

GEODESIC CURRENTS AND LENGTH COMPACTNESS FOR AUTOMORPHISMS OF FREE GROUPS

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ABSTRACT. We prove a compactness theorem for automorphisms of free groups. Namely, we show that the set of automorphisms keeping the length of the uniform current bounded is compact (up to conjugation). This implies that the spectrum of the length of the images of the uniform current is discrete, proving a conjecture of I. Kapovich.

1. INTRODUCTION

The aim of this paper is to investigate the action of the automorphisms of a free group F on the Cayley graph of F . In particular, we are interested in understanding how automorphisms can stretch geodesics. One can define the *length* of an automorphism as the generic stretching factor (see [KKS05]) which is, roughly speaking, the average of the stretching ratios, taken over all geodesics (Definition 3.7). Intuitively, for $\Phi \in \text{Aut}(F)$, one can think of its length as the limit, as n tends to infinity, of $\|\Phi(w_n)\|/n$, where w_n is a “random” cyclically reduced word in F of length n , and $\|\cdot\|$ denotes the cyclically reduced length.

The length function on $\text{Aut}(F)$ is invariant under conjugation, so it descends to a length function on $\text{Out}(F)$. Our main result is the following:

Theorem 1.1 (Length compactness theorem). *Let Φ_n be a sequence of automorphisms of F . Then, up to passing to subsequences, there exists a sequence $v_n \in F$ such that the automorphisms Ψ_n defined by $x \mapsto v_n \Phi_n(x) v_n^{-1}$ satisfy one (and only one) of the following:*

- Ψ_n converges to an automorphism Φ (that is, it has a constant subsequence $\Psi_{n_i} = \Phi$).
- $L(\Phi_n)$ goes to ∞ .

A reformulation of this theorem is the following:

The set of automorphisms of bounded length is compact up to conjugation. That is, for any $M \in \mathbb{R}$, the set $\{[\Phi] \in \text{Out}(F) : L(\Phi) < M\}$ is finite.

Equivalently: *For a sequence of automorphisms Φ_n , if there is a word w such that the cyclically reduced length of $\Phi_n(w)$ goes to ∞ , then $L(\Phi_n) \rightarrow \infty$.*

All the work pivots on the fact that the Cayley graph of F is a hyperbolic object. Therefore, the boundary at infinity ∂F of F is well defined, and encodes enough information about the dynamic of the action of $\text{Aut}(F)$. The main idea is

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that length controls attractors: if Φ_n is a sequence of automorphisms of bounded length, then, up to conjugation, there are no attractors for the action of Φ_n on ∂F . Using this fact we prove that the sequence Φ_n keeps the cyclically reduced length of any element of F bounded, and this implies that Φ_n has a constant subsequence.

As a consequence of Theorem 1.1, we get the following result.

Corollary 1.2. *The spectrum of the length function is discrete. That is, the set*

$$\{L(\Phi) : \Phi \in \text{Aut}(F)\}$$

is a discrete subset of \mathbb{R} .

We remark that, while the length of an automorphism depends on the free basis chosen for the Cayley graph, the spectrum of the length function is intrinsic (it depends in fact on the current we use to define it, which in our case is the uniform current).

Corollary 1.2 was conjectured to be true by I. Kapovich, inspired by computational evidence and partial results. For example, in [KKS05] V. Kaimanovich, I. Kapovich and P. Shupp proved (among other results) that an automorphism of length one must be simple (see below) and estimated the “first gap” of the length function.

A consequence of Corollary 1.2 is that one can use inductive arguments on the length. For example, we get the following result, which can be viewed as an *Ideal Whitehead Algorithm* (see [Kap06a, Conjecture 5.3]). Recall that, given a free basis Σ of F , an automorphism τ is simple if it is either a permutation of Σ or an inner automorphism, while it is a Whitehead automorphism of the second kind if there is $a \in \Sigma$ such that $\tau(x) \in \{x, xa, a^{-1}x, a^{-1}xa\}$ for all $x \in \Sigma$ (see for example [LS77] for more details).

Theorem 1.3. *Let $\Phi \in \text{Aut}(F)$ be a non-simple automorphism. Then there exists a factorisation*

$$\Phi = \tau_n \tau_{n-1} \cdots \tau_1 \sigma$$

where $n \geq 1$, the automorphism σ is simple, each τ_i is a Whitehead automorphism of the second kind, and

$$L(\tau_{i-1} \cdots \sigma) < L(\tau_i \tau_{i-1} \cdots \sigma), \quad i = 1, \dots, n-1.$$

Let us say a few words about Theorem 1.3. The *automorphism problem* for a free group F asks, given two arbitrary elements $u, v \in F$, whether there exists $\Phi \in \text{Aut}(F)$ such that $\Phi(u) = v$. In [Whi36] Whitehead gave an algorithm solving that problem. The first part of the algorithm is to maximally reduce the lengths of u and v via Whitehead automorphisms. Then, given two *minimal* elements one proves that they are in the same $\text{Aut}(F)$ -orbit if and only if they are related via a sequence of minimal elements, each one obtained from the preceding via a Whitehead automorphism. Roughly speaking, Theorem 1.3 is an *averaged* version of the first part of the Whitehead algorithm. We refer the reader to [Kap06a] for a more detailed discussion on the matter. We only notice that, as our proof is “typically hyperbolic,” one may expect that it could be adapted to a more general setting, like that of hyperbolic groups, for which the automorphism problem is still not completely solved.

The main tool we use is the theory of *geodesic currents*. These are locally finite F -invariant Borel measures on the space of geodesics lines in the Cayley graph of

F . Geodesic currents were introduced by F. Bonahon [Bon86] in the setting of hyperbolic manifolds, and turned out to be very useful in group theory (see for example [Mar95, Kap06b, Kap06a]).

We also consider measures on the space of geodesic rays (that is, half geodesics) in the Cayley graph of F . This is the space of *frequency measures* (see also [Kap05]).

Such spaces are in fact homeomorphic, but each one has peculiar characteristics that are well-adapted to different situations, and we will jump from one to the other depending on the calculations needed. Roughly speaking, the action of $\text{Aut}(F)$ on currents is “more natural”, while frequency measures are “more compact”.

As mentioned earlier, the length function on $\text{Aut}(F)$ depends on the choice of a current. We use the uniform current, which is the analogous of the Liouville current for the geodesic flow of a surface, but, *ça va sans dire*, length compactness should hold for many other currents. In the last section we discuss some generalisations of Theorem 1.1 in such a direction.

