is collapsed to the node.

Next suppose by induction that for some $g \geq 3$ we have proved the Lemma for all genera up to $g-1$. In particular, p_2 is the period of a marked abelian differential ω_2 on a genus $g - 1 \geq 2$ smooth curve. By Lemma 5.15 we can find an embedded arc β with distinct endpoints in ω_2 of length $p(b_1)$. Glue the endpoints and mark the obtained nodal curve to guarantee that the period character of the stable form is p and the class a is pinched to the node. This finishes the proof of the Lemma. □

By smoothing the non-separating node of the form obtained in Lemma 5.16 we obtain a form of periods p on a smooth curve. This finishes the proof of the inductive step in Theorem 2.9.

Lemma 5.16 motivates the following

Definition 5.17. Let V be a symplectic module and $p : V \to \mathbb{C}$ be a Haupt homomorphism. We say that a primitive element $a \in V$ is pinched by p if $a \in \mathrm{Ker}(p)$ and p_a is a Haupt homomorphism.

Corollary 5.18. *Let $p : H_1(\Sigma_g) \to \mathbb{C}$ be a Haupt homomorphism, $H_1(\Sigma_g) = V_1 \oplus \ldots \oplus V_k$ a p-admissible decomposition. Then there exists an abelian differential with periods p and $k-1$ separating nodes such that each $p_{|V_j}$ corresponds to the periods of a component of its normalization.*

Proof. We have $0 < \mathrm{vol}_p(V_j) < \mathrm{vol}(p)$ for each factor V_j of the decomposition. Denote $2g_j$ the rank of V_j and choose isomorophisms $m_j : H_1(\Sigma_{g_j}) \cong V_j$. For each j an abelian differential ω_j on a curve C_j of genus g_j and periods $p \circ m_j \in H^1(\Sigma_{g_j}, \mathbb{C})$. Any form of the type $\omega = \omega_1 \vee \cdots \vee \omega_k$ on $C_1 \vee \cdots \vee C_k$ marked by $m = m_1 \oplus \cdots \oplus m_k$ has periods p. By construction all those forms ω share the $k-1$ separating nodes we have introduced. □

Corollary 5.19. *Let $g \geq 2$, $p \in \mathcal{H}_g$. Then $\mathcal{Z}_p$ is non-empty. An element $V \in \mathcal{Z}$ belongs to $\mathcal{Z}_p$ if and only if $V \oplus V^{\perp}$ is a p-admissible decomposition or $V = \mathbb{Z}a$ where a is pinchable for p.*

Proof. By Theorem 2.9, $\mathcal{S}(p)$ is nonempty. By proposition 4.3 there exists some element in the boundary of $\overline{\mathcal{S}}'(p)$, so $\mathcal{Z}_p \neq \emptyset$. For the characterization part, apply Lemma 5.16 for pinchable classes or Corollary 5.18 for p-admissible decompositions. □

Corollary 5.19 shows in particular that there are many components of the boundary of $\Omega\overline{\mathcal{M}}_g$ that are not accumulated by leaves of the isoperiodic foliation: for instance, any symplectic $V \in \mathcal{Z}$ such that $\mathrm{vol}_p(V) < 0$ is not accumulated by the leaf associated to $p \in \mathcal{H}_g$.

6. Connecting different isoperiodic boundary components

In this section we prove that the different boundary components of $\overline{\mathcal{S}}'(p)$ can be connected by paths in $\overline{\mathcal{S}}'(p)$.

Proposition 6.1 (Connecting different boundary components)**.** *Let $g \geq 2$ and $p \in \mathcal{H}_g$. Suppose that for all $V \in \mathcal{Z}$, $V'(p)$ is connected. Then every pair $V_1, V_2 \in \mathcal{Z}_p$ lie in the same connected component of $\overline{\mathcal{S}}'(p)$.*

Remark that if $\overline{\mathcal{S}}'(p)$ is connected then the claim is obvious. In particular there is nothing to prove if $g = 2$ or 3. For the proof we introduce some notation:

Definition 6.2. Let $p \in \mathcal{H}_g$ with $g \geq 4$. Given two connected $V_1, V_2 \in \mathcal{Z}_p$, we say that V_1 and V_2 are equivalent and write $V_1 \sim V_2$ if $V_1'(p)$ and $V_2'(p)$ lie in the same connected component of $\overline{\mathcal{S}}'(p)$.

To prove Proposition 6.1 we need to show that there is only one equivalence class in $\mathcal{Z}_p$. Remark that if $V_2 = V_1^{\perp}$ we have $V_1'(p) = V_2'(p)$ and hence $V_1 \sim V_2$. More generally, if

$V_1, V_2 \in \mathcal{Z}_p$ are factors appearing in the same p-admissible decomposition, say $V_1 \oplus V \oplus V_2$ of $H_1(\Sigma_g)$ we can construct a form ω_{12} of periods p with two simple nodes by Corollary 5.18, that shares a node with an element of $V_1'(p)$ and another with an element of $V_2'(p)$. By smoothing the nodes of ω_{12} we obtain an abelian differential ω on a smooth curve that is connected to a point of $V_1'(p)$ and also to a point in $V_2'(p)$. This implies $V_1 \sim V_2$.

In particular, if two p-admissible decompositions share a factor, then all factors appearing in any of the decompositions are equivalent.

6.1. **Proof of Proposition 6.1 in the injective case.** Let $g \geq 4$ and $p \in \mathcal{H}_g$ be injective. Then there are no abelian differentials with non-separating nodes having periods p. Therefore in $\mathcal{Z}_p$ there are only symplectic submodules. Thanks to the results in Section 5 the proof is purely algebraic. Along this subsection the primes will be used to distinguish submodules, not to denote forms with simple nodes.

We first prove that every element $V \in \mathcal{Z}_p$ is equivalent to another $V_1 \in \mathcal{Z}_p$ of rank two. Indeed, if V is of rank strictly between 2 and $2g-2$ then $p_{|V}$ is a Haupt homomorphism and by Corollary 5.19 there exists a $p_{|V}$-admissible decomposition $V_1 \oplus V_2$ of V with a factor V_1 of rank 2. Therefore $V_1 \oplus V_2 \oplus V^{\perp}$ is a p-admissible decomposition and $V \sim V^{\perp} \sim V_1$.

Recall that by Corollary 5.19, $\mathcal{V}_p$ is precisely the set of modules $V \in \mathcal{Z}_p$ of rank two.

Proposition 6.3. *If $g \geq 4$, $p \in \mathcal{H}_g$ is injective and $V, V' \in \mathcal{V}_p$, we have $V \sim V'$.*

We will prove it in several Lemmas :

Lemma 6.4. *Let $g \geq 2$ and $p \in \mathcal{H}_g$ be injective. If $V, V' \in \mathcal{V}_p$ satisfy $V \cap V' \neq 0$, then*

$$V \sim V'.$$

Proof. The cases $g = 2, 3$ are evident by the connectedness of $\overline{\mathcal{S}}'(p)$.

Let $g \geq 4$. If $V = V'$ we are done. Suppose that $V \neq V'$

First step: *there is a symplectic basis $a_1, b_1, \ldots, a_g, b_g$ such that $V = \mathbb{Z}a_1 + \mathbb{Z}b_1$ and $V' = \mathbb{Z}a_1 + \mathbb{Z}(b_1 + m_2' a_2)$ for a certain integer m_2'.*

Proof. The intersection $V \cap V'$ is a primitive submodule of H_1, since both V and V' are primitive. Being of rank 1, we have $V \cap V' = \mathbb{Z}a_1$ with a_1 primitive. Let $b_1 \in V$ (resp. $b_1' \in V'$) such that $a_1 \cdot b_1 = 1$ (resp. $a_1 \cdot b_1' = 1$). These elements exist since V and V' are unimodular. The element $b_1' - b_1$ belongs to $a_1^{\perp}$. For a certain integer n, the element $b_1' + na_1 - b_1$ is also orthogonal to b_1. Change b_1 to $b_1 + na_1$ if necessary. We then have that $b_1' - b_1$ is orthogonal to $V = \mathbb{Z}a_1 + \mathbb{Z}b_1$. Write $b_1 - b_1 = m_2' a_2$ where a_2 is a primitive element of $V^{\perp}$. Completing a_2 into a symplectic basis $a_2, b_2, \ldots, a_g, b_g$ of $V^{\perp}$ gives the desired statement. □

Second step: *If the periods of $(V + V')^{\perp}$ do not lie in a real line of $\mathbb{C}$, there exists a symplectic rank two submodule $W \subset H_1$ such that $V \perp W$, $V' \perp W$ and*

$$0 < \mathrm{vol}_p(W) < \inf(\mathrm{vol}(p_{|V^{\perp}}), \mathrm{vol}(p_{|(V')^{\perp}})). \tag{10}$$

In particular, $V \sim W \sim V'$.

