Stefano FRANCA VIGLIA

Constructing equivariant maps for representations

<http://aif.cedram.org/item?id=AIF_2009__59_1_393_0>
CONSTRUCTING EQUIVARIANT MAPS FOR REPRESENTATIONS

by Stefano FRANCAVIGLIA (*)

Abstract. — We show that if $\Gamma$ is a discrete subgroup of the group of the isometries of $\mathbb{H}^k$, and if $\rho$ is a representation of $\Gamma$ into the group of the isometries of $\mathbb{H}^n$, then any $\rho$-equivariant map $F : \mathbb{H}^k \rightarrow \mathbb{H}^n$ extends to the boundary in a weak sense in the setting of Borel measures. As a consequence of this fact, we obtain an extension of a result of Besson, Courtois and Gallot about the existence of volume non-increasing, equivariant maps. Then, we show that the weak extension we obtain is actually a measurable $\rho$-equivariant map in the classical sense. We use this fact to obtain measurable versions of Cannon-Thurston-type results for equivariant Peano curves. For example, we prove that if $\Gamma$ is of divergence type and $\rho$ is non-elementary, then there exists a measurable map $D : \partial \mathbb{H}^k \rightarrow \partial \mathbb{H}^n$ conjugating the actions of $\Gamma$ and $\rho(\Gamma)$. Related applications are discussed.

1. Introduction

Let $\Gamma$ be a discrete subgroup of $\text{Isom}(\mathbb{H}^k)$ and let $\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^n)$ be a representation. Then, it is easy to construct a piecewise smooth map...
$D : \mathbb{H}^k \to \mathbb{H}^n$ which is $\rho$-equivariant, that is $D(\gamma x) = \rho(\gamma)D(x)$ for all $\gamma \in \Gamma$ and $x \in \mathbb{H}^k$, and the problem arises of whether such a map continuously extends to the boundaries of the hyperbolic spaces (this is a key step in the proofs of rigidity results for hyperbolic manifolds, see for example \cite{29, 3, 12, 13, 16}). In some cases, for example if the representation is not discrete, such an extension is not possible. Moreover, in general it is hard even to construct a $\rho$-equivariant map between the boundaries with some regularity properties like continuity and measurability. If $\Gamma < \text{Isom}(\mathbb{H}^2)$ is a surface group and if $\rho : \Gamma \to \text{Isom}(\mathbb{H}^3)$ is an isomorphism such that $\mathbb{H}^3/\rho(\Gamma)$ is an hyperbolic 3-manifold, under certain assumptions, Cannon and Thurston \cite{9} and Minsky \cite{20}, proved the existence of a continuous, $\rho$-equivariant, surjective map $\partial \mathbb{H}^2 \to \partial \mathbb{H}^3$, and Soma \cite{25} proved that, outside zero-measure sets, such a map is a homeomorphism (see also the recent works \cite{17, 18}).

The starting point of this paper is the existence, for any $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$, of a measurable, $\rho$-equivariant map from the limit set of $\Gamma$ to the set of probability measures on $\partial \mathbb{H}^n$, that can be stated as follows (see Section 2 for precise definitions).

**Theorem 1.1 (Existence of developing measures).** — Let $\Gamma < \text{Isom}(\mathbb{H}^k)$ be an infinite, non-elementary discrete group. Let $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ be a representation. Then, a family of developing measures for $\rho$ exists.

The fact that we work in the world of measure with the weak-* topology introduces a lot of compactness, so that we will able to establish results of existence and convergence for equivariant maps. The main applications of our technique belong to the framework of *Barycentric maps* and the one of *Cannon Thurston maps*.

**Barycentric maps.** — First of all, we get a generalisation of the celebrated Theorem of Besson, Courtois and Gallot on existence of natural maps.

**Theorem 1.2 (Existence of B-C-G-natural maps).** — Let $\Gamma < \text{Isom}(\mathbb{H}^k)$ be an infinite discrete group. Let $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ be a representation whose image is non-elementary. Then, there exists a map $F : \mathbb{H}^k \to \mathbb{H}^n$, called natural map, such that:

1. $F$ is smooth.
2. $F$ is $\rho$-equivariant, i.e., $F(\gamma x) = \rho(\gamma)F(x)$ for all $x \in \mathbb{H}^k$ and $\gamma \in \Gamma$.
3. For all $p \geq 3$, $\text{Jac}_p F(x) \leq \left( \frac{\delta(\Gamma)}{p-1} \right)^p$. 