Remark 1.4. The space of automorphisms of a free group is discrete. Thus, compactness is equivalent to finiteness, and to say that a sequence converges is equivalent to saying that it is finite (and hence eventually constant). Nevertheless, we prefer to speak about compactness and convergence because we think that this is closer to the spirit of the paper, in which we used “more hyperbolicity than discreteness”. (Even if discreteness is necessary, as, for example, in Corollary 1.2.)

2. NOTATION

For the remainder of the paper, we fix the following notation:

- F is a free group of rank k , with a fixed free basis Σ . We set $A = \Sigma \cup \Sigma^{-1}$. Any element of F corresponds to a unique *freely reduced* word in the alphabet A , that is, a word not containing sub-words of the form aa^{-1} with $a \in A$. We identify F with the set of freely reduced words. We denote by 1 the neutral element of F (the empty word). A word w is *cyclically reduced* if all the cyclic permutations of w are freely reduced. For v a freely reduced word, $|v|$ denotes its length, and $\|v\|$ denotes its *cyclically reduced length*, that is, the length of the cyclically reduced word obtained by cyclically reducing v .
- The Cayley graph of F corresponding to A will shortly be denoted simply by a *Cayley graph*. We denote by 1 the base point of the Cayley graph corresponding to $1 \in F$.
- ∂F is the *boundary at infinity* of F , identified with the set of geodesic rays in the Cayley graph, that is, freely reduced, right-infinite words in the alphabet A . The boundary ∂F is endowed with the Cantor-set topology. Namely, for each $v \in F$ we denote by $\text{Cyl}(v)$ the set of rays having v as an initial segment. We set $\text{Cyl}(1) = \partial F$. Then, a basis for the topology of ∂F is given by the sets $\{\text{Cyl}(v) : v \in F\}$.
- $\partial^2 F$ is the set $\{(x, y) \in (\partial F)^2 : x \neq y\}$. We identify $\partial^2 F$ with the set of oriented bi-infinite geodesics in the Cayley graph. We define the *base-ball* B of $\partial^2 F$ as the set of geodesics passing through 1.
- For any $x \neq y \in F$, the cylinder $\text{Cyl}([x, y])$ is defined as the subset of $\partial^2 F$ of geodesics passing through the oriented segment joining x and y in the Cayley graph (with the correct orientation). We set $\text{Cyl}([1, 1]) = B$.

- We denote by $T : \partial F \rightarrow \partial F$ the shift operator deleting the first letter of a ray. It turns out that T is a continuous map.
- Given a topological space M , we identify the space of Borel measures on M with the dual of $\mathcal{C}_0(M)$ (the space of continuous functions on M with compact support) endowed with the weak-* topology. Namely, measures μ_i converge to μ if and only if $\int \varphi d\mu_i \rightarrow \int \varphi d\mu$ for all $\varphi \in \mathcal{C}_0(M)$. If μ is a Borel measure on M , N is a topological space, and $f : M \rightarrow N$ is a measurable, proper map, we denote by $f_*\mu$ the push-forward of μ , that is, the measure on N such that $\int_N \varphi d(f_*\mu) = \int_M \varphi \circ f d\mu$ for all $\varphi \in \mathcal{C}_0(N)$ (see for example the first chapters of [Fed69] or [AFP00] for good introductions to geometric measure theory).

3. DEFINITIONS AND PRELIMINARY FACTS

In this section, we define the space of geodesic currents and of frequency measures, and we show that such spaces are homeomorphic. We introduce the *length* of a current, which is the analog of the length of a cyclically reduced word. We define the *uniform frequency measure* and the *uniform current*, which we use to define the *length* of automorphisms.

First of all, in order to describe the action of $\text{Aut}(F)$ on currents, we need the following classical result, whose proof can be found in [Coo87].

Theorem 3.1. *Let Φ be an automorphism of F . Then it extends to a homeomorphism of ∂F (which we still denote by Φ).*

Since Φ is a homeomorphism of ∂F , the map $\Phi \times \Phi$ is continuous and proper on $\partial^2 F$. It follows that any automorphism Φ acts on the space of Borel measures on $\partial^2 F$ via $(\Phi \times \Phi)_*$. The inclusion of F in $\text{Aut}(F)$ given by inner automorphisms induces an action of F on the space of Borel measures on $\partial^2 F$. By abuse of notation, if η is a Borel measure, we will denote by $\Phi\eta$ the measure $(\Phi \times \Phi)_*\eta$. We can now give the definition of currents and frequency measures. Our definitions are a little different from those introduced in [Kap06b, Kap05], as we do not require measures to be normalised to probability measures. This is because the quantities we are interested in (lengths of automorphisms) depend on the total mass of the measures with which we work.

Definition 3.2 (Geodesic currents and their lengths). The space of *geodesic currents* is the space of locally finite (*i.e.* finite on compact sets) F -invariant non-negative Borel measures on $\partial^2 F$. The *length* $L(\eta)$ of a current η is the measure $\eta(B)$ of the base-ball B of $\partial^2 F$.

The length of a current η is explicitly given by

$$L(\eta) = \sum_{x \in A} \eta(\text{Cyl}([1, x])).$$

Definition 3.3 (Frequency measures). The space of *frequency measures* is the set of finite-mass T -invariant non-negative Borel measures on ∂F (where T -invariant means that $T_*\mu = \mu$). The total mass of a measure μ will be denoted by $\|\mu\|$.

The unitary ball of the frequency measures, that is to say, the set of probability T -invariant measures on ∂F , is sometimes called the *frequency space* of F in the literature (see [Kap05]).

We will use the letter η primarily to denote a current, and the letter μ to denote a frequency measure. We refer the reader to Appendix A for some basic facts about currents and measures.