Proof. In the coordinates of the first step, we have $(V+V')^\perp = \mathbb{Z}a_2 + X$ where $X := \sum_{k\geq 3} \mathbb{Z}a_k + \mathbb{Z}b_k$. We apply Proposition 5.7 to $p_{|X}$. If the restriction of p to X belongs to case (1) of that proposition, we are done. If it belongs to case (2), we use the

Proposition 6.5. *Let X be a unimodular symplectic module of rank 4. For every Lagrangian subspace $L \subset X$, there exists a symplectic rank two submodule $Y \subset X$ such that $L \cap Y = \{0\}$.*

Proof. We can assume that $L = \mathbb{Z}a + \mathbb{Z}a'$ is primitive. Let $a_1 = a$ and b_1 be an element of X such that $a_1 \cdot b_1 = 1$. We have $a' = m_1 a_1 + c$ where $c \in (\mathbb{Z}a_1 + \mathbb{Z}b_1)^\perp$ and $m_1 \in \mathbb{Z}$. Up to replacing a' by $a' - m_1 a_1$, we can assume that $m_1 = 0$. Since L is primitive, so is c, so that we can extend the family $a_1, b_1, a_2 = c$ to a symplectic basis of X. The symplectic submodule $Y = \mathbb{Z}(a_1 + b_2) + \mathbb{Z}b_1$ has the desired properties. □

From now on a greek letter will denote the period of the corresponding latin letter. Let $l = \mathbb{R}\alpha_2$, and $L = X \cap p^{-1}(l)$. This space is either $\{0\}$ or a Lagrangian subspace of X since we assume the restriction of p to X is in case (2). By the preceding proposition, there exists a symplectic rank two submodule $Y \subset X$ such that $Y \cap p^{-1}(l) = \{0\}$. Let a', b' be a symplectic basis of Y, and let

$$a = a' + Aa_2, \qquad b = b' + Ba_2,$$

for some $A, B \in \mathbb{Z}$. We have $a \cdot b = a' \cdot b' = 1$, and the volume of $W = \mathbb{Z}a + \mathbb{Z}b$ is given by

$$\mathrm{vol}_p(W) = \Im((\beta' + B\alpha_2)\overline{(\alpha' + A\alpha_2)}) = \Im(\beta'\overline{\alpha'}) + \Im((A\beta' - B\alpha')\overline{\alpha_2}).$$

By construction none of the cycles of Y are mapped by p to an element of the line $l = \mathbb{R}\alpha_2$, so the linear form $(A, B) \in \mathbb{Z}^2 \mapsto \Im(A\beta' - B\alpha')\overline{\alpha_2}) \in \mathbb{R}$ is injective, and thus the volume of W can approximate any real value. Since W is orthogonal to both V and V', this gives the solution to step 2.

□

Step 3. *Assume that the periods of $(V+V')^\perp$ lie on a real line $l \subset \mathbb{C}$. Then $V \sim V'$.*

Proof. We can suppose that $l = \mathbb{R}$ for simplicity. Recall that $X = \sum_{i\geq 3} \mathbb{Z}a_i + \mathbb{Z}b_i \subset (V+V')^\perp$. Let $c \in X$ and let $V'' = \mathbb{Z}a_1 + \mathbb{Z}(b_1 + c)$. The volume of V'' is given by

$$\mathrm{vol}_p(V'') = \mathrm{vol}_p(V) + \Im(\gamma\overline{\alpha_1}).$$

So V'' is admissible as soon as

$$-\mathrm{vol}_p(V) < \Im(\gamma\overline{\alpha_1}) < \mathrm{vol}(p) - \mathrm{vol}_p(V).$$

This equation has an infinite number of solutions $c \in X \setminus \{0\}$. We fix one of them.

We claim that $V'' \sim V$. Indeed, the space $(V + V'')^\perp$ contains the element b_2. Observe that the period β_2 of b_2 is not real, since otherwise all the periods of $V^\perp$ would be real, and so we would have $\mathrm{vol}_p(V) = \mathrm{vol}(p)$ which contradicts $V \in \mathcal{V}_p$. On the other hand, the submodule $c^\perp \cap X$ has rank ≥ 3 and is contained in $(V + V')^\perp$. Since the periods of X are real, this proves that some periods of $(V + V'')^\perp$ are real. We can thus apply Step 2 to the couple (V, V'') to infer $V'' \sim V$.

To prove that $V'' \sim V'$, we observe similarly that $b_2 + m_2' a_1$ belongs to $(V' + V'')^\perp$ and that $\beta_2 + m_2'\alpha_1$ is not real, since otherwise all the periods of $(V')^\perp$ would be real. Then the same argument as before shows that we can apply step 2 to the couple (V', V''). □

□

Lemma 6.4, allows to reduce the equivalence relation $\sim$ on submodules in $\mathcal{V}_p$ to an equivalence relation on the elements that belong to those submodules.

Definition 6.6. Let $p \in \mathcal{H}_g$. A primitive element $w \in H_1(\Sigma_g)$ is said to be p-admissible if it is contained in some module $V \in \mathcal{V}_p$. Two p-admissible elements w, w' are equivalent and denoted $w \sim w'$ if there exist $V, V' \in \mathcal{V}_p$ containing w and w' respectively such that $V \sim V'$.

The transitivity of this relation is proven by the use of Lemma 6.4.

In particular, we already know that if $V \cap V' \neq 0$ then any pair of primitive elements in $V \cup V'$ are equivalent.

If V and W belong to $\mathcal{V}_p$ and there exists some elements $v \in V$ and $w \in W$ such that $v \sim w$, then $V \sim W$. Indeed, we can find $V', W' \in \mathcal{V}_p$ such that $v \in V'$, $w \in W'$ and $V' \sim W'$. By Lemma 6.4 $V \sim V'$ and $W \sim W'$, so $V \sim W$.

Let us analyze the p-admissible elements.

Lemma 6.7. *Given $w_1, w_2, w_3 \in H_1(\Sigma_g)$ such that*

(1) $w_i \cdot w_{i+1} = 1$ for $i = 1, 2$,
(2) $p(w_3) \notin \mathbb{R}p(w_1)$ and
(3) for every real line $\ell \subset \mathbb{C}$ containing 0

$$\mathit{rank}(p^{-1}(\ell) \cap w_1^\perp \cap w_3^\perp) < 2g - 3.$$

Then there exists $w_2' \in H_1(\Sigma_g)$ such that $w_1 \cdot w_2' = w_2' \cdot w_3 = 1$ and $\mathbb{Z}w_1 \oplus \mathbb{Z}w_2'$ and $\mathbb{Z}w_2' \oplus \mathbb{Z}w_3$ belong to $\mathcal{V}_p$. Therefore w_1 and w_3 are p-admissible and $w_1 \sim w_3$.

Proof. Write $w_2' = w_2 + z$ where $z \in w_1^\perp \cap w_3^\perp$. If we show that the image of the map

$$w_1^\perp \cap w_3^\perp \to \mathbb{R}^2$$

defined by $z \mapsto (\mathrm{vol}_p(\mathbb{Z}w_1 \oplus \mathbb{Z}(w_2 + z)), \mathrm{vol}_p(\mathbb{Z}(w_2 + z) \oplus \mathbb{Z}w_3))$ has a point in the square $(0, \mathrm{vol}(p)) \times (0, \mathrm{vol}(p))$ we will be done. The previous map is affine, with linear part

$$\varphi(z) = (\Im(p(z)\overline{p(w_1)}), \Im(p(z)\overline{p(w_3)})).$$

Since $p(w_1)$ and $p(w_3)$ are not $\mathbb{R}$-collinear, $\text{Ker}(\varphi) = 0$ and therefore $\text{rank}(\text{Im}\varphi) = 2g-2$. The topological closure of $\text{Im}\varphi$ in $\mathbb{R}^2$ is either $\mathbb{R}$, $\mathbb{Z}\times\mathbb{R}$ or $\mathbb{R}^2$. Suppose it is not $\mathbb{R}^2$. Then there exists a submodule $H \subset w_1^\perp \cap w_3^\perp$ such that $\varphi(H) \subset \ell \subset \mathbb{R}^2$ for some real line ℓ passing through the origin and $\text{rank}H \geq (2g-2)-1 = 2g-3$. Write $\ell = \{(x,y) : \alpha x + \beta y = 0\}$ and then for each $z \in H$,

$$\Im(p(z)\overline{(\alpha p(w_1) + \beta p(w_3))} = 0.$$

Hence $p(H) \subset \mathbb{R}(\alpha p(w_1) + \beta p(w_3))$ is a submodule of rank at least $2g-3$ and we reach a contradiction with the rank hypothesis. □

Remark that the rank condition of Lemma 6.7 is automatically satisfied if $r(p) < 2g-3$. Also, for $g \geq 4$ and $r(p) \geq 2g-3$ we have to check the rank condition only for ℓ_{max}. In case $r(p) = 2g-1$ we cannot apply the lemma directly.