(4) If \( \|d_x F(u_1) \wedge \cdots \wedge d_x F(u_p)\| = \left(\frac{\delta(\Gamma)}{p-1}\right)^p \) for an orthonormal \( p \)-frame \( u_1, \ldots, u_p \) at \( x \in \mathbb{H}^k \), then the restriction of \( d_x F \) to the subspace generated by \( u_1, \ldots, u_p \) is a homothety.

Theorem 1.2 was proved by Besson Courtois and Gallot in the special case in which \( \rho \) is discrete and faithful and both \( \Gamma \) and \( \rho(\Gamma) \) are convex co-compact ([6]).

In our setting, Theorem 1.2 will follow directly from Theorem 1.1 and a modification of the construction of Besson, Courtois and Gallot (see [4, 5, 6]).

In [6], for \( \varepsilon > 0 \) the authors construct a smooth \( \rho \)-equivariant map \( F_\varepsilon : \mathbb{H}^k \to \mathbb{H}^n \) such that for all \( p \geq 3 \), \( \text{Jac}_p F_\varepsilon(x) \leq \left(\frac{\delta(\Gamma)(1 + \varepsilon)}{p-1}\right)^p \). We call such maps \( \varepsilon \)-natural maps. We will see that the natural map we construct is actually the limit of a sequence of such \( \varepsilon \)-natural maps.

**THEOREM 1.3.** — In the hypotheses of Theorem 1.2, there exists a family \( \{F_\varepsilon\} \) of \( \varepsilon \)-natural maps (constructed as in [6]) and a sequence \( \varepsilon_i \to 0 \) such that \( F_{\varepsilon_i} \) converges to the natural map \( F \).

The proofs of Theorems 1.1 and 1.3 both start with a \( \rho \)-equivariant map \( D : \mathbb{H}^k \to \mathbb{H}^n \). While the natural map \( F \) does not depend on \( D \), the maps \( F_\varepsilon \)'s can be constructed in such a way to keep memory of \( D \). More precisely, the construction of the \( \varepsilon \)-natural maps depends on the choice of a probability measure on a fundamental domain for the action of \( \Gamma \), and they depends on the restriction of \( D \) to the support of such measure. This is useful to study non-compact manifolds. For example, suppose that \( f : M \to N \) is a proper map between complete non-compact hyperbolic manifolds. If \( D \) is the lift of \( f \) to the universal covers, then the natural maps \( F_\varepsilon \) descend to maps \( f_\varepsilon : M \to N \), and one can show that, if one used a suitable measure to construct such maps, then they are proper.

Such results can be used to prove rigidity results for representations. The following theorem (whose proof will be sketched in Section 6 and completely described in [14]) is an example of applications of Theorems 1.2 and 1.3.

**THEOREM 1.4 (Rigidity of representations).** — Let \( M \) be a complete hyperbolic \( k \)-manifold of finite volume. Let \( \rho : \pi_1(M) \to \text{Isom}(\mathbb{H}^n) \) be an irreducible representation. Then \( \text{vol}(\rho) \leq \text{vol}(M) \) and equality holds if and only if \( \rho \) is a discrete and faithful representation into the group of isometries of a \( k \)-dimensional hyperbolic subspace of \( \mathbb{H}^n \).
Measurable Cannon-Thurston maps. — A Cannon-Thurston map for $\rho$ is a continuous, $\rho$-equivariant map from the limit set of $\gamma$ to $\partial \mathbb{H}^n$. It is quite difficult to show the existence of such maps (and in general there are obstructions to continuity, and one needs to impose geometric constraints.) In fact, the existence problem has been solved for $k = 2$ and $n = 3$ with geometric hypotheses (see for instance [9, 19, 20, 17, 18]).

