Maximal volume representations are fuchsian

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Abstract

We prove a volume-rigidity theorem for fuchsian representations of fundamental groups of hyperbolic k-manifolds into $\mathrm{Isom}(\mathbb{H}^n)$. Namely, we prove that the volume of a representation is less or equal than the volume of the manifold, and it is maximum if and only if the representation is discrete, faithful and "k-fuchsian".

1 Introduction

The main result of this paper is a generalization and streamlined proof of a result which is often referred to as the "representation volume rigidity" theorem, in particular we prove

Theorem 1.1. *Let M be an oriented, connected, complete hyperbolic k-manifold of finite volume, with $k \geq 3$. Let $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation of its fundamental group into the group of isometries of the hyperbolic n-space. Then the volume of ρ is less than or equal to the volume of M, and equality holds if and only if ρ is k-fuchsian, i.e., a discrete and faithful representation into the group of isometries of a k-dimensional subspace of $\mathbb{H}^n$.*

In the case $k = n = 3$ this result was proved, following the original ideas of W. Goldman, M. Gromov, and W. Thurston [10, 12], by N. Dunfield [6] in the compact case and by the authors [8, 11] in the finite-volume case.

The new ingredient in the proof presented in this paper is the use of natural maps (or barycenter method), a technique introduced and developed by G. Besson, G. Courtois, and S. Gallot in the 90's. (See, for example [2, 3, 4].) Indeed, a key step in this proof is a first author's generalization of the B-C-G method to construct natural maps for representations [7].

We remark that in addition to the new proof offered here, the "volume-rigidity of representations" has been generalized, in that it deals with the case when the target dimension is greater than that of the domain.

We notice that Besson, Courtois and Gallot in a recent work [5] proved a similar result.

The paper is organized as follows. In Section 2, we give the necessary background and definitions, including the definition of volume of a representation. In Section 3, we give the proof of Theorem 1.1 in the compact case. Finally, in Section 4, we complete the proof of Theorem 1.1 proving the claim for non-compact, complete, finite-volume manifolds.

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2 Definitions and notation

Throughout Sections 2 and 3, M will denote a closed, oriented hyperbolic k-manifold, where $k \geq 3$. We suppress a choice of basepoint in M and let $\Gamma = \pi_1(M)$ denote the fundamental group of M. We let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ denote a representation of Γ into the group of isometries of $\mathbb{H}^n$.

By a *pseudo-developing map* for ρ, we mean a piecewise smooth map $D = D_\rho : \widetilde{M} \to \mathbb{H}^n$ which is ρ-equivariant, i.e., such that

$$D(\gamma \cdot x) = \rho(\gamma) \cdot D(x),$$

for every $x \in \widetilde{M} = \mathbb{H}^k$, the universal cover of M, and for every $\gamma \in \Gamma$.

Given any representation ρ, one can construct a pseudo-developing map for ρ as follows: lift a smooth triangulation for M to $\widetilde{M}$ and then recursively define the map D on the i-skeleta, $0 \leq i \leq k$, by choosing images for a complete system of orbit representatives for the ith skeleta, and then extending the map equivariantly.

We introduce now the notion of the volume of a representation.

Let h denote the hyperbolic metric on the target $\mathbb{H}^n$ and let D be a pseudo-developing map for ρ. The pullback D^*h of h along D is a (possibly degenerate) pseudo-metric on $\widetilde{M}$. The pullback D^*h induces a k-form $\tilde{\omega}_D = |\det D^*h|$ on $\widetilde{M}$. Since the map D is ρ-equivariant, D^*h and hence $\tilde{\omega}_D$ are Γ-invariant. Hence the k-form $\tilde{\omega}_D$ descends to a k-form ω_D on M.

Definition 2.1 (Volume of a pseudo-developing map). The *volume* $\mathrm{vol}(D)$ *of a pseudo-developing map* D for a representation ρ is defined by

$$\mathrm{vol}(D) = \int_M \omega_D.$$

We can now make the following

Definition 2.2 (Volume of a representation). The *volume* $\mathrm{vol}(\rho)$ *of a representation* ρ is defined by

$$\mathrm{vol}(\rho) = \inf_D\{\mathrm{vol}(D)\},$$

where the infimum is taken over the set of all pseudo-developing maps D for ρ.