Lemma 6.8. *For $g \geq 2$, given primitive $w_1, w_4 \in H_1(\Sigma_g)$ such that $w_1 \cdot w_4 = 0$, there exists $w_2, w_3 \in H_1(\Sigma_g)$ such that*

$$w_i \cdot w_{i+1} = 1 \text{ for } i = 1, \ldots, 3$$

Proof. Let b_1 satisfy $w_1 \cdot b_1 = 1$ and b_4 satisfy $b_4 \cdot w_4 = 1$. Since $w_1 \in w_4^\perp$ for any k we have $(b_4 + kw_1)\cdot w_4 = 1$. Choose k as to have $(b_4 + kw_1)\cdot b_1 = 0$ and define $w_3 = b_4 + kw_1$. It is primitive and we can take b_3 satisfying $b_3 \cdot w_3 = 1$. Since $b_1 \in w_3^\perp$ there exists l such that $w_2 = (b_3 + lb_1)$ satisfies $w_1 \cdot w_2 = 1$. □

Lemma 6.9. *If $g \geq 4$, $p : H_1(\Sigma_g) \to \mathbb{C}$ is an injective homomorphims of positive volume and $r(p) < 2g-2$, for any pair of primitive v, w we can find $V, W \in \mathcal{V}_p$ such that $V \sim W$ and $v \in V$, $w \in W$.*

Proof. By taking $z \in v^\perp \cap w^\perp$ and applying twice Lemma 6.8 we can consider a sequence $w_0, w_1, \ldots, w_6 \in H_1(\Sigma_g)$ such that $w_i \cdot w_{i+1} = 1$ for $i = 0, \ldots, 5$, $w_0 = v$, $w_3 = z$ and $w_6 = w$. We claim that there exist $c_2 \in w_1^\perp \cap w_3^\perp$ and $c_4 \in w_3^\perp \cap w_5^\perp$ such that for $w_2' = w_2 + c_2$ and $w_4' = w_4 + c_4$ we have

$$\Im(p(w_2')\overline{p(w_0)}) \neq 0, \quad \Im(p(w_4')\overline{p(w_2')}) \neq 0, \quad \Im(p(w_6')\overline{p(w_4')}) \neq 0 \tag{11}$$

Suppose the first of the inequalities is false for all $c_2 \in w_1^\perp \cap w_3^\perp$. It means that all the periods of the submodule $w_1^\perp \cap w_3^\perp$ of rank $\geq 2g-2$ are contained in a line, contradicting the line rank hypothesis on p. Hence we can already take a solution w_2' to the first inequality. On the other hand if the other pair of inequalities do not hold always, the map

$$w_3^\perp \cap w_5^\perp \to \mathbb{R}^2$$

defined by $c_4 \mapsto (\Im(p(c_4)\overline{p(w_2')}), \Im(p(w_6)\overline{p(c_4)}))$ has its image – a submodule of $\mathbb{R}^2$ of rank $2g-2$ – contained in the pair of real axis $x = 0$ and $y = 0$ where (x,y) are coordinates of $\mathbb{R}^2$. Hence it is contained in a single axis and we reach a contradiction with the hypothesis on the line rank of p.

Since $r(p) < 2g-2$ and the elements $p(w_{2i})$ and $p(w_{2(i+1)})$ are not alligned, all the hypotheses of Lemma 6.7 are satisfied for each of the triples w_i, w_{i+1}, w_{i+2} for $i = 0, 2, 4$. Therefore $w_0 \sim w_6$.

□

The proof of Proposition 6.3 is done for the case of $r(p) < 2g-2$. For the other cases we use the next lemmas:

Lemma 6.10. *Let $g \geq 4$ and $p : H_1(\Sigma_g) \to \mathbb{C}$ an injective Haupt homomorphism with $r(p) \geq 2g-2$. Write $I = p^{-1}(\ell_{max})$ and suppose $v, w \in H_1(\Sigma_g) \setminus I$ are primitive elements such that $v \cdot w = 0$ and $[w] \in v^{\perp}/\mathbb{Z}v$ is also primitive. Then $v \sim w$.*

Proof. Without loss of generality we can suppose $\ell_{max} = \mathbb{R}$. Choose $b \in w^{\perp}$ such that $v \cdot b = 1$. We claim that, up to changing b by $b+e$ for some $e \in I \cap w^{\perp} \cap v^{\perp}$ we can suppose that $V = \mathbb{Z}v \oplus \mathbb{Z}b$ belongs to $\mathcal{V}_p$. Indeed, since $p(e) \in \mathbb{R}$,

$$\mathrm{vol}(\mathbb{Z}v \oplus \mathbb{Z}(b+e)) = \mathrm{vol}(\mathbb{Z}v \oplus \mathbb{Z}b) + p(e)\Im(\overline{p(v)}).$$

By hypothesis the rank of $p(I \cap w^{\perp} \cap v^{\perp})$ is at least $2g-4 \geq 4$ for $g \geq 4$, so the value of the volume of V can be chosen arbitrarily close to any desired value.

Next take $c \in V^{\perp}$ such that $w \cdot c = 1$. Given $f \in w^{\perp} \cap V^{\perp} \cap I$, we have

$$\mathrm{vol}(\mathbb{Z}w \oplus \mathbb{Z}(c+f)) = \mathrm{vol}(\mathbb{Z}w \oplus \mathbb{Z}c) + p(f)\Im(\overline{p(w)}).$$

Again, since the rank of $w^{\perp} \cap V^{\perp} \cap I$ is at least $2g-5 \geq 3$ for $g \geq 4$, we can suppose that c is chosen so that $W = \mathbb{Z}w \oplus \mathbb{Z}c$ belongs to $\mathcal{V}_{p_{|V^{\perp}}}$. By construction $V \perp W$ and $0 < \mathrm{vol}(V) + \mathrm{vol}(W) < \mathrm{vol}(p)$. Therefore $V \sim W$ and also $v \sim w$. □

Lemma 6.11. *Let $g \geq 4$ and $p : H_1(\Sigma_g) \to \mathbb{C}$ an injective Haupt homomorphism with $r(p) \geq 2g-2$. Define $I = p^{-1}(\ell_{max})$. If $v, w \in H_1(\Sigma_g) \setminus I$ are primitive such that*

$$v^{\perp} \cap w^{\perp} \nsubseteq I$$

then $v \sim w$.

Proof. Take a symplectic basis a_i, b_i of $H_1(\Sigma_g)$ such that $a_1 = v$ and

$$w = m_1 a_1 + n_1 b_1 + m_2 a_2.$$

Let $X := \mathbb{Z}a_3 \oplus \mathbb{Z}b_3 \oplus \cdots \oplus \mathbb{Z}a_g \oplus \mathbb{Z}b_g$. If $X \nsubseteq I$ then choose $z \in X \setminus I \cap X$ primitive. By Lemma 6.10 $v \sim z$ and $w \sim z$ therefore $v \sim w$. If $X \subset I$ take $c \in (v^{\perp} \cap w^{\perp}) \setminus I$ and write $c = c_X + c_{\perp}$ where $c_{\perp} \in X^{\perp}$. Then $c_{\perp} \notin I$ and $c_{\perp} \cdot v = c_{\perp} \cdot w = 0$. By changing X by $X' = \mathbb{Z}(a_3 + c_{\perp}) \oplus \mathbb{Z}b_3 \oplus \cdots \oplus \mathbb{Z}a_g \oplus \mathbb{Z}b_g$ and choosing an appropriate basis of $X'^{\perp}$ we fall in one of the previous cases. □

Lemma 6.12. *Under the same hypothesis and notation of Lemma 6.11. Let $v \in H_1(\Sigma_g) \setminus I$ and define*

$$I_v = \{z \in H_1(\Sigma_g) : z^{\perp} \cap v^{\perp} \subset I\}.$$

Then there exists a proper submodule $J \nsubseteq H_1(\Sigma_g)$ such that $I_v \subset J$.