If η is a geodesic current, and $x, y \in F$, by F -invariance, the value $\eta(\text{Cyl}([x, y]))$ depends only on the label $x^{-1}y \in F$. The F -invariance of currents plays the role of T -invariance for frequency measures. With this in mind, we can construct an isomorphism between the space of geodesic currents and the space of frequency measures as follows:

$$\eta \leftrightarrow \mu \quad \text{if and only if} \quad \eta(\text{Cyl}([x, y])) = \mu(\text{Cyl}(x^{-1}y)).$$

More precisely, one can prove (see also [Kap05, Kap06a]):

Theorem 3.4. *The map α from the space of frequency measures to the space of geodesic currents, defined by*

$$\alpha(\mu)(\text{Cyl}([x, y])) = \mu(\text{Cyl}(x^{-1}y))$$

for all $x, y \in F$, is a homeomorphism with respect to the weak- $*$ topologies. Moreover, under this correspondence, the total mass corresponds to length, that is,

$$L(\alpha(\mu)) = \|\mu\|.$$

Proof. We only sketch the proof. The fact that α is well-defined and bijective can easily be proved using F - and T -invariance. The weak- $*$ continuity follows from Proposition A.1, while the last claim is a straightforward computation. \square

The identification between currents and frequency measures induces an action of $\text{Aut}(F)$ on the frequency measures given by

$$\Phi\mu = \alpha^{-1} \circ (\Phi \times \Phi)_* \circ \alpha\mu.$$

Note that the action on frequency measures is not just the push-forward via Φ because the push-forward does not commute with T .

The fact that the length of a current corresponds to the total mass of a frequency measure will be the first ingredient of the proof of the compactness result: bounded length \rightarrow bounded norm \rightarrow weak compactness.

Next, we briefly discuss relations between currents and cyclically reduced words, referring the reader to [Kap06b] for more details.

There is a natural embedding of the space of cyclically reduced words in the space of geodesic currents (or frequency measures). Namely, if w is a cyclically reduced word, we denote by $w^{+\infty}$ the ray $www \dots$, by $w^{-\infty}$ the ray $w^{-1}w^{-1}w^{-1} \dots$, and by γ_w the geodesic joining $w^{-\infty}$ and $w^{+\infty}$, that is, $\gamma_w = (w^{-\infty}, w^{+\infty}) \in \partial^2 F$. Then one can associate at each word w the current

$$\eta_w = \sum_{v \in [w]} \delta_{\gamma_v},$$

where $[w]$ is the conjugacy class of w in F and δ_{γ_v} denotes the Dirac measure concentrated on γ_v . In the literature, such currents are often referred to as *rational currents*. Note that if w is not a proper power, then $\|w\| = L(\eta_w) = \|\alpha^{-1}(\eta_w)\|$.

Definition 3.5 (Uniform current and uniform measure). The *uniform current* η_A and the *uniform frequency measure* μ_A are defined as follows. For all $v \in F$ we set

$$\mu_A(\text{Cyl}(v)) = \frac{1}{2k(2k-1)^{|v|-1}} \quad \text{and} \quad \eta_A = \alpha(\mu_A).$$

Note that $L(\eta_A) = 1$ and $\|\mu_A\| = 1$.

Remark 3.6. The uniform current is not the product $\mu_A \times \mu_A$ on $(\partial F)^2$. Indeed, η_A is a measure on $\partial^2 F \neq (\partial F)^2$, and F -invariance implies that neighbourhoods of the diagonal have infinite measure.

Nevertheless, the current η_A is not so different from $\mu_A \times \mu_A$. Indeed, we can *disintegrate* η_A with measures that are in the same density class as μ_A . This means that if we cut a slice S_x of $\partial^2 F$ at the point x , namely $S_x = \{x\} \times \{\partial F \setminus x\}$, then there exists a continuous function φ on $\{\partial F \setminus x\}$ such that the measure μ_x induced on S_x by η_A is $\varphi \cdot \mu_A$. A precise version of this fact is proved in Lemma A.2.

Definition 3.7 (Length of automorphisms). For any automorphism Φ of F we define the length of Φ as the length of the image of the uniform current, that is,

$$L(\Phi) = L(\Phi\eta_A) = \eta_A(\Phi^{-1}(B)).$$

Because of F -invariance of currents, $L(\Phi)$ depends only on the class $[\Phi] \in \text{Out}(F)$. We set $L([\Phi]) = L(\Phi)$.

Intuitively speaking, for $\Phi \in \text{Aut}(F)$, one can think of the length of Φ as the limit, as n tends to infinity, of $\|\Phi(w_n)\|/n$, where w_n is a “random” cyclically reduced word in F of length n .

4. PROOFS OF THE MAIN RESULTS

Sketch of the proof of Theorem 1.1. Let $\{\Phi_n\}$ be a sequence of automorphisms of bounded length. The strategy of the proof can be summarised as follows.

Step 1. The bounded length hypothesis, together with compactness of frequency measures, implies that the currents $\Phi_n\eta_A$ have a limit η_∞ (Lemma 4.1).

Step 2. The core of the proof. We study the action of Φ_n on $\partial^2 F$ and on ∂F . The main idea is that unbounded lengths of Φ_n correspond to the fact that the maps Φ_n accumulate all the boundary on some points (the *attractors*). The key point is that, on the one hand, the bounded length hypothesis excludes the presence of attractors, while on the other hand, the existence of an element of $f \in F$ such that $\|\Phi_n(f)\|$ is unbounded implies the presence of attractors (Lemma 4.6 and Lemma 4.8). Therefore, the maps Φ_n keep bounded the cyclically reduced length of all elements of F .

Step 3. If Φ_n keep bounded the cyclically reduced length of all elements of F , then Φ_n has a subsequence that converges, *i.e.* is eventually constant (Lemma 4.10).

We now proceed to work out the details of the proof.

Lemma 4.1. *Up to passing to a subsequence, the currents $\Phi_n\eta_A$ have a limit η_∞ which is a geodesic current.*

Proof. Since the lengths $L(\Phi_n\eta_A)$ are bounded, the total mass of the corresponding frequency measure $\Phi_n\mu_A$ is bounded. The set of non-negative measures with bounded mass on a compact space is weakly-* compact. Since ∂F is compact, up to passing to subsequences, $\Phi_n\mu_A$ has a limit μ_∞ . Such a limit is T -invariant because the push-forward via a continuous map is weak-* continuous. Thus, the limit is a frequency measure, which therefore corresponds to a geodesic current η_∞ . By continuity of the correspondence α between frequency measures and geodesic currents, it follows that $\Phi_n\eta_A \rightarrow \eta_\infty$. \square

Remark 4.2. Although it is not relevant for the sequel, we note that the limit η_∞ is actually a non-zero current because $L(\Phi) \geq 1$ for any Φ .