Theorem 1.1 is a weak existence result, and its proof use a kind of weak extension result; namely, the family of developing measures weakly extends the orbit of a point. The main result here, is the proof of a stronger measurable extension result, which will be the base for the study of measurable versions of the Cannon-Thurston map (the measures we consider on $\partial \mathbb{H}^k$ are the Patterson-Sullivan measures, see Section 2.) The strong extension theorem can be stated in his general form as follows.

**Theorem 1.5** (Existence and uniqueness of measurable extensions). — Let $\Gamma < \text{Isom}(\mathbb{H}^k)$ be a discrete group and let $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ be a representation whose image is non-elementary. Suppose that there exists a family $\{\lambda_z\}_{z \in \partial \mathbb{H}^k}$ of developing measures for $\rho$ such that for almost all $z$, the measure $\lambda_z$ is not the sum of two Dirac deltas with equal weights. Then, the natural map $F$ constructed using $\{\lambda_z\}$ extends to the conical limit set. More precisely, there exists a measurable, $\rho$-equivariant map $\overline{F} : \mathbb{H}^k \to \mathbb{H}^n$ which agrees with $F$ on $\mathbb{H}^k$ and such that, for almost all $\omega$ in the conical limit set, if $\xi \in \mathbb{H}^k$ and \{\gamma_n \xi\} is a sequence conically converging to $\omega$, then $\overline{F}(\gamma_n x) \to \overline{F}(\omega)$ for all $x \in \mathbb{H}^k$.

Moreover, the map $\overline{F}$ is unique. More precisely, if $\overline{F}_1$ and $\overline{F}_2$ are two measurable, $\rho$-equivariant maps from $\Lambda(\Gamma)$ to the set of probability measures on $\partial \mathbb{H}^n$, then $\overline{F}_1$ and $\overline{F}_2$ are in fact ordinary functions, that is to say, they map almost every point of $\Lambda(\Gamma)$ to a Dirac delta concentrated on a point of $\partial \mathbb{H}^n$. Moreover, they agree almost everywhere on $\partial \mathbb{H}^k$.

Theorem 1.5 in particular applies to the case of fundamental groups of hyperbolic manifolds. Indeed, we will prove that in this case the developing measures are almost never the sum of two Dirac deltas with equal weights. In particular, this gives the following theorem.

**Theorem 1.6** (Existence and uniqueness of Cannon-Thurston maps). Let $\mathbb{H}^k/\Gamma$ be a complete hyperbolic manifold of finite volume and let $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ be a representation whose image is not elementary. Then there exists a measurable $\rho$-equivariant map $\overline{F} : \partial \mathbb{H}^k \to \partial \mathbb{H}^n$ Such a map is the extension of a natural map, and two such maps are equal almost everywhere.
In particular when a classical Cannon-Thurston map exists, it coincides with the extension provided by Theorem 1.6. An immediate consequence of these facts is the following result.

**Theorem 1.7** (Inverse of Cannon-Thurston maps). — Let $\Gamma < \text{Isom}(\mathbb{H}^k)$ and $\Gamma' < \text{Isom}(\mathbb{H}^n)$ be non-elementary discrete groups such that they diverge respectively at $\delta(\Gamma)$ and $\delta(\Gamma')$. Let $\rho: \Gamma \to \Gamma'$ be an isomorphism.

Then, there exist measurable maps $\overline{F}: \partial \mathbb{H}^k \to \partial \mathbb{H}^n$ and $\overline{G}: \partial \mathbb{H}^n \to \partial \mathbb{H}^k$ which are respectively $\rho$ and $\rho^{-1}$ equivariant. Moreover, almost everywhere $\overline{F} \circ \overline{G} = \text{Id}_{\mathbb{H}^k}, \overline{G} \circ \overline{F} = \text{Id}_{\mathbb{H}^n}$.

As noticed above, working with measures helps when one has to deal with convergence problems. For example, an application of the above techniques provides an answer to the following question.

Suppose that $\rho_i \to \rho$ is a converging sequence of non-elementary representations. Do the corresponding Cannon-Thurston maps converge?