Proof. If $I_v = \emptyset$, the module $J = 0$ does the job. Otherwise take $z \in I_v$. Then $z^\perp \cap v^\perp \subset v^\perp \cap I$. We also have

$$2g - 2 \leq \mathrm{rank}(v^\perp \cap I) < \mathrm{rank}(v^\perp) = 2g - 1$$

where the strict inequality comes from the fact that $v \notin I$ and I is a primitive module. Therefore $\mathrm{rank}(v^\perp \cap I) = 2g - 2$. Its primitive submodule $z^\perp \cap v^\perp$ has also rank $2g - 2$ so the only possibility is that $z^\perp \cap v^\perp = v^\perp \cap I$. Therefore $z \in (v^\perp \cap I)^\perp =: J$ □

Lemma 6.13. *Let $g \geq 4$ and $p : H_1(\Sigma_g) \to \mathbb{C}$ an injective Haupt homomorphism with $r(p) \geq 2g - 2$. Then for any $V, W \in \mathcal{V}_p$ we have $V \sim W$.*

Proof. Again we suppose $\ell_{max} = \mathbb{R}$ and define $I = p^{-1}(\mathbb{R})$. Since V and W are of positive volume we can find primitive elements $v \in V \cap I^c$ and $w \in W \cap I^c$. If $w^\perp \cap v^\perp \not\subset I$ we have $v \sim w$ by Lemma 6.11. Therefore $V \sim W$.

If $w^\perp \cap v^\perp \subset I$ we can consider the union $I \cup I_v \cup I_w$. Since by Lemma 6.12 it is contained in a *union* of proper submodules, it cannot cover the whole of $H_1(\Sigma_g)$. Take $z \in H_1(\Sigma_g) \setminus (I \cup I_v \cup I_w)$. Then by Lemma 6.11 $v \sim z \sim w$, which as before implies that $V \sim W$. □

Proposition 6.3 is now proven for all possible ranks of p. This finishes the proof of Proposition 6.1 for injective periods.

6.2. **Proof of Proposition 6.1 in the non-injective case.** For non-injective Haupt homomorphism there are examples of periods that do not occur on curves of compact type (see Example 5.12). However there are always abelian differentials of non-compact type and non-injective periods $p \in \mathcal{H}_g$ (they correspond to modules $V \in \mathcal{Z}_p$ of rank one). The idea in the non injective case is to use degenerations of an abelian differential in the isoperiodic set to nodal forms of non-compact type, i.e. that pinch some primitive cycle $a \in \ker p \setminus 0$. From Proposition 4.3 we know that there are always nodal curves on the boundary of a connected component of $\overline{\mathcal{S}}'(p)$. The next result is crucial for this approach: it shows that in the boundary of any component of $\overline{\mathcal{S}}'(p)$ there are abelian differentials that pinch a non-trivial cycle in $H_1(\Sigma_g)$, i.e. on curves of non-compact type.

Proposition 6.14. *Let $g \geq 2$ and $p \in \mathcal{H}_g$ with $\ker p \neq 0$. Given $V \in \mathcal{Z}_p$, there exists a pinchable $a \in \ker p$ such that $V \sim \mathbb{Z}a$.*

Proof. By Lemma 5.14 there exist pinchable classes $a \in \ker p \setminus 0$. The claim is obvious for $g = 2, 3$, since there is only one connected component of $\overline{\mathcal{S}}'(p)$.

Let $V_1 \in \mathcal{Z}_p$ be of rank > 1 and define $V_2 = V^\perp$ and $p_i = p_{|V_i}$. Then $V_1 \oplus V_2$ is a p-admissible decomposition and $V_1 \sim V_2$.

If one of the restrictions, say p_1, is not injective, there exists a form η with a non-separating simple node that pinches $a \in V_1$ and periods p_1. On the other hand we can consider a form ω with periods p_2. By smoothing the nodes, the form with two simple nodes $\eta \vee \omega$ allows to connect a point of $V'(p)$ to a point having a non-separating node contained in the boundary component associated to $\mathbb{Z}a$.

If both p_1 and p_2 are injective, we can find p-admissible decompositions of V_1 and V_2 whose factors are of rank 2. Thus, up to changing V_1 by some equivalent element in $\mathcal{Z}_p$ we can suppose V_1 has rank 2, and both p_1 and p_2 are injective.

Next take a primitive element $a \in \ker p$ and write $a = m_1a_1 + m_2a_2$ for primitive $a_i \in V_i$ and coprime $m_1, m_2 \in \mathbb{N}^*$. If $a_2 \in V_2$ is contained in a factor of a p-admissible decomposition $V_2 = V \oplus V^\perp$ of V_2, say $a_2 \in V$, then there exists a form with two separating nodes and associated decomposition $V_1 \oplus V \oplus V^\perp$. On the other hand the factor $V_1 \oplus V$ has nontrivial kernel, since $a \in V_1 \oplus V$. By smoothing the initial separating node to the last form, we are reduced to the case where one of the p_i's has some element in the kernel. In particular this argument works if a_2 is p_2-admissible.

The remaining case is when p_2 is injective and $a_2 \neq 0$ is not p_2-admissible. Then by Lemma 5.6 applied to p_2, the rank of $p_2^{-1}(\mathbb{R}p(a_2)) \cap a_2^\perp)$ is at least $2g-4$. Completing a_2 into a symplectic basis $a_2, b_2, a_3, b_3, \ldots, a_g, b_g$ of V_2, and denoting $V_3 = \mathbb{Z}a_3 + \mathbb{Z}b_3 + \ldots + \mathbb{Z}a_g + \mathbb{Z}b_g$, we conclude that $H = p^{-1}(\mathbb{R}p(a_2)) \cap V_3$ is either V_3 or a corank one primitive submodule of V_3.

In the latter case, by considering an element $w \in V_3$ such that $p(w)$ does not belong to $\ell := \mathbb{R}p(a_2)$, we apply Lemma 5.6 to ℓ and V_3 to construct a symplectic rank two submodule $W \subset V_3$ containing w with $0 < \mathrm{vol}_p(W) < \mathrm{vol}_p(V_2)$. Since p_2 is injective, this implies that $W \in \mathcal{V}_{p_2}$. The splitting $V_1 \oplus W \oplus (W^\perp \cap V_2)$ of $H_1(\Sigma_g)$ is p-admissible. On the other hand $a \in W^\perp \cap \ker p$ so the restriction $p_{|W^\perp}$ is non-injective and we are done.

It remains to treat the case where $H = V_3$, namely $p(V_3) \subset \mathbb{R}p(a_2)$. Up to composing p with a $\mathbb{R}$-linear equivalence from $\mathbb{C}$ to $\mathbb{C}$, we can assume that

$$p(a_1) = n_2,\ \Im p(b_1) = 1,\ p(a_2) = -n_1,\ \Im p(b_2) < 0,$$

and that

$$p(a_k) = \alpha_k \in \mathbb{R},\ p(b_k) = \beta_k \in \mathbb{R} \text{ for } k \geq 3.$$

Also, since the period p_2 is injective, the numbers $\alpha_3, \ldots, \alpha_g, \beta_3, \ldots, \beta_g$ are linearly independent over $\mathbb{Q}$. The strategy in that case is to find a p_2-admissible decomposition $V_2 = V_1' \oplus V_2'$ of V_2 such that V_1' falls in one of the previous cases. Then, $V_1 \oplus V_1' \oplus V_2$ is a p-admissible decomposition and $V_1 \sim V_1' \sim \mathbb{Z}a$ for some pinchable $a \in \ker p$. Such submodules V_1' satisfy $0 < \mathrm{vol}_p(V_1') < \mathrm{vol}_p(V_2)$. In such a situation, one can write the decomposition of a as a sum $a = n_1''a_1'' + n_2''a_2''$ with $a_1'' \in V_1'$ and $a_2'' \in V_2' = (V_1')^\perp$ (the reason for this notation a_i'' instead of a_i' will become clear hereafter), and we claim that it is possible to find V_1' so that either a_2'' is $p_{V_2'}$-admissible or p_2' is not injective. We already explained that this would conclude the proof.

We are going to look for the module V_1' as being generated by the elements a_1' and b_1', where

$$a_1' = a_2 + \sum_{k\geq 3} m_k a_k + n_k b_k,\ b_1' = b_2,\ a_2' = a_1,\ b_2' = b_1$$

and for $k \geq 3$

$$a'_k = a_k + n_k b_2,\ b'_k = b_k - m_k b_2.$$

Here m_k, n_k are integers that have to be determined for $k \geq 3$. We have

$$\mathrm{vol}_p(V'_1) = -\Im(p(b_2))(n_1 - \sum_{k\geq 3} m_k\alpha_k + n_k\beta_k).$$

Observe also that $\mathrm{vol}_p(V_2) = -n_1\Im(p(b_2))$. We will choose m_k, n_k multiple of n_1, so we write $m_k = m'_k n_1$, $n_k = n'_k n_1$ with m'_k, n'_k integers. We then have

$$\mathrm{vol}_p(V'_1) = \varepsilon \mathrm{vol}_p(V_2) \text{ with } \varepsilon = 1 - \sum_{k\geq 3} m'_k\alpha_k + n'_k\beta_k.$$

Since p_2 is injective, V'_1 is p_2-admissible iff $0 < \varepsilon < 1$. Because of the rational independence of the α_k, β_l's we make a choice of m_k, n_k's so that ε belongs to $(0, 1)$. Further conditions will be imposed later on ε.