Note that $\mathrm{vol}(D)$ and hence $\mathrm{vol}(\rho)$ are non-negative real numbers. Note also that $\mathrm{vol}(D)$ is not invariant under ρ-equivariant homotopy. Hence the volumes of two pseudo-developing maps for a given representation can be different. We remark that we use the above definition of representation-volume because we must deal with the case $n \neq k$. (Compare the definition of representation-volume and the consequent property of invariance under homotopy in [6, 8].)

We also notice that, in the non-compact case, the definition of volume of a representation is a little different (see Section 4).

3 The compact case

The proof of Theorem 1.1 goes as follows. First, we invoke a result of the first author [7] which says that there is a pseudo-developing map F for ρ with $\mathrm{vol}(\rho) \leq \mathrm{vol}(M)$.

Next, we use the hypothesis that $\mathrm{vol}(\rho) = \mathrm{vol}(M)$, elementary Riemannian geometry, and the properties of the pseudo-developing map F to show that it is a Riemannian isometry from $\mathbb{H}^k$ to a k-dimensional hyperbolic subspace of $\mathbb{H}^n$. (Note that this reduces the remainder of the proof to the case $k = n$.)

It is then easy to conclude that F is a covering map onto its image; it then follows that ρ is discrete and faithful. Finally, we show the (easier) converse, namely that if ρ is a discrete, faithful representation into the group of isometries of a k-dimensional hyperbolic subspace of $\mathbb{H}^n$, then $\mathrm{vol}(\rho) = \mathrm{vol}(M)$.

The following result is proved in [7].

Lemma 3.1. *Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation whose image is a non-elementary group. Then, for $k \geq 3$, there exists a smooth pseudo-developing map $F : \mathbb{H}^k \to \mathbb{H}^n$ such that for all $x \in \mathbb{H}^k$,*

$$|\mathrm{Jac}\, F(x)| \leq 1; \tag{1}$$

moreover, equality holds at x if and only if $dF_x : T_x\mathbb{H}^k \to T_{F(x)}\mathbb{H}^n$ is an isometry.

Assuming the image of ρ is non-elementary, Lemma 3.1 implies the inequality of Theorem 1.1: by the definition of volume of a representation and the inequality in the lemma, it follows immediately that

$$\mathrm{vol}(\rho) \leq \mathrm{vol}(F) \leq \mathrm{vol}(M). \tag{2}$$

If the image of ρ is elementary, then it is easy to check that the $\mathrm{vol}(\rho) = 0$, and thus the desired inequality holds.

We now suppose that $\mathrm{vol}(\rho) = \mathrm{vol}(M)$ and show that the image of the pseudo-developing map F is contained in a k-dimensional hyperbolic subspace of $\mathbb{H}^n$.

Since $\mathrm{vol}(\rho) = \mathrm{vol}(M)$, each of the inequalities of (2) is an equality. Hence, for each x in $\mathbb{H}^k$, the inequality in (1) is equality. Thus the map F is a Riemannian isometry.

Now, we need to recall some ideas from Riemannian geometry. We refer the reader to [9] for notation and details about these facts. We notice that, in what follows, C^2-regularity is enough.

Let X denote a Riemannian manifold. A submanifold N of X is called *minimal* if it is a critical point of the volume function. A submanifold is *locally minimal* if, for each point x of N, there exists a neighborhood A of x such that all perturbations of N with support in A does not decrease the volume of N. A submanifold N of X is *totally geodesic* if for any two points x and y in N, the geodesic joining x and y in X is contained in N. We denote by R^X and ∇^X (resp., R^N and ∇^N) the curvature tensor and the connection of X (resp., N).

For any two vector fields U and V in N, we denote by $\Pi(U,V)$ the second fundamental form of the submanifold N. Equivalently, if $\{\nu_1 \ \ldots, \nu_r\}$ denotes an orthonormal frame of the orthogonal complement of TN in TX, and if $l_i(U,V)$ denotes the real-valued fundamental form corresponding to ν_i, then

$$\nabla^X_U V - \nabla^N_U V = -\sum_{i=1}^{r} l_i(U,V)\nu_i = \Pi(U,V).$$

What we'll do is prove that the image of the map f is a minimal submanifold of $\mathbb{H}^n$, and from this conclude that the image of f is contained in a k-dimensional subspace of $\mathbb{H}^n$. To do this, we need the following standard results ([9, Chapter V]).