Next, we proceed to study the attractors. As the next lemma shows, attractors correspond to singularities of the limit current η_∞ . More precisely, a Borel measure σ_1 is said to be *absolutely continuous* w.r.t. the Borel measure σ_2 if for any Borel set C , $\sigma_2(C) = 0$ implies $\sigma_1(C) = 0$. We say that a current η has a *part concentrated on a set C* if $\eta_A(C) = 0$ and $\eta(C) > 0$. Similarly, we say that a frequency measure μ has a *part concentrated on a set C* if $\mu_A(C) = 0$ and $\mu(C) > 0$.

Lemma 4.3. *Let Ψ_n be a sequence of automorphisms such that the currents $\Psi_n \eta_A$ have a limit current η . Let $p, q \in \partial F$ be two distinct points. Suppose that there exist cylinders $P_n = \text{Cyl}(p_n), Q_n = \text{Cyl}(q_n) \subset \partial F$ such that $p_n \rightarrow p$ and $q_n \rightarrow q$, and such that there is a positive constant c for which $\eta_A(\Psi_n^{-1}(P_n \times Q_n)) > c$. Then, the current η has a part concentrated on (p, q) .*

Proof. Any cylinder $C \subset \partial^2 F$ containing (p, q) , contains $P_n \times Q_n$ for all sufficiently large n . Therefore, by definition of push-forward and by hypothesis, we get (for all sufficiently large n):

$$\Psi_n \eta_A(C) > \Psi_n \eta_A(P_n \times Q_n) = \eta_A(\Psi_n^{-1}(P_n \times Q_n)) > c.$$

By Proposition A.1 it follows that the limit current satisfies $\eta(C) \geq c$ for any cylinder C containing (p, q) . This implies that $\eta((p, q)) \geq c$ while $\eta_A((p, q)) = 0$, that is, η has a part concentrated on (p, q) . \square

The following is a standard fact, which says that the only currents that can have a part concentrated on a geodesic are essentially the rational currents (recall that ∂F and $\partial^2 F$ are identified with the set of geodesic rays and of bi-infinite geodesics in the Cayley graph, respectively).

Lemma 4.4. *Any frequency measure (and hence the limit μ_∞) cannot have a part concentrated on a non-periodic ray. Therefore any current (and hence the limit η_∞) cannot have part concentrated on a non-periodic geodesic.*

Proof. Any frequency measure μ has finite mass and is T -invariant. Hence if it has a part concentrated on a point x , it has a part concentrated on each point of the T -orbit of x , with the same weight. It follows that such an orbit must be finite, forcing x to be periodic. \square

We now prove a small lemma that will be used in Lemma 4.6. We isolated this result from the proof of Lemma 4.6 because it is the only point in which we crucially use the properties of the uniform current (this will be further discussed in Section 5).

Lemma 4.5. *For any $f \in F$, there exists a positive constant c , depending only on f , such that whenever two disjoint Borel-subsets E, S of ∂F satisfy*

$$f(\partial F \setminus E) \subset S, \quad f^{-1}(\partial F \setminus S) \subset E,$$

then we have

$$\eta_A(E \times S) \geq c(1 - \mu_A(E))(1 - \mu_A(S)).$$

In particular, $\eta_A(E \times S) = 0$ if and only if either E or S has full-measure with respect to μ_A .

We notice that the hypothesis that E and S are disjoint is redundant. We formulated the lemma in that way for reasons of compatibility with Theorem 5.3.

Proof of Lemma 4.5. The hypothesis that E and S are disjoint will not be used in this proof. By hypothesis we have $\mu_A(S) \geq \mu_A(f(\partial F \setminus E))$ and $\mu_A(E) \geq \mu_A(f^{-1}(\partial F \setminus S))$, and by Lemma A.4

$$\mu_A(E) \geq \frac{1 - \mu_A(S)}{(2k - 1)^{|f|}}$$

and

$$\mu_A(S) \geq \frac{1 - \mu_A(E)}{(2k - 1)^{|f|}}$$

from which we get

$$\mu_A(E)\mu_A(S) \geq \frac{(1 - \mu_A(E))(1 - \mu_A(S))}{(2k - 1)^{2|f|}}.$$

By Lemma A.3

$$\eta_A(E \times S) \geq \mu_A(E)\mu_A(S) \geq \frac{(1 - \mu_A(E))(1 - \mu_A(S))}{(2k - 1)^{2|f|}},$$

and setting $c = \frac{1}{(2k-1)^{2|f|}}$ completes the proof. \square

Now the aim is to prove that the maps Φ_n keep bounded the lengths of all elements of F , so that we can apply Lemma 4.10. We do it in the following two lemmata. Namely, in Lemma 4.6 we show that if this is not the case, then there are no attractors in $\partial^2 F$ and at most a unique attractor in ∂F . Lemma 4.8 will show that, up to conjugation, we can avoid the presence of a unique attractor in ∂F .

Lemma 4.6. *Suppose that $\Phi_n \eta_A$ has bounded length (uniformly on n). Suppose that there exists an element $f \in F$ such that the cyclically reduced length $\Phi_n(f)$ goes to ∞ . Then, after possibly passing to a subsequence, Φ_n (as maps of ∂F) pointwise converge almost everywhere to a constant. That is to say, there exists $y \in \partial F$ such that for μ_A -almost all $x \in \partial F$, $\Phi_n(x) \rightarrow y$.*

Proof. By Lemma 4.1, without loss of generality we can suppose that $\Phi_n \eta_A$ has a limit η_∞ .

Recall that we consider $\Phi_n(f)$ and $\Phi_n(f^{-1})$ as freely reduced words. Let v_n be the maximal initial segment shared by $\Phi_n(f)$ and $\Phi_n(f^{-1})$, and let Ψ_n be the map $x \mapsto v_n^{-1} \Phi_n(x) v_n$. Note that $\Psi_n \eta_A = \Phi_n \eta_A$ and that $\Psi_n(f)$ is cyclically reduced. Up to passing to a subsequence, $\Psi_n(f)$ and $\Psi_n(f^{-1})$ have limits, which we denote by r_+ and r_- , in ∂F . Since $\Psi(f)$ is cyclically reduced, $r_+ \neq r_-$. Note that this also implies that r_+ and r_- have no common initial segment, that is, the geodesic (r_-, r_+) passes through 1, the base-point of the Cayley graph.