Now it remains to understand how the class a decomposes according to the decomposition $V'_1 + (V'_1)^{\perp}$: it is given by $a = n_2 a''_1 + n_1 a''_2$ with $a''_1 = a'_1$ and

$$a''_2 = a'_2 - n_2 \sum_{k\geq 3} m'_k a'_k + n'_k b'_k.$$

Hence, it suffices to see that either a''_2 is p'_2-admissible, or p'_2 is not injective, where $p'_2 = p_{|V'_2}$ and $V'_2 = (V'_1)^{\perp}$. Assuming p'_2 to be injective, the volume of the symplectic rank two submodule $\mathbb{Z}a''_2 + \mathbb{Z}b'_2 \subset V'_2$ (containing a''_2) is

$$\mathrm{vol}_p(\mathbb{Z}a''_2 + \mathbb{Z}b'_2) = (\Im(p(b'_2)\overline{p(a''_2)}) = n_2\big(1 - \sum_{k\geq 3} m'_k\alpha_k + n'_k\beta_k\big) = n_2\varepsilon,$$

while

$$\mathrm{vol}_p(V'_2) = \mathrm{vol}_p - \mathrm{vol}_p(V'_1) = \mathrm{vol}_p - \varepsilon\mathrm{vol}_p(V_2).$$

Hence, as soon as $0 < \varepsilon < \frac{\mathrm{vol}_p}{n_2 + \mathrm{vol}_p(V_2)}$ one concludes that a''_2 is p'_2-admissible. The proof of the proposition is complete since by rational independance of the α_k, β_l's one can make choices of m'_k, n'_l's such that ε satisfies this condition.

□

Given a unimodular symplectic module V of rank $2g$ and a Haupt homomorphism $p : V \to \mathbb{C}$ we say that a primitive $a \in \ker p$ is pinched by p if the map induced by p on $a^{\perp}/\mathbb{Z}a$ is Haupt. Equivalently, up to identifying V with $H_1(\Sigma_g)$ there exists an abelian differential of periods p that pinches a (see Lemma 5.16).

Remark 6.15. Let $p : V \to \mathbb{C}$ be a Haupt homomorphism and $W \subset \ker p$ a symplectic submodule. Then

(1) All primitive elements in W are pinched by p;
(2) p is a Haupt homomorphism if and only if $p_{|W^{\perp}}$ is a Haupt homomorphsim;
(3) $a \in \ker(p) \cap W^{\perp}$ is pinched by p if and only if it is pinched by $p_{|W^{\perp}}$.

Next we prove that abelian differentials that pinch distinct classes in $\ker p$ are equivalent.

Proposition 6.16. *Let $g \geq 4$ and $p \in \mathcal{H}_g$ with $\ker p \neq 0$. Then for every pair of pinchable $a, a_1 \in \ker p$, we have $\mathbb{Z}a \sim \mathbb{Z}a_1$.*

Proof. To abridge notations, we say that two pinchable classes a, a_1 are equivalent and denote it by $a \sim a_1$ if $\mathbb{Z}a \sim \mathbb{Z}a_1$.

The next lemma will be useful for the proof.

Lemma 6.17. *Let $g \geq 4$ and $p \in \mathcal{H}_g$ with $\ker(p) \neq 0$.*

(1) If $W \subset \ker p$ is a symplectic submodule of rank 2, all pinchable elements in $W \cup W^{\perp}$ are equivalent.
(2) If $a_1, a_2 \in \ker p$ are pinchable elements that belong to a symplectic basis and $a_1 \cdot a_2 = 0$, then all pinchable elements in $\mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ are equivalent.
(3) Suppose $H_1(\Sigma_g) = V_1 \oplus V_2$ is a p-admissible decomposition. Pinchable elements of $p_{|V_i}$ are pinchable for p. All elements pinchable by $p_{|V_1}$ or $p_{|V_2}$ are equivalent.

Proof. (1) The restriction $p_{W^{\perp}}$ is a Haupt homomorphism. Fix an abelian differential ω of periods $p_{|W^{\perp}}$ with a node and a simple zero elsewhere (we use $g-1 \geq 3$ here). Take a pair of embedded twins with distinct endpoints and glue the endpoints. Given any primitive $a \in W$ Choose $b \in W$ such that $a \cdot b = 1$. Mark the obtained form by pinching a and associating b to the loop obtained from the twins. Any pair of such marked abelian differentials share a node: the node of ω. This shows that all elements in W are equivalent. Next suppose $a_1 \in W^{\perp}$ is pinchable. Then, since $W \subset \ker p$ is symplectic, a_1 is also pinchable for $p_{|W^{\perp}}$. It thus suffices to take the form ω in the previous argument pinching a_1 to conclude that a_1 is equivalent to any primitive element $a \in W$.

(2) Let a_i, b_i $i = 1, \ldots, g$ denote a symplectic basis, $W_i = \mathbb{Z}a_i \oplus \mathbb{Z}b_i$ and $V = W_3 \oplus \cdots \oplus W_g$.

If $p_{|V}$ is not Haupt, without loss of generality we can suppose $\mathrm{vol}(W_3) > 0$ and $W_4 \oplus \cdots \oplus W_g \subset \ker p$. By (1) the pinchable elements of $W_4 \cup W_4^{\perp}$ are all equivalent. Since $W_1 \oplus W_2 \subset W_4^{\perp}$, all pinchable elements in $W_1 \oplus W_2$ are equivalent.

Since $a_1, a_2 \in \ker p$, we have $\mathrm{vol}(p_{|V}) = \mathrm{vol}\, p > 0$. If $p_{|V}$ is a Haupt homomorphism, then any primitive element in $\mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ is pinchable (see Lemma 5.5). Take a form ω with periods $p_{|V}$ and two simple zeroes. In ω choose embedded paths with distinct endpoints of lengths $p(b_1)$ and $p(b_2)$ respectively that do not intersect. This can always be done if $p(b_1) = p(b_2) = 0$ by taking pairs of short twins at distinct zeroes of ω. If only one of them is non-zero, we can take a very short pair of twins to realize the zero period so as to avoid the path of non-zero length. If both are non-zero and we have initially taken two paths that intersect, we change one of the paths in its homotopy class with fixed endpoints to avoid the intersections. Gluing the endpoints of the said paths and marking the form by pinching a_1 and a_2 and associating b_i to the corresponding loop we obtain a form that pinches a_1 and a_2. Therefore $a_1 \sim a_2$.

In the module $W_1 \oplus W_2$ we can consider the new symplectic basis

$$a_1' = a_1 - a_2, b_1' = b_1, a_2' = a_2, b_2' = b_1 + b_2.$$

Since a_1' is pinchable, we have already shown that $a_1' \sim a_2'$ and thus $a_1 - a_2 \sim a_2$. We can equivalently show that $a_1 + a_2 \sim a_1$. Therefore in the set of ordered basis (v_1, v_2) of $\mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ the transformations $(u, v) \mapsto (v, u)$ and $(u, v) \mapsto (u \pm v, v)$ send a basis of equivalent elements to another basis of equivalent elements, preserving the equivalence class. By Gauss algorithm any primitive element $a \in \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ can be obtained as a member of such a pair after a finite number of applications of these transformations to the pair (a_1, a_2). Hence $a \sim a_1$.

(3) Let $a_i \in V_i$ a $p_{|V_i}$ pinchable element. Let ω_i denote a form of periods $p_{|V_i}$ that pinches a_i. Then $\omega_1 \vee \omega_2$ pinches a_1 and a_2 and therefore $a_1 \sim a_2$. If $a_1, a_2 \in V_1$, we can consider ω_1, ω_2 of periods $p_{|V_1}$ pinching a_1 and a_2 respectively and a form η of periods $p_{|V_2}$. The forms $\omega_1 \vee \eta$ and $\omega_2 \vee \eta$ share a separating node. Therefore $a_1 \sim a_2$. □

Case 1: There exists a symplectic submodule $W \subset \ker p$ containing one of a_1 or a.