Lemma 3.2. *Let N be a submanifold of a Riemannian manifold X. Then*

1. *N is minimal if and only if the traces of all the real-valued second fundamental forms vanish (see [9, p.228]);*
2. *N is totally geodesic if and only if the second fundamental form vanishes (see [9, p. 220]).*

Lemma 3.3. *The image $F(\mathbb{H}^k)$ of the map F is contained in a locally minimal submanifold of $\mathbb{H}^n$.*

Proof. Suppose not. Then by a perturbation of F in a small ball B of $\mathbb{H}^k$, we can decrease the volume of F. Indeed, by ρ-equivariantly perturbing F in the Γ-orbit of B, we can find a pseudo-developing map $F' : \mathbb{H}^k \to \mathbb{H}^n$ with a strictly smaller volume than that of F. But then $\mathrm{vol}(M) = \mathrm{vol}(\rho) \le \mathrm{vol}(F') < \mathrm{vol}(F) = \mathrm{vol}(M)$, a contradiction. □

Lemma 3.4. *Let N be a locally minimal k-submanifold of a Riemannian $(k+r)$-manifold X. If, for all vector fields $U, V, W,$ and T we have*

$$R^N(U,V,W,T) = R^X(U,V,W,T),$$

then N is totally geodesic.

Proof. By (2) of Lemma 3.2, it suffices to show that the second fundamental form of N vanishes. We again let $\{\nu_1, \ldots, \nu_r\}$ be an orthonormal frame of the orthogonal complement TN of TX, and for each index i, we let $l_i(\cdot, \cdot)$ denote the real-valued fundamental form corresponding to ν_i. By Gauss's theorem (see for example [9, Chapter V]), we conclude that for any point $p \in N$ and for any u, v, w, and t in T_pN,

$$R^N(u, v, w, t) = R^X(u, v, w, t) + \sum_{i=1}^{r}(l_i(u, w)l_i(v, t) - l_i(u, t)l_i(v, w)).$$

It then follows that for any u, v, w, and t in T_pN,

$$\sum_{i=1}^{r}(l_i(u, w)l_i(v, t) - l_i(u, t)l_i(v, w))) = 0.$$

By hypothesis, N is a locally minimal submanifold; therefore, by (1) of Lemma 3.2, we have that $\mathrm{tr}(\mathrm{l_i}) = 0$ for each $i = 1, \ldots, r$.

Now let $e_1, \ldots, e_k$ denote an orthonormal basis of T_pN. Setting $u = t = e_j$ in the above equality, we have

$$\sum_{i=1}^{r}(l_i(e_j, w)l_i(v, e_j) - l_i(e_j, e_j)l_i(v, w))) = 0.$$

Setting $w = v$ and summing over the index j, we get

$$\sum_{j=1}^{k}\sum_{i=1}^{r} l_i^2(e_j, w) - \sum_{i=1}^{r} tr(l_i)l_i(w, w)) = 0.$$

Whence, by the vanishing trace condition of Lemma 3.2, we conclude that for any p in N and w in T_pN,

$$\sum_{i,j} l_i^2(e_j, w) = 0.$$

It now follows that $l_i(e_j, w) = 0$ for any i, j, and w, and hence that $l_i \equiv 0$ for $1 \leq i \leq r$. This shows that the second fundamental form vanishes at each point p in N, which completes the proof of the lemma. □

We now apply Lemma 3.4 with $N = F(\mathbb{H}^k)$ and $X = \mathbb{H}^n$. Since F is a Riemannian isometry, the hypothesis that $R^N = R^X$ is satisfied. By Lemma 3.3, N is a locally minimal submanifold of X. Hence by Lemma 3.4, N is totally geodesic. Therefore the map F is an isometry from $\mathbb{H}^k$ to a k-dimensional subspace H of $\mathbb{H}^n$, and it follows that the image of ρ is contained in the group of isometries of H.

We claim now that $F : \mathbb{H}^k \to H$ is a covering map. Indeed, note that there exists an $r > 0$ such that for any $x \in \mathbb{H}^k$, the restriction map $F|_{B(x,r)}$ is an isometry onto its image. This easily implies the claim.

Since $\mathbb{H}^k$ is simply connected, the covering $F : \mathbb{H}^k \to H$ is a homeomorphism. Thus F is a ρ-equivariant global isometry of $\mathbb{H}^k$. It follows that the representation ρ is discrete and faithful.