We now show that Ψ_n , as maps of ∂F , converge μ_A -almost everywhere either to r_- or to r_+ .

Next, cut $\Psi_n(f)$ into two segments of equal length. More precisely, we set

$$\Psi_n(f) = s_n e_n^{-1}$$

where the starting segment s_n and the ending one e_n both have length $|\Psi_n(f)|/2$ (approximated to the nearest integers). We have $s_n \rightarrow r_+$ and $e_n \rightarrow r_-$. In

particular, for large enough n , $\text{Cyl}(e_n) \cap \text{Cyl}(s_n) = \emptyset$, which implies $\text{Cyl}([e_n, s_n]) = \text{Cyl}(e_n) \times \text{Cyl}(s_n)$. For large n , let C_n be such a cylinder:

$$C_n = \text{Cyl}([e_n, s_n]) = \text{Cyl}(e_n) \times \text{Cyl}(s_n)$$

and set $E_n = \Psi_n^{-1}(\text{Cyl}(e_n))$ and $S_n = \Psi_n^{-1}(\text{Cyl}(s_n))$. Note that $E_n \cap S_n = \emptyset$.

For all $x \in \partial F$, either $\Psi_n(x) \in \text{Cyl}(e_n)$ or $\Psi_n(fx) \in \text{Cyl}(s_n)$, so either $x \in E_n$ or $fx \in S_n$, whence

$$f(\partial F \setminus E_n) \subset S_n.$$

Similarly, either $\Psi_n(x) \in \text{Cyl}(s_n)$ or $\Psi_n(f^{-1}x) \in \text{Cyl}(e_n)$ and

$$f^{-1}(\partial F \setminus S_n) \subset E_n.$$

Thus, by Lemma 4.5

$$\eta_A(E_n \times S_n) \geq \frac{(1 - \mu_A(E_n))(1 - \mu_A(S_n))}{(2k - 1)^{2|f|}}.$$

By the definition of push-forward

$$\Psi_n \eta_A(C_n) = \eta_A(\Psi_n^{-1}(C_n)) = \eta_A(E_n \times S_n),$$

and putting together these (in)equalities, we get

$$\Psi_n \eta_A(C_n) \geq \frac{[1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(e_n)))] [1 - \mu_A(\Psi_n^{-1}(\text{Cyl}(s_n)))]}{(2k - 1)^{2|f|}}.$$

If $\Psi_n \eta_A(C_n) \rightarrow 0$, then either $\mu_A(\Psi_n^{-1}(\text{Cyl}(e_n)))$ or $\mu_A(\Psi_n^{-1}(\text{Cyl}(s_n)))$ converges to 1 and therefore, up to passing to subsequences, Ψ_n converges almost everywhere either to r_- or to r_+ , and we are done.

We now show that the bounded length hypothesis excludes the possibility that $\Psi_n \eta_A(C_n)$ stays bounded away from zero. Indeed, suppose that there exists a constant c such that $\Psi_n \eta_A(C_n) > c$, uniformly on n . Then, by Lemma 4.3, η_∞ has a part concentrated on the geodesic (r_-, r_+) , which is therefore periodic by Lemma 4.4; let w be its period. We must have

$$(1) \quad r_- = w^{-\infty} \quad \text{and} \quad r_+ = w^{+\infty}.$$

We may assume that the element f is not a proper power. Since Ψ_n is an automorphism of F , it follows that $\Psi_n(f)$ is not a proper power either. Therefore, by (1), for all large enough n , we can write $\Psi_n(f)$ as

$$\Psi_n(f) = w^{i(n)} u_n w^{j(n)}$$

where u_n neither starts nor ends with w , and the exponents $i(n)$ and $j(n)$ are non-negative and go to infinity as n does. Without loss of generality, we can suppose $i(n) \leq j(n)$, so that s_n starts with $w^{i(n)}$.

For any $0 \leq h \leq i(n)$, let C_n^h be the cylinder

$$C_n^h = \text{Cyl}([w^{-h} e_n, w^{-h} s_n]).$$

Note that the C_n^h 's are pairwise disjoint, because u_n neither starts nor ends with w . Moreover, the condition $0 \leq h \leq i(n) \leq j(n)$ implies that the geodesic segment from $w^{-h} e_n$ and $w^{-h} s_n$ passes through 1, thus $C_n^h \subset B$ for all $0 \leq h \leq i(n)$. By F -invariance of currents, we have

$$\Psi_n \eta_A(C_n^h) = \Psi_n \eta_A(C_n) > c$$

uniformly on n . It follows that

$$L(\Phi_n) = L(\Psi_n) = \Psi_n \eta_A(B) > ci(n)$$

which goes to infinity as n does, contradicting the bounded length hypothesis.

Thus we have proved that, after passing to a subsequence, the maps $\Psi_n : x \mapsto v_n^{-1} \Phi_n(x) v_n$ μ_A -almost everywhere converge to a map which is constant (either to r_- or to r_+). Up to possibly passing to a subsequence, v_n converges to a limit v_∞ , which is either an element of F or of ∂F . Since the elements v_n were the maximal initial segments shared by $\Phi_n(f)$ and $\Phi_n(f^{-1})$, the words $v_n \Psi_n(f)$ and $v_n \Psi_n(f^{-1})$ are freely reduced. It follows that the maps Φ_n converge almost everywhere, up to passing to the same subsequence, to a constant – which is either v_∞ (if v_∞ is an element of ∂F) or $v_\infty r_-$ or $v_\infty r_+$ (if $v_\infty \in F$). \square

Remark 4.7. The proof of Lemma 4.6 can be adapted to prove the following more general fact. If we replace the hypothesis “ $\Phi_n \eta_A$ has bounded length η_∞ ” with “ $\frac{\Phi_n}{L(\Phi_n)} \eta_A$ has a limit η_∞ ” – which is always true up to passing to a subsequence – then, we get that η_∞ does not have a part concentrated on a geodesic. Indeed, if η_∞ has a part concentrated on a geodesic γ , then there exists a positive constant c such that for any cylinder C containing γ we have $\eta_A((\Phi_n \times \Phi_n)^{-1}(C)) \sim cL(\Phi_n)$. As in the argument above, we must have $\gamma = (w^{-\infty}, w^{+\infty})$ for some $w \in F$, and conjugating Φ_n by a suitable power of w , we reach a contradiction. Indeed, if X denotes the set $(\Phi_n \times \Phi_n)^{-1}(C)$, then $(\Phi_n \times \Phi_n)^{-1}(wC) = \Phi_n^{-1}(w)X$ which is contained in the set $(\Phi_n \times \Phi_n)^{-1}(B)$, whose η_A -measure is $L(\Phi_n)$ by definition. If a geodesic belongs to $X \cap \Phi_n^{-1}(w)X$, then it passes through 1, whence

$$\eta_A(X \cap \Phi_n^{-1}(w)X) \leq 1.$$

Since $\eta_A(X) \sim cL(\Phi_n)$, we can conjugate by w approximately at most $1/c$ times, while if $\gamma = (w^{-\infty}, w^{+\infty})$, then we can do that infinitely many times.