Without loss of generality we can suppose W has rank 2 and $a_1 \in W$. Write $a = m_1 a_1' + m_2 a_2$ where $a_1' \in W$, $a_2 \in W^\perp$ is a primitive element. If $m_2 = 0$, Lemma 6.17 item (1) gives $a \sim a_1$. Next suppose $m_2 \neq 0$. Since $a, a_1 \in \ker p$ we have $a_2 \in \ker p$. We also have $p_{|W^\perp}$ is a Haupt homomorphism by item (2) in Remark 6.15.

If a_2 is pinchable for $p_{|W^\perp}$, then it is pinchable for p by item (3) in Remark 6.15. Item (2) in Lemma 6.17 guarantees that $a \in \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ is equivalent to a_1. Otherwise $p_{|W^\perp}$ is not Haupt and by Proposition 2.7 has as kernel a symplectic submodule of rank $2g - 4$. Take $W_3 \subset a_2^\perp \cap W^\perp \cap \ker p$ a symplectic submodule of rank 2. By Lemma 6.17 item (1) the elements of $W_3 \cup W_3^\perp$ are equivalent. Hence $a \sim a_1$.

Case 2: Neither a_1 nor a belong to any symplectic submodule $W \subset \ker p$.

Remark that if a symplectic submodule $V \subset H_1(\Sigma_g)$ satisfies $\mathrm{vol}(V) > 0$ and contains a or a_1, then by Proposition 2.7, $p_{|V}$ is a Haupt homomorphism.

Choose b_1 such that $a_1 \cdot b_1 = 1$. Since a_1 is pinchable, the symplectic submodule $W_1 = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1$ satisfies that $p_1 = p_{|W_1^\perp}$ is a Haupt homomorphism. By hypothesis $p(b_1) \neq 0$. Write

$$a = m_1 a_1 + n_1 b_1 + m_2 a_2 \tag{12}$$

where $a_2 \in W_1^\perp$ is primitive.

Subcase 2.1:If $a \cdot a_1 = 0$.

Then $n_1 = 0$, $a = m_1 a_1 + m_2 a_2$ where m_1 and m_2 are coprime. We have $p(a_2) = 0$. If a_2 is p_1-pinchable, by Remark 5.13 there exists a symplectic decomposition $W_1^\perp = W_2 \oplus V$ where $p_{|V}$ is a Haupt homomorphism and $a_2 \in W_2$. In particular $\mathrm{vol}_p(W_1 \oplus V) > 0$ and $a_1 \in W_1 \oplus V$. Therefore $p_{W_1 \oplus V}$ is a Haupt homomorphism and since $W_2^\perp = W_1 \oplus V$, $a_2 \in W_2$ is pinchable for p. By item (2) in Lemma 6.17 $a \sim a_1$. If a_2 is not p_1-pinchable, then there exists a rank two symplectic submodule $W_3 \subset a_2^\perp \cap W_1^\perp \cap \ker p$ such that $a_1, a \in W_3^\perp$ and we conclude by item (1) in Lemma 6.17.

Remark that with this last argument we have treated all the cases where $a \cdot a_1 = 0$.

Subcase 2.2 If $a \cdot a_1 \neq 0$.

Then $n_1 \neq 0$, $p(a_2) \neq 0$ and $\mathbb{R}p(a_2) = \mathbb{R}p(b_1)$. Remark that $W_1^\perp$ has rank at least 6. There are two subcases

Subsubcase 2.2.1: $\operatorname{rank}(\ker p_1) \geq 2$.

This implies that $\ker p \cap W_1^\perp$ has rank at least two and hence there exists a primitive element $a_3 \in a_2^\perp \cap \ker p \cap W_1^\perp$. If a_3 is p_1-pinchable, then for every symplectic decomposition $W_1^\perp = W_3 \oplus V$ where W_3 is a rank two symplectic module containing a_3, $p_{|V}$ is Haupt. Since $W_1 \oplus V$ is of positive volume and contains a_1, $p_{|W_1 \oplus V}$ is a Haupt homomorphism and thus a_3 is also pinchable for p. Since $a_3 \cdot a_1 = a_3 \cdot a = 0$ we have $a \sim a_3 \sim a_1$.

Subsubcase 2.2.2: $\operatorname{rank}(\ker p_1) \leq 1$.

If a_2 is p_1-admissible, i.e., it belongs to a symplectic module $V_2 \in \mathcal{V}_{p_1}$, there exists a p-admissible decomposition $W_1^\perp = V_2 \oplus V_3$. Then $a_1, a \in V_4 := W_1 \oplus V_2$ and $\operatorname{vol}_p(V_4) = \operatorname{vol}_p(V_2) > 0$. Since $a_1 \in V_4$, $p_{|V_4}$ is a Haupt homomorphism on a rank four symplectic module. By Lemma 5.14 applied for $g = 2$, all elements in $\ker p_{|V_4}$ are pinchable. The decomposition $H_1(\Sigma_g) = V_4 \oplus V_3$ is p-admissible. By item (3) in Lemma 6.17, $a \sim a_1$.

By Lemma 5.7 applied to $W_1^\perp$, there exists a proper submodule $I \subset W_1^\perp$ containing all elements that are not p_1-admissible, i.e. that do not belong to some symplectic module $V_2 \in \mathcal{V}_{p_1}$. So if $a_2 \notin I$ we are done. If $a_2 \in I$ denote $\ell = \mathbb{R}p(a_2)$. We know by Lemma 5.6 that $I = p^{-1}(\ell) \cap W_1^\perp$ has rank at least $2g - 4$ and for every other real line ℓ', $\ell' \cap p(W_1^\perp)$ has rank at most 2. Since $p(a_1), p(b_1) \in \ell$, $p^{-1}(\ell)$ has rank at least $2g - 2 \geq 6$. On the other hand, for every other real line $\ell' \subset \mathbb{C}$ containing 0 we have $\operatorname{rank}(p^{-1}(\ell') \cap W_1^\perp) \leq 2 + \operatorname{rank}(\ker p_1) \leq 3$. Therefore $p^{-1}(\ell')$ has rank at most 5. If we manage to find a decomposition as in equation (12) where the image of the a_2 is outside ℓ we will be done. We are going to show that, up to changing the initial b_1, we can suppose that we fall in this case or one of the previous cases.

Given $w \in a_1^\perp$ define $b_1' = b_1 + w$ and $W_1' = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1'$. Then

$$a = m_1 a_1 + n_1 b_1' + m_2' a_2'$$

where $m_2' a_2' = m_2 a_2 - n_1 w$. If we manage to guarantee that

- $a_2' \in W_1'^\perp$, or equivalently $0 = -n_1(b_1 \cdot w) + m_2(w \cdot a_2)$
- $p(a_2') \notin \ell = \mathbb{R}p(a_2)$ or equivalently $p(w) \notin \ell$,

we will be done: $a_2' \in W_1'^\perp$ will be $p_{|W_1'^\perp}$-admissible.

If there exists $w \in a_2^\perp \cap W_1^\perp \setminus p^{-1}(\ell)$, it constitutes a solution. Otherwise $a_2^\perp \cap W_1^\perp \subset p^{-1}(\ell) \cap W_1^\perp$, and since $W_1^\perp$ has positive volume, any $b_2 \in W_1^\perp$ satisfying $a_2 \cdot b_2 = 1$ satisfies $p(b_2) \notin \ell$. In this case the element $w = m_2 a_1 + n_1 b_2$ provides a solution. □

6.3. **End of proof of the inductive step of Theorem 1.2.** We fix some $g \geq 4$ and suppose that all the fibers of Per of genus up to $g - 1$ are connected. Take $p \in \mathcal{H}_g$. By Proposition 4.3 every connected component of $\overline{\mathcal{S}}'(p)$ has points in the boundary and points in $\mathcal{S}(p)$. By Proposition 3.1 each boundary set $V'(p)$ defined by some $V \in \mathcal{Z}$ is connected. By Proposition 6.1 every pair of distinct non-empty boundary components $V_1'(p)$ and $V_2'(p)$

lie in the same connected component of $\overline{\mathcal{S}}'(p)$. Therefore $\overline{\mathcal{S}}'(p)$ is connected. By Lemma 4.5 this implies $\mathcal{S}(p)$ is connected.

7. Proof of theorem 1.3

Given any integer period $\alpha + i\beta$ as in Theorem 1.3 the set of marked abelian differentials having periods $\alpha + i\beta$ is a smooth connected complex manifold, by our Theorem 1.2. A form of this subset is the pull-back of the form dz on the Gaussian elliptic curve $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ by a degree d ramified covering. The set of those forms whose associated ramified covering has distinct critical values, is still connected, since it is Zariski dense. The monodromy representations $\rho : \pi \to \mathbb{S}_d$ of these branched coverings of the torus, send peripherals to transpositions, and are well-defined up to precomposition by an element of the braid group of the torus on $2g - 2$ braids. Observe that this class of representations does not depend on the class of $\alpha + i\beta$ modulo precomposition by an element of $\mathrm{Sp}(2g, \mathbb{Z})$.