Miyachi proved [21] that if $\Gamma$ is a surface-group without parabolics, and if the injectivity radius is bounded away from zero along the whole sequence, then the answer is "yes" (in fact, one has uniform convergence,) and it is conjectured that the same holds in the case of cusped surfaces with a uniform bound on the injectivity radius outside the cusps (see for example [24, Conjecture 5.2].) In general, as Example 9.1 shows, one can construct sequences having no uniformly converging sub-sequence (the condition on injectivity radii is violated.) We prove here that in the general case (no geometric hypotheses, no bounds on dimensions) the answer is "yes" for the point-wise convergence almost everywhere.

**Theorem 1.8** (Convergence of Cannon-Thurston maps). — Let $\Gamma < \text{Isom}(\mathbb{H}^k)$ be a discrete group that diverges at its critical exponent $\delta(\Gamma)$. Let $\rho_i : \Gamma \to \text{Isom}(\mathbb{H}^n)$ be a sequence of representations with non-elementary images. Suppose that $\rho_i$ converges to a representation $\rho$ whose image is non-elementary. Let $f_i$ and $f$ be the corresponding measurable Cannon-Thurston maps for $\rho_i$ and $\rho$ respectively. Then, $f_i$ converges to $f$ almost everywhere with respect to the Patterson-Sullivan measures on the limit set of $\Gamma$.

**Acknowledgements.** I need to thank Jeff Brock, Ben Klaff, Carlo Mantegazza, Joan Porti, Juan Souto and Xavi Tolsa for their fundamental help.

2. Definitions, notation and preliminary facts

First of all, we recall some definitions.
**Definition 2.1.** — A subgroup $\Gamma \subset \text{Isom}(\mathbb{H}^k)$ is said non-elementary if any $\Gamma$-invariant, non-empty set $A \subset \partial \mathbb{H}^k$ contains at least three points. Otherwise, $\Gamma$ is said elementary.

**Definition 2.2.** — Let $F : \mathbb{H}^k \to \mathbb{H}^n$ be a smooth map. The $p$-Jacobian $\text{Jac}_p F$ of $F$ is defined by

$$\text{Jac}_p F(x) = \sup \| d_x F(u_1) \wedge \ldots \wedge d_x F(u_p) \|_{\mathbb{H}^n},$$

where $\{u_i\}_{i=1}^p$ varies on the set of orthonormal $p$-frames at $x \in \mathbb{H}^k$.

**Definition 2.3.** — Let $(X,g)$ be a complete Riemannian manifold and let $\Gamma$ be a group of isometries of $X$. We denote by $\delta(\Gamma)$ the critical exponent of the Poincaré series of $\Gamma$, that is:

$$\delta(\Gamma) = \inf \{ s > 0 : \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x)} < +\infty \}$$

where $d(\cdot, \cdot)$ denotes the distance induced by $g$ on $X$ and $x$ is a point of $X$.

It is readily checked that $\delta(\Gamma)$ does not depend on $x$. We notice that, when $X = \mathbb{H}^k$, the critical exponent is the Hausdorff dimension of the conical limit set of $\Gamma$ ([7]). Moreover, $\delta(\Gamma)$ can be computed by

$$\delta(\Gamma) = \lim_{R \to \infty} \frac{1}{R} \log(\# \{ \gamma \in \Gamma : d(\gamma O, O) < R \}).$$

We refer to [22, 28, 30] for details.

**Definition 2.4.** — Let $(X,g)$ be a complete Riemannian manifold and let $\Gamma$ be a group of isometries of $X$. We say that $\Gamma$ diverges at $\delta(\Gamma)$ if

$$\lim_{s \to \delta(\Gamma)^+} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x)} = +\infty.$$

The following lemma is an immediate consequence of [22, Theorem 1.6.1], [22, Theorem 1.6.3] and [22, Corollary 3.4.5].

**Lemma 2.5.** — Let $\Gamma$ be a infinite, non-elementary discrete subgroup of $\text{Isom}(\mathbb{H}^k)$, then

$$0 < \delta(\Gamma) \leq k - 1.$$

Moreover, if $\mathbb{H}^k/\Gamma$ has finite volume, then $\delta(\Gamma) = k - 1$ and $\Gamma$ diverges at $\delta(\Gamma)$.