After Lemma 4.6, it remains to deal with the case where Φ_n converges almost everywhere to a constant. What is the behavior of such a sequence? An example can be constructed by taking a fixed Φ and conjugating with elements v_n whose length goes to infinity. The next lemma shows that more or less this is the only possibility.

Lemma 4.8. *Let Φ_n be a sequence of automorphisms of F . Then, there exists $v_n \in F$ such that, up to possibly passing to a subsequence, the maps $x \mapsto v_n^{-1} \Phi_n(x) v_n$ have no subsequence converging to a constant μ_A -almost everywhere.*

Proof. The rough idea is that, via conjugations, we can force the “barycentre of Φ_n ” to stay in a fixed compact.

For any freely reduced word w of length M , define $B_n(w)$ as the set of rays x such that $\Phi_n(x)$ starts by w , namely

$$B_n(w) = \{x \in \partial F : \Phi_n(x) \in \text{Cyl}(w)\} = \Phi_n^{-1}(\text{Cyl}(w)).$$

Obviously $B_n(1) = \partial F$. Moreover, for each n we have

$$(2) \quad \lim_{M \rightarrow \infty} \sup_{|w|=M} \mu_A(B_n(w)) = 0.$$

Indeed, otherwise for all M there exists $w_M \in F$ of length M such that $\mu_A(B_n(w_M)) > c > 0$. Up to subsequences, w_M converges to a ray R , and $\Phi_n(B_n(w_M))$

$= \text{Cyl}(w_M) \rightarrow R$, contradicting the fact that Φ_n is a homeomorphism of ∂F (in this argument n is fixed).

Now, let v_n be a freely reduced word of maximal length such that $\mu_A(B_n(v_n)) \geq \frac{1}{2}$. Let $\tilde{\Phi}_n$ be the map $x \mapsto v_n^{-1}\Phi(x)v_n$ and let

$$\tilde{B}_n(w) = \{x \in \partial F : \tilde{\Phi}_n(w) \in \text{Cyl}(w)\}.$$

Let $l_n \in A$ be the last letter of v_n . Since $\mu_A(B_n(v_n)) \geq \frac{1}{2}$ we get $\mu_A(\tilde{B}_n(l_n^{-1})) \leq \frac{1}{2}$. On the other hand, for any $a \in A$, different from l_n^{-1} , maximality of the length of v_n implies

$$\mu_A(\tilde{B}_n(a)) \leq \frac{1}{2}.$$

Hence, such an inequality holds for all $a \in A$. It follows that the sequence $\tilde{\Phi}_n$ cannot have any subsequence converging to a constant almost everywhere. \square

Since conjugations do not affect the length of cyclically reduced words, Lemma 4.6 and Lemma 4.8 can be summarised as follows (recall that for $f \in F$, $|f|$ denotes its length, while $\|f\|$ denotes the length of the cyclically reduced word obtained from f).

Corollary 4.9. *Let Φ_n be a sequence of automorphisms. If there is M such that $L(\Phi_n) < M$, then for each $f \in F$ there exists $M(f)$ such that $\|\Phi_n(f)\| < M(f)$.*

As the experts know, Corollary 4.9 implies Theorem 1.1. We include the proof of the following Lemma 4.10 for completeness.

Lemma 4.10. *Let $\{\Phi_n\}$ be a sequence of automorphisms such that for each $f \in F$ there is an $M(f)$ such that $\|\Phi_n(f)\| < M(f)$. Then, there exist elements $v_n \in F$ such that a subsequence of $\{v_n^{-1}\Phi_n v_n\}$ converges to an automorphism (i.e. $\{v_n^{-1}\Phi_n v_n\}$ has a constant subsequence).*

Proof. By a diagonal argument, up to passing to a subsequence, the maps Φ_n , as maps from F to itself, pointwise converge to a map Φ_∞ (up to conjugation). In particular, there exists a map $\Phi_\infty : A \rightarrow F$ and maps $w_n : A \rightarrow F$ such that $\Phi_\infty(f)$ is cyclically reduced and, up to passing to a subsequence, for all $f \in A$ we have

$$\Phi_n(f) = w_n(f)\Phi_\infty(f)w_n(f)^{-1}.$$

Choose an element $a \in A$. Up to conjugation we can suppose that $w_n(a) = 1$, that is, Φ_n really converges as an automorphism on the subgroup generated by a . Let $G \subset A$ be a maximal set of generators g such that $|\Phi_n(g)|$ stays bounded. If $G = A$ we are done, because, up to subsequences, Φ_n converges on A , whence on F . Otherwise, there exists $f \in A$ such that the length of $w_n(f)$ goes to infinity. Since

$$\Phi_n(af) = \Phi_\infty(a)\Phi_n(f) = \Phi_\infty(a)w_n(f)\Phi_\infty(f)w_n(f)^{-1}$$

has bounded cyclically reduced length, and since $\Phi_\infty(a)$ has finite length, we get that, for large enough n , $w_n(f)$ must start either with $\Phi_\infty(a)$ or with $\Phi_\infty(a)^{-1}$. Iterating this argument we get that $w_n(f)$ is the product of a power of $\Phi_\infty(a)$ and a word of bounded length. Thus, up to subsequences, we get

$$(3) \quad w_n(f) = \Phi_\infty(a)^m u$$

for some $m \in \mathbb{Z}$ with $|m| \rightarrow \infty$ as $n \rightarrow \infty$, and u a finite word (which depends on f).