Finally, we suppose that ρ is a discrete and faithful representation into the group of isometries of a k-dimensional subspace H of $\mathbb{H}^n$, and show that $\mathrm{vol}(\rho) = \mathrm{vol}(M)$. First, note that it is not restrictive to work with pseudo-developing maps for ρ whose images are contained in H. Thus, after identifying H with $\mathbb{H}^k$, we may assume that $n = k$.

Let $N = \mathbb{H}^k/\rho(\Gamma)$. By Mostow rigidity, the hyperbolic k-manifolds M and N are isometric, and in particular, $\mathrm{vol}(N) = \mathrm{vol}(M)$. Now let D be any pseudo-developing map for ρ. Since D is ρ-equivariant, it induces a map $g : M \to N$, and by definition, $\mathrm{vol}(D) = \int_M |g^*\omega|$, where ω is the hyperbolic volume form of N. Hence

$$\mathrm{vol}(N) = \mathrm{vol}(M) = |\int_M g^*\omega| \leq \int_M |g^*\omega| = \mathrm{vol}(D).$$

It follows that

$$\mathrm{vol}(M) \leq \mathrm{vol}(\rho) = \inf_D\{\mathrm{vol}(D)\},$$

and we have already shown (see inequality (2) after Lemma 3.1) that the reverse inequality also holds. This completes the proof of Theorem 1.1 when M is a compact manifold.

4 The finite-volume case

In this section we complete the proof of Theorem 1.1, proving the claim in the finite-volume case. The main difference with respect to the compact case, is that, as our manifolds are no longer compact, we need to work with proper maps and, since we work at the level of universal coverings, we need an equivariant notion of properness. We keep here all the notation and definitions of previous sections, except that in the sequel M will denote an oriented, complete, non-compact hyperbolic k-manifold of finite volume with $k \geq 3$. Also, we need to modify the definition of volume of a representation.

The manifold M is diffeomorphic to the interior of a compact manifold $\overline{M}$ whose boundary consist of Euclidean $(k-1)$-manifolds (see for example [1]). In particular, for each boundary component $T \subset \partial \overline{M}$ the group $\pi_1(T) < \pi_1(M) = \Gamma < \mathrm{Isom}(\mathbb{H}^\mathrm{k})$ is an Abelian parabolic group. The following lemma is easy to check.

Lemma 4.1. *Let G be an Abelian group of isometries of a hyperbolic space $\mathbb{H}^m$. Then the set* $\mathrm{Fix(G)} \subset \overline{\mathbb{H}^\mathrm{m}}$ *of points which are fixed by G is not empty.*

We notice that G may have no fixed point in $\partial \mathbb{H}^m$ (for example if $G < \mathrm{Isom}(\mathbb{H}^3)$ is the dihedral group generated by two rotation of angle π around orthogonal axes).

Any boundary subgroup of $\pi_1(M)$ has a unique fixed point, which lies in $\partial \mathbb{H}^k$. Thus, for each $T \subset \partial \overline{M}$, each conjugate of $\pi_1(T)$ in $\pi_1(M) \subset \mathrm{Isom}(\mathbb{H}^\mathrm{k})$ corresponds to its fixed point in $\partial \mathbb{H}^k$.

We are now ready to give the definition of properly ending map.

Definition 4.2 (Properly ending maps). Let $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n)$ be a representation, and let $D : \mathbb{H}^k \to \mathbb{H}^n$ be a ρ-equivariant map. We say that D *properly ends* if for each $T \subset \partial \overline{M}$, if $\xi = \mathrm{Fix}(\pi_1(\mathrm{T}))$ and $\alpha(t)$ is a geodesic ray ending at ξ, then all limit points of $D(\alpha(t))$ lie either in $\mathrm{Fix}(\rho(\pi_1(\mathrm{T}))) \subset \overline{\mathbb{H}^\mathrm{n}}$ or in a finite union of $\rho(\pi_1(T))$-invariant geodesics.

Definition 4.3 (Volume of a representation). The *volume* $\mathrm{vol}(\rho)$ *of a representation* ρ is defined by

$$\mathrm{vol}(\rho) = \inf_D \{\mathrm{vol}(D)\},$$

where the infimum is taken over the set of all properly ending pseudo-developing maps D for ρ.