**Notation.** — For the rest of the paper we fix the following notation. $\Gamma$ will be an infinite discrete subgroup of $\text{Isom}(\mathbb{H}^k)$, and

$$\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$$
will be a representation. We fix base-points in \( \mathbb{H}^k \) and \( \mathbb{H}^n \), both denoted by \( O \). We denote by \( B_K \) and \( B_N \) the Busemann functions, respectively of \( \mathbb{H}^k \) and \( \mathbb{H}^n \), normalised at \( O \). Namely, for \( x \in \mathbb{H}^k \) and \( \theta \in \partial \mathbb{H}^k \) (resp. \( \mathbb{H}^n \) and \( \partial \mathbb{H}^n \)) we set

\[
B_K(x, \theta) = \lim_{t \to \infty} \left( d_{\mathbb{H}^k}(x, \gamma_\theta(t)) - t \right),
\]

where \( \gamma_\theta \) is the geodesic ray from \( O \) to \( \theta \), parametrised by arc length. We denote by \( \pi_K \) (respectively \( \pi_N \)) the projection of \( \mathbb{H}^k \times \mathbb{H}^n \) to \( \mathbb{H}^k \) (resp. \( \mathbb{H}^n \)):

\[
\pi_K : \mathbb{H}^k \times \mathbb{H}^n \to \mathbb{H}^k.
\]

Finally, we fix a continuous piecewise smooth \( \rho \)-equivariant map

\[
D : \mathbb{H}^k \to \mathbb{H}^n.
\]

We notice that such a map can be easily constructed by triangulating a fundamental domain for \( \Gamma \) and then arguing by induction on the \( i \)-skeleta.

**Patterson-Sullivan measures.** — A fundamental tool for our purpose is the family of Patterson-Sullivan measures. We recall the main results we need, which we summarise in Theorem 2.6, referring to [6], [22, Chap. 3, 4] and [30, 31] for proofs and details.

**Theorem 2.6.** — Let \( \Gamma \) be an infinite, non-elementary discrete subgroup of Isom(\( \mathbb{H}^k \)) with critical exponent \( \delta(\Gamma) \). For all \( x \in \mathbb{H}^k \) there exists a positive Borel measure \( \mu_x \) of finite, non-zero mass such that, for all \( x, y \in \mathbb{H}^k \) and \( \gamma \in \Gamma \):

1. The measures \( \mu_x \) and \( \mu_y \) are in the same density class of measures and are concentrated on \( \partial \mathbb{H}^k \).
2. The measure \( \mu_x \) satisfies
   \[
   d\mu_x(\theta) = e^{-\delta(\Gamma)B_K(x, \theta)} d\mu_O(\theta)
   \]
   where \( \theta \in \partial \mathbb{H}^k \).
3. The measures \( \mu_x \) are \( \Gamma \)-equivariant, that is
   \[
   \mu_{\gamma x} = \gamma_* \mu_x.
   \]

**Proof.** — We only sketch the proof, recalling the construction of the Patterson-Sullivan measures because we will explicitly use it in the following.

For all \( s > \delta(\Gamma) \) let

\[
c(s) = \sum_{\gamma \in \Gamma} e^{-sd(O, \gamma O)}
\]
where \( d(\cdot, \cdot) \) denotes the hyperbolic distance of \( \mathbb{H}^k \). For simplicity, here we stick to the case that \( \Gamma \) diverges at \( \delta(\Gamma) \) (this happens for example if \( \Gamma \) is geometrically finite, see [22, p. 87]). For any \( x \in \mathbb{H}^k \) and \( s > \delta(\Gamma) \), define

\[
\mu_x^s = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma O)} \delta_{\gamma O}
\]

where \( \delta_{\gamma O} \) is the Dirac measure concentrated on \( \gamma O \). For \( s > \delta(\Gamma) \), \( \mu_x^s \) is a well-defined positive Borel measure on \( \mathbb{H}^k \subset \mathbb{H}^k \). It can be shown that for \( s \to \delta(\Gamma)^+ \), the measures \( \mu_x^s \) weakly converge to a positive Borel measure \( \mu_x \) on \( \mathbb{H}^k \). More precisely, for all \( x \in \mathbb{H}^k \) and \( \varphi \in C(\mathbb{H}^k) \)

\[
\int_{\mathbb{H}^k} \varphi \, d\mu_x^s \to \int_{\mathbb{H}^k} \varphi \, d\mu_x.
\]

Moreover, the fact that \( \lim_{s \to \delta(\Gamma)^+} c(s) = +\infty \) easily implies that \( \mu_x \) is concentrated on the boundary (in fact \( \mu_x \) is concentrated on the limit set of \( \Gamma \)).