It follows that, up to conjugating Φ_n by $\Phi_\infty(a)^m$, we can suppose that G has at least two elements a, b and that Φ_n is eventually constant on the subgroup generated by a and b . If $G \neq A$, let f be as above. As in (3), we get

$$\begin{aligned} w_n(f) &= \Phi_\infty(a)^m u, \\ w_n(f) &= \Phi_n(b)^l v \end{aligned}$$

for some exponents m, l such that $|m|, |l|$ go to infinity as n does, and fixed words u, v (depending on f, a, b). Therefore, as n goes to infinity, we get that the unoriented geodesics $(\Phi_\infty(a)^{-\infty}, \Phi_\infty(a)^{+\infty})$ and $(\Phi_n(b)^{-\infty}, \Phi_n(b)^{+\infty})$ coincide.

This implies that $\Phi_n(b)$ is cyclically reduced. In particular, we get $\Phi_n(b) = \Phi_\infty(b)$, and therefore Φ_∞ is an automorphism on the group generated by a and b . Moreover, the above inequalities imply that

$$\Phi_\infty(b)^{|\Phi_\infty(a)|} = \Phi_\infty(a^{\pm 1})^{|\Phi_\infty(b)|},$$

whence

$$b^{|\Phi_\infty(a)|} = a^{\pm 1|\Phi_\infty(b)|},$$

which is impossible because F is free. Thus $G = A$, and hence there exists a subsequence of $\{\Phi_n\}$ which converges. \square

By Lemma 4.8, up to conjugation, the sequence Φ_n does not sub-converge almost everywhere to the same point; by Lemma 4.6 we can apply Lemma 4.10, and the proof of Theorem 1.1 is complete. \square

Proof of Corollary 1.2. We have to prove that the spectrum of the length function is discrete. Suppose not, and take a sequence Φ_n of automorphisms such that $L(\Phi_n)$ has a limit λ , with $L(\Phi_n) \neq \lambda$ for all n . By Theorem 1.1 there exist elements v_n and a subsequence n_i such that the maps $\Psi_{n_i} : x \mapsto v_{n_i} \Phi_{n_i}(x) v_{n_i}^{-1}$ converge to an automorphism Ψ . Thus, the sequence Ψ_{n_i} is eventually constant, and therefore the sequence of lengths $L(\Psi_{n_i})$ is also eventually constant. But $L(\Psi_{n_i}) = L(\Phi_{n_i})$ is therefore eventually equal to λ , a contradiction. \square

Proof of Theorem 1.3. This immediately follows from Corollary 1.2 and [Kap06a, Proposition 5.2]. Indeed, I. Kapovich proved that for any non-simple automorphism Φ there exists a Whitehead automorphism τ such that

$$1 \leq L(\tau\Phi) < L(\Phi),$$

and the claim follows by an inductive argument on the length. \square

5. GENERALISATIONS

In this section we give a (partial) answer to the question: *For which currents does Theorem 1.1 hold?* The idea is that length-compactness is true (for any action on metric trees and) for any current for which Lemma 4.5 holds.

Definition 5.1 (η -length of automorphisms). Let η be a geodesic current and let $\Phi \in \text{Aut}(F)$. We define the η -length of Φ as

$$L_\eta(\Phi) = L(\Phi\eta).$$

The proof of Theorem 1.1 can be followed step by step in this new setting, obtaining:

Theorem 5.2. *Let η be a geodesic current and let μ be its corresponding frequency-measure. Suppose there exists a constant $c > 0$ and that for each $f \in F$ there is $b(f) > 0$ such that for any disjoint Borel sets $E, S \subset \partial F$,*

$$\mu(fE) \geq b(f)\mu(E) \quad \text{and} \quad \eta(E \times S) \geq c\mu(E)\mu(S).$$

Then Theorem 1.1 holds for η . That is to say, any sequence of automorphisms $\Phi_n \in \text{Aut}(F)$ with bounded η -length has, after possibly conjugating, a convergent subsequence (i.e. a constant subsequence).

Proof. The hypotheses on η guarantee that Lemma 4.5 holds for η . Moreover, our assumptions imply that η is not concentrated on a single geodesic. This implies that Lemma 4.8 can be rewritten, with the difference being that equation (2) of page 170 becomes (notation as in Lemma 4.8)

$$\lim_{M \rightarrow \infty} \sup_{|w|=M} \mu(B_n(w)) < C < 1$$

for a certain constant C , so that we have to consider a word v_n of maximal length such that $\mu(B_n(v_n)) \geq C$.

The proof now is exactly as in Theorem 1.1 because there are no other places in the proof of Theorem 1.1 where the specific properties of the uniform current can be used. □

We now give a criterion for a current to satisfy the hypotheses of Theorem 5.2, formulated in terms of the corresponding frequency measure.

Theorem 5.3. *Let η be a geodesic current and μ be its corresponding frequency measure. Suppose that for each $a \in A$ there exist two strictly positive constants $C_1(a)$ and $C_2(a)$ such that for any $E \subset \partial F \setminus \text{Cyl}(a^{-1})$ we have*

$$C_1(a)\mu(E) \leq \mu(aE) \leq C_2(a)\mu(E).$$

For all freely reduced words $w = a_0 \dots a_k$, set $C_i(w) = C_i(a_0) \dots C_i(a_k)$, $i = 1, 2$.

If there is a constant M such that

$$\inf_{w \in F} \frac{C_1(w^{-1})}{C_2(w)} \geq \frac{1}{M}, \quad \sup_{w \in F} C_2(w) \leq M,$$

then the hypothesis of Theorem 5.2 is satisfied. Namely, there exists a constant $c > 0$ and for each $f \in F$ there is $b(f) > 0$ such that for any $E, S \subset \partial F$

$$\mu(fE) \geq b(f)\mu(E) \quad \text{and} \quad \eta(E \times S) \geq c\mu(E)\mu(S).$$

In particular, length compactness holds for η .

Proof. By Proposition A.1, it suffices to prove the claims when E and S are cylinders, and, regarding the first claim, it suffices to prove it for generators. Let $E \subset \partial F$ be a cylinder and $a \in A$. Set $E_0 = E \cap \text{Cyl}(a^{-1})$ and $E_1 = E \setminus E_0$. Then

$$\mu_A(aE_0) \geq \frac{1}{C_2(a^{-1})}\mu_A(E_0), \quad \mu_A(aE_1) \geq C_1(a)\mu_A(E_1),$$

and the first claim follows setting $b(a) = \inf\{C_1(A), \frac{1}{C_2(a^{-1})}\}$.