Remark 4.4. It is easy to construct properly ending pseudo-developing map (see for example [6, 8]). We need to work with such maps because otherwise, using the collapse of M to any of its spine one can easily construct (non properly ending) pseudo-developing maps with volume zero. Finally, we notice that this definition of volume "extends" the previous one given for compact manifolds. Indeed, if M is compact, then any pseudo-developing map properly ends.

Now, we need to recall the definition and properties of the barycentre of measures in $\overline{\mathbb{H}^n}$, referring to [4, 7] for details (the reader which is familiar with such construction may skip directly to Lemma 4.5.) Let β be a probability Borel measure on $\partial\mathbb{H}^n$. We define a function $\mathcal{B}_\beta : \mathbb{H}^n \to \mathbb{R}$ by

$$\mathcal{B}_\beta(y) = \int_{\partial\mathbb{H}^n} B(y,\theta)\, d\beta(\theta)$$

where $B(y,\theta)$ is the Busemann function of $\mathbb{H}^n$. Then:

1. If β is not concentrated in two points, then $\mathcal{B}_\beta$ is strictly convex (because its Hessian is the β-average of the Hessians of the Busemann functions $B(y,\cdot)$) and goes to ∞ as y goes to $\partial\mathbb{H}^n$.

2. If β is not the sum of two Dirac deltas with the same weight, then $\mathcal{B}_\beta$ has a unique minimum (possibly $-\infty$) in $\overline{\mathbb{H}^n}$. Such minimum is attained in $\partial\mathbb{H}^n$ if and only if β has an atom of weight greater that $\frac{1}{2}$. The point $\mathrm{bar}(\beta)$ where $\mathcal{B}_\beta$ attains its minimum is called *barycentre* of β.

3. If β is the sum $\frac{1}{2}(\delta_{\theta_1} + \delta_{\theta_2})$ of two Dirac deltas concentrated in θ_1 and θ_2, then $\mathcal{B}_\beta$ is convex and constant on the geodesic joining θ_1 and θ_2, where it attains its minimum.

4. If β is a probability measure on $\mathbb{H}^n$, its barycentre is defined making convolution with the family of visual measures as follows. Let $\nu_{O'}$ be the standard probability measure on $\partial\mathbb{H}^n \simeq \mathbb{S}^{n-1}$ in the disc model with center O'. For all $y \in \mathbb{H}^n$ define $\nu_y = \psi_*\nu_{O'}$ where ψ is any isometry mapping O' to y (note that this is well-defined because $\nu_{O'}$ is $\mathrm{Stab}(O')$-invariant.) Then define $\bar\beta$ a probability measure on $\partial\mathbb{H}^n$ by

$$\int_{\partial\mathbb{H}^n} \varphi(\theta)\, d\bar\beta(\theta) = \int_{\mathbb{H}^n} \left(\int_{\partial\mathbb{H}^n} \varphi(\theta)\, d\nu_y(\theta) \right) d\beta(y).$$

   The barycentre of β is defined as the barycentre of $\bar\beta$.

5. The barycentre is defined in the same way for non-negative measures of finite, non-zero mass. For any positive constant c we have $\mathrm{bar}(c\beta) = \mathrm{bar}(\beta)$.

6. The barycentre is continuous w.r.t. the weak-$*$ convergence of measures, that is, if $\{\beta_i\}$ is a sequence of measures having barycentre and converging to a measure β having barycentre, then $\mathrm{bar}(\beta_i) \to \mathrm{bar}(\beta)$.

7. The barycentre is equivariant by isometries, that is $\mathrm{bar}(\gamma_* \beta) = \gamma(\mathrm{bar}(\beta))$ for any isometry γ (where $\gamma_* \beta$ denotes the push-forward via γ of the measure β).

What we need to complete the proof of Theorem 1.1 is the following fact (compare with Lemma 3.1.)

Lemma 4.5. *For any $\varepsilon > 0$ there exists a map $F^\varepsilon : \mathbb{H}^k \to \mathbb{H}^n$ such that:*

1. *The map F^ε is smooth and ρ-equivariant.*
2. $|\mathrm{Jac}\, F^\varepsilon(x)| \leq 1 + \varepsilon$, *and equality holds if and only if $dF^\varepsilon_x : T_x\mathbb{H}^k \to T_{F^\varepsilon(x)}\mathbb{H}^n$ is a homothety.*
3. $\lim_{\varepsilon \to 0} F^\varepsilon = F$, *where F is the map of Lemma 3.1.*
4. *The map F^ε properly ends.*

Before proving Lemma 4.5 we show how it implies Theorem 1.1. The inequality directly follows from points $(1), (2)$ and (4). If $\mathrm{vol}(\rho) = \mathrm{vol}(M)$, then by point (3) one gets that $\mathrm{vol}(F) = \mathrm{vol}(M)$ (note that a priori the map F of Lemma 3.1 does not end properly). The proof now follows exactly as in the compact case.