**Barycentre of a measure.** — We recall now the definition and the main properties of the barycentre of a measure. We refer the reader to [11] and [4] for complete proofs and details.

Let \( \beta \) be a positive Borel measure on \( \partial \mathbb{H}^n \) of finite mass. Define the function \( B_\beta : \mathbb{H}^n \to \mathbb{R} \) by

\[
B_\beta : y \mapsto \int_{\partial \mathbb{H}^n} B_N(y, \theta) \, d\beta(\theta).
\]

Since we are working in the hyperbolic space, the Busemann functions are convex. Thus, its \( \beta \)-average \( B_\beta \) is strictly convex, provided that \( \beta \) is not the sum of two deltas. Moreover, one can show that \( B_\beta(y) \to \infty \) as \( y \) approaches \( \partial \mathbb{H}^n \). It follows that \( B \) has a unique minimum in \( \mathbb{H}^n \).

**Definition 2.7.** — For any positive Borel measure \( \beta \) on \( \partial \mathbb{H}^n \) of finite mass which is not concentrated on two points, we define the barycentre \( \text{bar}(\beta) \) of \( \beta \) as the unique minimum point of the function \( y \mapsto \int_{\partial \mathbb{H}^n} B_N(y, \theta) \, d\beta(\theta) \).

We refer to [11], [4] and [5] for a proof of the following lemma.

**Lemma 2.8.** — The barycentre of a measure \( \beta \) satisfies the following properties:

1. The barycentre is characterised by the equation

\[
\int_{\partial \mathbb{H}^n} dB_N(\text{bar}(\beta), \theta)(\cdot) \, d\beta(\theta) = 0.
\]
The barycentre is Isom($\mathbb{H}^n$)-equivariant, that is, for any $g \in \text{Isom}(\mathbb{H}^n)$
\[ \text{bar}(g_\ast \beta) = g(\text{bar}(\beta)). \]

The barycentre is continuous respect the weak convergence of measures. That is, if $\beta_i \rightharpoonup \beta$, then $\text{bar}(\beta_i) \to \text{bar}(\beta)$.

The first property follows from the definition after differentiating the function $B_\beta$. The equivariance follows from the properties of the Busemann functions. The continuity can be easily proved using that, if $\beta_i \rightharpoonup \beta$, then $B_{\beta_i}$ and $dB_{\beta_i}$ point-wise converge to $B_\beta$ and $dB_\beta$ respectively.

Remark 2.9. — If $\beta = a\delta_{\theta_1} + b\delta_{\theta_2}$, with $0 < a < b$, then it can be checked that the minimum of $B_\beta$ is the point $\theta_2 \in \partial \mathbb{H}^n$. Thus, one can define the barycentre of a measure $\beta$ whenever $\beta$ is not the sum of two deltas with the same weights. Note that, since the barycentre of a measure concentrated on two points belongs to $\partial \mathbb{H}^n$, equation 1 of Lemma 2.8 makes no sense for such measures.

Developing measures. — We now introduce the notion of family of developing measures for $\rho$, which extends the one of $\rho$-equivariant map. We recall that $\{\mu_x\}$ is the family of Patterson-Sullivan measures.