The inverse operation is clear: having a representation ρ from a $2g-2$-punctured torus group as in Theorem 1.3, we can equip the torus with the structure of the Gaussian elliptic curve, and define $\alpha + i\beta$ as the periods of the pull-back of dz on the branched degree d covering having monodromy ρ, equipped with any marking.

8. Appendix: dynamics of the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{H}_g$

In this section, we review Misha Kapovich's remarkable note [9], which unfortunately remains unpublished. This will enable us to prove Theorem 1.1 assuming the transfer principle, namely assuming Theorem 1.2. To this end, we need to understand the dynamics of the linear action of $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ on $\mathbb{C}^{2g}$, or, more precisely on the set of periods $p \in \mathbb{C}^{2g}$ of positive volume. The volume of $p \in \mathbb{C}^{2g}$ is by definition the number $V(p) := \sum_{1 \le k \le g} \Im(p_{2k+1}\overline{p_{2k}})$.

If we introduce the symplectic form ω on $\mathbb{R}^{2g}$ defined by $\omega(x, y) = \sum_{1 \le k \le g} x_{2k}y_{2k+1} - x_{2k+1}y_{2k}$, the volume of a period can be expressed as $V(p) = \omega(\Re p, \Im p)$. In particular, the subspace $W = \mathbb{R}\Re p + \mathbb{R}\Im p \subset \mathbb{R}^4$ is symplectic. In the sequel, we denote by $\Lambda(p)$ the $\mathbb{Z}$-submodule of $\mathbb{C}$ generated by the entries of p. Notice that it is invariant under the action of $\mathrm{Sp}(2g, \mathbb{Z})$.

Proposition 8.1. *Assume $g > 2$. For any $p \in \mathbb{C}^{2g}$ of positive volume, we have the trichotomy*

- *W is defined over $\mathbb{Q}$. In this case, $\Lambda(p)$ is discrete and p is the period of a ramified cover of the abelian differential $(\mathbb{C}/\Lambda, dz)$.*
- *W is not rational but contains a rational line. In this case, $\overline{\Lambda(p)}$ is affinely equivalent to $\mathbb{R} + i\mathbb{Z}$, and $\overline{\Gamma \cdot p}$ is the set of periods $q \in \mathbb{C}^{2g}$ of volume $V(q) = V(p)$ whose imaginary part are integer valued and primitive.*
- *W does not contain any rational subspace of positive dimension. In this case, $\overline{\Lambda(p)} = \mathbb{C}$, and $\overline{\Gamma \cdot p}$ is the set of periods $q \in \mathbb{C}^{2g}$ such that $V(q) = V(p)$.*

In genus $g = 2$, another case shows up, the one associated to the Hilbert modular manifolds (see [3] *and* [16, Case 3. of Theorem 5.1]*):*

- *W is defined over a quadratic field K, and $W^\perp = W^\sigma$, where σ is the Galois involution of K. In this case $\overline{\Gamma \cdot p}$ is the set of periods q which differs from p by post composition by an affine automorphism of $\mathbb{R}^2$, and by pre-composition by an element of $Sp(4, \mathbb{Z})$.*

Proof. Since the action of Γ is linear, and that the volume is multiplicative, namely $V(\lambda p) = |\lambda| V(p)$ for every $\lambda \in \mathbb{C}$ and $p \in \mathbb{C}^{2g}$, we can restrict our attention to the action of Γ on the subset $X \subset \mathbb{C}^{2g}$ whose elements have volume 1. Recall that this means that $\omega(\Re p, \Im p) = 1$. Since the simple real Lie group $G = \mathrm{Sp}(2g, \mathbb{R})$ acts transitively on the set of couples $(x, y) \in (\mathbb{R}^{2g})^2$ such that $\omega(x, y) = 1$, and that the stabilizer of the couple $\big((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0)\big)$ is the group

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \mathrm{Sp}(2g-2, \mathbb{R}) \end{pmatrix}$$

that we will denote by U in the sequel, our set X is isomorphic to the homogeneous space G/U. The linear action of Γ on X is under the isomorphism $X \simeq G/U$ given by left multiplication on G/U.

Since the group G is simple, that U is generated by unipotent elements, and that Γ is a lattice in G, Ratner's theorem [20] tells us that the closure of the Γ-orbits on X are homogeneous in the following sense

Theorem 8.2 (Ratner)**.** *For every $p \in X$ of the form $p = gU$, there exists a closed subgroup H of G containing $U^g = gUg^{-1}$, such that $\Gamma \cap H$ is a lattice in H, and such that $\overline{\Gamma \cdot p} = \Gamma H p$.*

Notice that $U^g = I_{|W} \oplus \mathrm{Sp}(W^\perp) \simeq \mathrm{Sp}(2g-2, \mathbb{R})$. Let H_0 be the connected component of H containing the identity: then $\Gamma \cap H_0$ is still a lattice in H_0, and U^g is contained in H_0. Kapovich observes that such subgroups H_0 fall into two categories

- (Semi-simple case) H_0 is of the form $S \oplus \mathrm{Sp}(W^\perp)$, where S is a Lie subgroup of $\mathrm{Sp}(W)$.
- (Non semi-simple case) H_0 preserves a line $L \subset W$.

The proof of this dichotomy can be found in [9, p. 12], and is based on Dynkin's classification of maximal connected complex Lie subgroups of $\mathrm{Sp}(2g, \mathbb{C})$, see [4]. Let L be a maximal complex Lie subgroup of $\mathrm{Sp}(2g, \mathbb{C})$ which contains H_0. If $H_0 \neq \mathrm{Sp}(2g, \mathbb{R})$, its Zariski closure in the complex domain is a strict subgroup of $\mathrm{Sp}(2g, \mathbb{C})$, so it is contained in a maximal complex Lie (strict) subgroup of $\mathrm{Sp}(2g, \mathbb{C})$. It satisfies one of the following properties (see [6, Ch. 6, Thm 3.1, 3.2]):

(1) $L = \mathrm{Sp}(V) \oplus \mathrm{Sp}(V^\perp)$ for some complex symplectic subspace $V \subset \mathbb{C}^{2g}$,

(2) L is conjugated to $\mathrm{Sp}(s,\mathbb{C})\otimes \mathrm{SO}(t,\mathbb{C})$ where $2g = st$, $s \geq 2$, $t \geq 3$, $t \neq 4$ or $t = 4$ and $s = 2$,
(3) L preserves a line of $\mathbb{C}^{2g}$.

Since H_0 contains U^g, L contains the complexification of U^g, which is nothing but $\mathrm{Id}_{W_{\mathbb{C}}} \oplus \mathrm{Sp}(W_{\mathbb{C}}^{\perp})$, where $W_{\mathbb{C}}$ denotes the complexification $W \otimes_{\mathbb{R}} \mathbb{C}$ of W. In case (1), the only possibility is that up to permutation of V and $V^{\perp}$, we have $W_{\mathbb{C}} = V$. In particular, H_0 is a subgroup of $\mathrm{Sp}(W)\oplus\mathrm{Sp}(W^{\perp})$. Since it contains $\mathrm{Id}_{|W}\oplus\mathrm{Sp}(W^{\perp})$, it must be of the form $S\oplus\mathrm{Sp}(W^{\perp})$, where S is a Lie subgroup of $\mathrm{Sp}(W)$. Case (2) cannot occur. In case (3), observe that the line L needs to be in $W_{\mathbb{C}}$, since the group $\mathrm{Id}_{|W_{\mathbb{C}}}\oplus\mathrm{Sp}(W_{\mathbb{C}}^{\perp})$ preserves this line. If L is defined over the reals, we are done. If not, both L and $\overline{L}$ (the image of L by the complex conjugation) are preserved by H_0, and thus H_0 is a subgroup of $\mathrm{Sp}(W)\oplus\mathrm{Sp}(W^{\perp})$. As before, because it contains $\mathrm{Id}_W\oplus\mathrm{Sp}(W^{\perp})$, it must be of the form $S\oplus\mathrm{Sp}(W^{\perp})$, where S is a Lie subgroup of $\mathrm{Sp}(W)$.

Semi-simple case. Since the group $H_0 = S\oplus\mathrm{Sp}(W^{\perp})$ contains a lattice, it must be unimodular. In particular, either S is the trivial group, or a 1-parameter subgroup, or the whole $\mathrm{Sp}(W)$.