Proof of Lemma 4.5. The maps F^ε are the so called ε-natural maps introduced by Besson, Courtois, and Gallot. We begin by recalling their construction. We give no details, referring to [4, 7, 2, 3] for a complete discussion on the construction of natural maps.

For any $\varepsilon > 0$ we set

$$s = (k-1)(1+\varepsilon).$$

Let O be a marked point in $\mathbb{H}^k$, and let $c(s) = \sum_{\gamma \in \Gamma} e^{-sd(O,\gamma O)}$. It turns out that $c(s) < \infty$ for any $s > k-1$.

For any $x \in \mathbb{H}^k$ we define μ^ε_x a positive Borel measure on $\mathbb{H}^k$ by

$$\mu^\varepsilon_x = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma O)} \delta_{\gamma O}$$

where $\delta_{\gamma O}$ denotes the Dirac measure concentrated on the point γO.

Then, define the measures η_x^ε on $\mathbb{H}^n$ and λ_x^ε on $\partial\mathbb{H}^n$ respectively as the equivariant push-forward of μ_x^ε and its convolution with the family $\{\nu_y\}$ of visual measures. Namely, choose a point $O' \in \mathbb{H}^n$ and define

$$\eta_x^\varepsilon = \frac{1}{c(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)} \delta_{\rho(\gamma)O'} \qquad \lambda_x^\varepsilon = \frac{1}{c(s)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)} \nu_{\rho(\gamma)O'}$$

The map F^ε is defined by

$$F^\varepsilon(x) = \mathrm{bar}(\eta_x^\varepsilon) = \mathrm{bar}(\lambda_x^\varepsilon) = \mathrm{bar}\left(\frac{\lambda_x^\varepsilon}{||\lambda_x^\varepsilon||}\right).$$

In the current hypotheses we have:

- (Besson, Courtois, Gallot [4, Théorème 1.10]) The map F^ε satisfies conditions (1) and (2) of Lemma 4.5.

- (Francaviglia [7, Proposition 1.5]) The maps F^ε satisfy condition (3) of Lemma 4.5.

Therefore, it remains only to prove that for each $\varepsilon > 0$ the map F^ε properly ends. Let $T \subset \overline{M}$ be a boundary component and let $\pi_1(T)$ be (one of) the corresponding parabolic subgroup of $\pi_1(M)$, and let $\xi = \mathrm{Fix}(\pi_1(T))$.

The idea is now the following. For $x \in \mathbb{H}^k$ we have:

$$\eta_x^\varepsilon = \frac{e^{-sd(x,O)}}{c(s)} \sum_{\gamma\in\Gamma} e^{-s(d(x,\gamma O)-d(x,O))} \delta_{\rho(\gamma)O'}$$

and by point (5) of page 9 we have

$$F^\varepsilon(x) = \mathrm{bar}(\eta_x^\varepsilon) = \mathrm{bar}\left(\frac{c(s)}{e^{-sd(x,O)}}\lambda_x^\varepsilon\right) = \mathrm{bar}\left(\sum_{\gamma\in\Gamma} e^{-s(d(x,\gamma O)-d(x,O))}\nu_{\rho(\gamma)O'}\right)$$

Now, let $\alpha(t)$ be a geodesic ray ending at ξ. As $t \to \infty$, we have

$$\sum_{\gamma\in\Gamma} e^{-s(d(\alpha(t),\gamma O)-d(\alpha(t),O))}\nu_{\rho(\gamma)O'} \overset{*}{\rightharpoonup} \sum_{\gamma\in\Gamma} e^{-sB(\xi,\gamma O)}\nu_{\rho(\gamma)O'}$$

where $B(\cdot,\cdot)$ denotes the Busemann function normalized at O. Thus, from point (6) of page 10 we would get that, as $t \to \infty$

$$F^{\varepsilon}(\alpha(t)) \to \mathrm{bar}\left(\sum_{\gamma\in\Gamma} e^{-sB(\xi,\gamma O)}\nu_{\rho(\gamma)O'}\right)$$

which should be fixed by the elements of $\rho(\pi_1(T))$ because the limit measure $\sum_{\gamma\in\Gamma} e^{-sB(\xi,\gamma O)}\nu_{\rho(\gamma)O'}$ is $\rho(\pi_1(T))$-invariant.