We now prove the second claim. Let $E, S \subset \partial F$ be two disjoint cylinders. Let

$$E = \text{Cyl}(v_0v), \quad S = \text{Cyl}(v_0w),$$

with $v_0v, v_0w, v^{-1}w$ freely reduced (that is, v_0 is the maximal initial segment shared by E and S). We set

$$\begin{aligned} E' &= \text{Cyl}(v) = v_0^{-1}E, & S' &= \text{Cyl}(w) = v_0^{-1}S, \\ E'' &= v^{-1}E', & S'' &= v^{-1}S' = \text{Cyl}(v^{-1}w). \end{aligned}$$

By F -invariance we have

$$\eta(E \times S) = \eta(E' \times S') = \eta(E'' \times S'') = \eta(\text{Cyl}([v^{-1}w])) = \mu(S'').$$

Then by induction on the lengths of v_0, v, w (and using the fact that $v_0v, v_0w, v^{-1}w$ are reduced) we get

$$\begin{aligned} \mu(S'') &\geq C\mu(S'')\mu(E'') \geq CC_1(v^{-1})\mu(S')\frac{1}{C_2(v)}\mu(E') \\ &= c\frac{C_1(v^{-1})}{C_2(v)}\mu(S')\mu(E') \geq C\frac{C_1(v^{-1})}{C_2(v)}\frac{\mu(S)}{C_2(v_0)}\frac{\mu(E)}{C_2(v_0)} \end{aligned}$$

where C is a suitable constant, and the claim follows by setting $c = C/M^3$. \square

APPENDIX A

Throughout the paper, we used some standard results about currents and measures. This section contains the proofs of some of these facts.

Proposition A.1. *Let m be a Borel measure on ∂F or $\partial^2 F$. Then, m is determined by its value on cylinders. Moreover, if $\{m_i\}$ is a sequence of Borel measures, then m_i converges to m if and only if for all cylinders C , $m_i(C) \rightarrow m(C)$.*

Proof. We restrict ourselves to the case where m is a Borel measure on ∂F ; an identical argument gives the $\partial^2 F$ -case. So assume m is a Borel measure on ∂F . The characteristic function of any cylinder belongs to $\mathcal{C}_0(\partial F)$, and the space V generated by the characteristic functions of cylinders is dense in $\mathcal{C}_0(\partial F)$ (the topology of $\mathcal{C}_0(\partial F)$ is given by uniform convergence). The first claim follows.

In addition, this implies that if m_i converges to m , then for any cylinder C , $m_i(C) \rightarrow m(C)$. On the other hand, suppose that $m_i(C) \rightarrow m(C)$ for all cylinders C . Then for any $\chi \in V$, $\int \chi dm_i \rightarrow \int \chi dm$. Therefore, for any $\varphi \in \mathcal{C}_0(\partial F)$, if $\{\chi_k\} \subset V$ is a sequence converging to φ , we have

$$\left| \int \varphi d(m_i - m) \right| \leq \left| \int |\varphi - \chi_k| dm_i \right| + \left| \int \chi_k d(m_i - m) \right| + \left| \int |\chi_k - \varphi| dm \right|$$

where the sum of the first and the last term is bounded by $\|\varphi - \chi_k\|(\|m_i\| + \|m\|)$, which goes to zero as $k \rightarrow \infty$, uniformly on i . The second term goes to zero for any k . \square

Lemma A.2. *For any $(x, y) \in \partial^2 F$ let $L(x, y)$ be the length of the maximal initial segment shared by x and y . Then we have*

$$\eta_A = 2k(2k-1)^{2L(x,y)-1} \mu_A \times \mu_A.$$

Proof. Let $D, E \subset \partial F$ be two disjoint cylinders. Since D and E are disjoint, there exist $v, w \in F$ such that $D = \text{Cyl}(v)$, $E = \text{Cyl}(w)$, and such that v is not the initial segment of w and vice versa. Let L be the length of the maximal initial segment shared by v and w (possibly $L = 0$). Now, let $D' = \text{Cyl}(v') \subset D$ and

$E' = \text{Cyl}(w') \subset E$ be two cylinders. We set $|v'| = L + a$ and $|w'| = L + b$. We have $D' \times E' = \text{Cyl}([v', w'])$ and, by definition (Definition 3.5)

$$\eta_A(\text{Cyl}([v', w'])) = \frac{1}{2k(2k-1)^{|(v')^{-1}w'| - 1}} = \frac{1}{2k(2k-1)^{a+b-1}}$$

which can be written as

$$\frac{2k(2k-1)^{2L-1}}{(2k(2k-1)^{L+a-1})(2k(2k-1)^{L+b-1})} = 2k(2k-1)^{2L-1} \mu_A(D') \mu_A(E)'.$$

So by Proposition A.1, the restriction of η_A to $D \times E$ is given by $2k(2k-1)^{2L-1} \mu_A \times \mu_A$. Since for each $(x, y) \in D \times E$ we have $L(x, y) = L$, we get that the restriction of η_A to $D \times E$ is given by

$$2k(2k-1)^{2L(x,y)-1} \mu_A \times \mu_A.$$

Since this holds for any D, E , the claim follows by Proposition A.1. \square

An immediate corollary of Lemma A.2 is the following.

Lemma A.3. *Let $E, D \subset \partial F$ be two Borel subsets of μ_A -positive measure. Then $\eta_A(E \times D) \geq \mu_A(E) \mu_A(D)$.*

Proof. Just apply the Fubini-Tonelli theorem, using Lemma A.2, and the fact that $L(x, y) \geq 0$ and that $\frac{2k}{2k-1} > 1$. \square

Lemma A.4. *Let E be a Borel subset of ∂F . Then for all $f \in F$*

$$\mu_A(fE) \geq \frac{\mu_A(E)}{(2k-1)^{|f|}}.$$

In particular, if E has μ_A -positive measure, then fE has μ_A -positive measure.

Proof. It suffices to prove the claim for $f \in A$. Let $E_0 = E \cap \text{Cyl}(f^{-1})$ and $E_1 = E \setminus E_0$. By definition of μ_A and Proposition A.1 we have

$$\mu_A(fE_0) = (2k-1)\mu_A(E_0), \quad \mu_A(fE_1) = \frac{\mu_A(E_1)}{2k-1},$$

and the claim follows. \square

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