If S is trivial, then $\Gamma_W = \Gamma\cap(\mathrm{Id}_W\oplus\mathrm{Sp}(W^{\perp}))$ is a lattice in $\mathrm{Id}_W\oplus\mathrm{Sp}(W^{\perp})$. Then, Γ_W would also act by the identity on W^{σ} for every Galois automorphism σ. The Zariski closure of Γ_W being $\mathrm{Id}_W\oplus\mathrm{Sp}(W^{\perp})$ (by Borel density theorem, see [25]), $\mathrm{Id}_W\oplus\mathrm{Sp}(W^{\perp})$ would act by the identity on W^{σ} as well. This implies that $W^{\sigma} = W$ for every σ, and so W is rational. So we are done in this case.

If S is 1-dimensional, $S\oplus\mathrm{Id}_{W^{\perp}}$ would be the radical of H_0, and a theorem of Wolf and Raghunathan, see [19], shows that it would intersect Γ in a lattice. In particular, the intersection of Γ with $\mathrm{Id}_W\oplus\mathrm{Sp}(W^{\perp})$ is also a lattice. Reasoning as above, this implies that W is rational, so we are done in this case as well.

Finally, it remains to treat the case where $S = \mathrm{Sp}(W)$. This case splits into two subcases, depending on the lattice $\Gamma\cap\mathrm{Sp}(W)\oplus\mathrm{Sp}(W^{\perp})$ being reducible or irreducible. If it is reducible, this implies that $\Gamma\cap(\mathrm{Id}_W\oplus\mathrm{Sp}(W^{\perp}))$ is a lattice, and then W must be rational by the above considerations. Assume now that we are in the irreducible case. Then $g = 2$, by a theorem of Margulis [12]. Assume W is not rational, otherwise we are done. Let σ be a Galois automorphism such that $W^{\sigma}\neq W$. The group $\Gamma\cap\mathrm{Sp}(W)\oplus\mathrm{Sp}(W^{\perp})$ preserves the decomposition $W^{\sigma}\oplus(W^{\sigma})^{\perp}$, since Γ and the symplectic form are defined over the rationals. Borel density theorem applied to the lattice $\Gamma\cap\mathrm{Sp}(W)\oplus\mathrm{Sp}(W^{\perp})$ shows that $\mathrm{Sp}(W)\oplus\mathrm{Sp}(W^{\perp})$ preserves the decomposition $W^{\sigma}\oplus(W^{\sigma})^{\perp}$. This implies that $W^{\sigma} = W^{\perp}$ and $(W^{\sigma})^{\sigma} = W$. This being true for every Galois automorphism, this means that W is defined over a totally real quadratic fields K, and we have $W^{\sigma} = W^{\perp}$ where σ is the Galois automorphism of K.

Non semi-simple case. We prove in this case that the periods p satisfies the second case of the proposition.

For this, we will first need to understand in detail the subgroup B of $\mathrm{Sp}(2g,\mathbb{R})$ whose elements stabilize the line L, see [9, p. 10]. To unscrew the structure of B, notice that any element of B stabilizes both L and $L^\perp$ so that we have an exact sequence

$$CH_{2g} \to B \to \mathrm{Sp}(L^\perp/L) \simeq \mathrm{Sp}(2g-2).$$

The group CH_{2g} is then the set of elements $M \in \mathrm{Sp}(2g)$ which induce the identity map on $L^\perp/L$.

We now have another exact sequence

$$H_{2g-1} \to CH_{2g} \to \mathrm{GL}(L) \simeq \mathbb{R}^*, \tag{13}$$

the last arrow being given by the restriction of an element $M \in CH_{2g}$ to the line L. Hence the subgroup $H_{2g-1} \subset CH_{2g}$ is the group of elements $M \in \mathrm{Sp}(2g)$ which act as the identity on L and on $L^\perp/L$. Such M are easily seen to be of the form $M_{\varphi,\alpha}$, for some $\varphi \in (L^\perp/L)^*$ and $\alpha \in \mathbb{R}$, where

- the restriction of $M_{\varphi,\alpha}$ to $L^\perp$ equals $id_{|L^\perp} + \varphi a_1$
- $M_{\varphi,\alpha}(b_1) = \alpha a_1 + b_1 + \sum_{k\geq 2} \varphi(b_k)a_k - \varphi(a_k)b_k$,

where $a_1, b_1, \dots, a_g, b_g$ is a symplectic basis such that $L = \mathbb{R}a_1$. The group structure on H_{2g-1} is then given by the following relation

$$M_{\varphi,\alpha}M_{\varphi',\alpha'} = M_{\varphi+\varphi',\alpha+\alpha'+\omega(\varphi,\varphi')}, \tag{14}$$

where $\omega(\varphi,\varphi')$ is the natural symplectic product induced by ω on $(L^\perp/L)^*$, namely

$$\omega(\varphi,\varphi') = \sum_{k\geq 2} \varphi(a_k)\varphi'(b_k) - \varphi'(a_k)\varphi(b_k).$$

Equation (14) is a straighforward computation. An equivalent formulation is that H_{2g-1} is the central extension

$$\mathbb{R} \to H_{2g-1} \to (L^\perp/L)^*,$$

defined by the 2-cocyle $(\varphi,\varphi') \mapsto \omega(\varphi,\varphi')$. The group H_3 is isomorphic to the classical Heisenberg group of upper triangular real matrices of size 3×3 with 1's on the diagonal.

Now CH_{2g} is a semi-direct product of $\mathbb{R}^*$ by H_{2g-1}, see (13). To understand its structure, we introduce for every λ, one of its lift $S_\lambda \in CH_{2g}$ defined by

$$S_\lambda(a_1) = \lambda a_1,\ S_\lambda(b_1) = \frac{1}{\lambda}b_1,\ S_\lambda(a_k) = a_k,\ S_\lambda(b_k) = b_k \text{ for } k \geq 2.$$

A trivial computation shows that for any $\lambda \in \mathbb{R}^*$, every $\varphi \in (L^\perp/L)^*$ and every $\alpha \in \mathbb{R}$, we have

$$S_\lambda M_{\varphi,\alpha} S_\lambda^{-1} = M_{\lambda\varphi,\lambda^2\alpha}.$$

This shows that CH_{2g} is not unimodular, and consequently does not contain any lattice.

We are now in a position to treat the non semi-simple case. By construction, our group H_0 is contained in B. We have an exact sequence $CH_{2g} \to B \to \mathrm{Sp}(L^\perp/L,\omega)$. The image of H_0 by the right arrow is onto since H_0 contains U^g, so that H_0 itself splits as an exact sequence $CH_{2g} \cap H_0 \to H_0 \to \mathrm{Sp}(L^\perp/L,\omega)$. The group $CH_{2g} \cap H_0$ is invariant under the action by conjugation of $\mathrm{Sp}(L^{\perp}/L,\omega)$, so this implies that one of the following possibilities occur

- $CH_{2g}\cap H_0 = CH_{2g}$, namely $H_0 = B$
- $CH_{2g}\cap H_0 = H_{2g-1}$, namely H_0 is a semi-direct product of $\mathrm{Sp}(L^{\perp}/L,\omega)$ by H_{2g-1}.
- $CH_{2g}\cap H_0 = A$ where A is a subgroup of the affine group.

The first and last case are impossible since CH_{2g} and A are not unimodular. It remains to treat the second one, namely the case $CH_{2g}\cap H_0 = H_{2g-1}$. In this latter, the theorem of Raghunathan and Wolf cited above tells us that $\Gamma\cap H_{2g-1}$ is a lattice in H_{2g-1}. As a consequence, we have that $L = \bigcap_{\gamma\in\Gamma\cap H_{2g-1}} \mathrm{Ker}(\gamma - I)$, and this shows that L is a rational subspace of $\mathbb{R}^{2g}$, since all the spaces appearing in the intersection are rational. Up to an real affine change of coordinates on $\mathbb{C}$, we can assume that the imaginary part of p generates L, and that it is a primitive element of $\mathbb{Z}^{2g}$. Since the group H_{2g-1} acts transitively on the set of vectors $v\in\mathbb{R}^{2g}$ such that $v\cdot\Im p = 1$, while fixing the period $\Im p$ fixed, we see that $H\cdot p$ already contains all the periods q such that $\Im q = \Im p$ and such that $V(q) = V(p) = 1$. Since, Γ acts transitively on the set of primitive elements of $\mathbb{Z}^{2g}$, we infer that $\Gamma H p = \overline{\Gamma\cdot p}$ contains all the periods q with volume $V(p) = 1$ and with a primitive integer imaginary part. Since any periods of $\overline{\Gamma\cdot p}$ is of this form, we deduce that this situation is exactly the second case of the proposition. The proof of this latter is now complete. □

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