Unfortunately, the limit measure $\sum_{\gamma\in\Gamma} e^{-sB(\xi,\gamma O)}\nu_{\rho(\gamma)O'}$ has no finite mass, whence its barycentre is not defined.

In order to overcome this difficulty some more work is required. For each x the measure $\lambda_x^{\varepsilon}/||\lambda_x^{\varepsilon}||$ is a probability measure on $\partial\mathbb{H}^n \simeq \mathbb{S}^{n-1}$. Since $\mathbb{S}^{n-1}$ is compact, the set of probability measures on $\partial\mathbb{H}^n$ is weak-$*$ compact. Therefore, up to pass to subsequences as $x \to \xi$ along the ray α, the measures $\lambda_x^{\varepsilon}/||\lambda_x^{\varepsilon}||$ converge to a probability measure λ_ξ on $\partial\mathbb{H}^n$ (the measure λ_ξ possibly depends on the chosen subsequence).

We show now that λ_ξ is $\rho(\pi_1(T))$-invariant. Let $\psi \in \pi_1(T) < \pi_1(M) = \Gamma < \mathrm{Isom}(\mathbb{H}^k)$. Since

$$\rho(\psi)_*\lambda_x^{\varepsilon} = \frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)}\nu_{\rho(\psi\gamma)O'} = \frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\psi^{-1}\gamma O)}\nu_{\rho(\gamma)O'}$$

we have

$$\rho(\psi)_*\lambda_x^{\varepsilon} - \lambda_x^{\varepsilon} = \frac{1}{c(s)}\sum_{\gamma\in\Gamma} e^{-sd(x,\gamma O)}(e^{-s(d(x,\psi^{-1}\gamma O)-d(x,\gamma O))} - 1)\nu_{\rho(\psi\gamma)O'}.$$

Using the hyperbolic law of sines on the triangles with vertices $x, \gamma O$ and $\psi^{-1}\gamma O$, one sees that there exists a function $E(x)$ such that $E(x) \to 0$ as $x \to \xi$ and

$$|e^{-s(d(x,\psi^{-1}\gamma O)-d(x,\gamma O))} - 1| < E(x),$$

whence

$$||\rho(\psi)_*\lambda_x^{\varepsilon} - \lambda_x^{\varepsilon}|| < E(x)||\lambda_x^{\varepsilon}||.$$

Since $||\lambda_x^{\varepsilon}|| = ||\rho(\psi)_*\lambda_x^{\varepsilon}||$, we have that $\lambda_x^{\varepsilon}/||\lambda_x^{\varepsilon}||$ and $\rho(\psi)_*\lambda_x^{\varepsilon}/||\rho(\psi)_*\lambda_x^{\varepsilon}||$ have the same limit λ_ξ. It follows that λ_ξ is $\rho(\pi_1(T))$-invariant.

Now we have two cases: either $\lambda_\xi = \dfrac{\delta_{\theta_1}+\delta_{\theta_2}}{2}$ or not. In the later case, by point (6) of page 10

$$F^{\varepsilon}(x) \to \mathrm{bar}(\lambda_\xi)$$

which, by point (7) of page 10, is fixed by the elements of $\rho(\pi_1(T))$.

In the former case, the barycentre of λ_ξ is not defined. Nevertheless, one can easily shows that the functions $\mathcal{B}_{\lambda_x^\varepsilon}(y)$ defined at page 9, converge to $\mathcal{B}_{\lambda_\xi}(y)$. Since $\mathrm{bar}(\lambda_x^\varepsilon)$ is the point of minimum of $\mathcal{B}_{\lambda_x^\varepsilon}$, it converges to a point of minimum of $\mathcal{B}_{\lambda_\xi}$ that, by point (3) of page 9, lies in the geodesic joining θ_1 and θ_2. Such geodesic is $\rho(\pi_1(T))$-invariant because the invariance of λ_ξ. $\square$

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