Definition 2.10. — A family of developing measures for $\rho$ is a set $\{\lambda_z\}_{z \in \partial \mathbb{H}^k}$ of positive Borel measures on $\partial \mathbb{H}^n$, of finite mass, and such that:

1. The measures $\lambda_z$’s are $\rho$-equivariant, that is, for $\mu_O$-almost all $z$ and all $\gamma \in \Gamma$
\[ \lambda_{\gamma z} = \rho(\gamma)_\ast \lambda_z \]

2. For any $\varphi \in C(\partial \mathbb{H}^n)$, the function
\[ z \mapsto \int_{\partial \mathbb{H}^n} \varphi(\theta) \, d\lambda_z(\theta) \]

is $\mu_O$-integrable (whence, by points 1 and 2 of Theorem 2.6, it is $\mu_x$-integrable for all $x$).

3. The function $z \mapsto \|\lambda_z\|$ belongs to $L^\infty(\partial \mathbb{H}^k, \mu_O)$.

4. For $\mu_O$-almost $z \in \partial \mathbb{H}^k$, $\|\lambda_z\| > 0$.

As an example, consider a $\mu_O$-measurable $\rho$-equivariant map $D : \partial \mathbb{H}^k \to \partial \mathbb{H}^n$. Then the family $\{\lambda_z = \delta_{D(z)}\}$, where $\delta_{D(z)}$ is the Dirac measure, is a family of developing measures for $\rho$. In this sense the notion of developing measures extends the one of equivariant map.
CONVOLUTIONS OF MEASURES. — Let $X, Y$ be topological spaces and let $\mu$ be a Borel measure on $X$. Let $\{\alpha_x\}_{x \in X}$ be a family of Borel measures on $Y$ such that for each $\varphi \in C_0(Y)$ the function $x \mapsto \int_Y \varphi \, d\alpha_x$ is $\mu$-integrable. The convolution $\mu * \{\alpha_x\}$ is the Borel measure on $Y$ defined by

$$
\int_Y \varphi(y) \, d(\mu * \{\alpha_x\}) = \int_X \left( \int_Y \varphi(y) \, d\alpha_x(y) \right) \, d\mu(x)
$$

for any $\varphi \in C_0(Y)$. Similarly, we define the product $\mu \times \{\alpha_x\}$ on $X \times Y$ by

$$
\int_Y \varphi(x, y) \, d(\mu \times \{\alpha_x\}) = \int_X \left( \int_Y \varphi(x, y) \, d\alpha_x(y) \right) \, d\mu(x).
$$

The measure $\mu * \{\alpha_x\}$ is the $\mu$-average of the $\alpha_x$’s. Moreover, if $\pi : X \times Y \to Y$ is the projection, then $\mu * \{\alpha_x\} = \pi_* (\mu \times \{\alpha_x\})$.

We say that a sequence of measures $\{\mu_i\}$ weakly converges to $\mu$ if, for any continuous function $f$ with compact support, $\int f \, d\mu_i \to \int f \, d\mu$. The proof of following lemmas are left to the reader.

**Lemma 2.11.** — Suppose that $\{\mu_i\}$ is a sequence of measures on $X$, weakly converging to $\mu$. If for each $\varphi \in C_0(Y)$ the function $x \mapsto \int_Y \varphi \, d\alpha_x$ belongs to $C_0(X)$, then the sequence $\mu_i * \{\alpha_x\}$ weakly converges to $\mu * \{\alpha_x\}$.

**Lemma 2.12.** — Let $Z$ be a topological space and let $\{\nu_y\}_{y \in Y}$ be a family of Borel measures on $Z$ such that for all $\psi \in C_0(z)$ the function $y \mapsto \int_Z \psi \, d\nu_y$ is $\alpha_x$-integrable for $\mu$-almost all $x$ and $x \mapsto \int_Y \int_Z \psi \, d\nu_y \, d\alpha_x$ is $\mu$-measurable. Then

$$
\mu * \{\alpha_x \ast \{\nu_y\}\} = (\mu * \{\alpha_x\}) \ast \{\nu_y\}.
$$

By Lemma 2.12 we can omit the parentheses and write $\mu * \{\alpha_x\} \ast \{\nu_y\}$.

**Some remarks on the hypotheses of Theorems 1.1, 1.2 and 3.1.** — First of all, we notice that the hypotheses of such theorems can be relaxed by replacing the spaces $\mathbb{H}^k$ and $\mathbb{H}^n$ with Riemannian manifolds with suitable bounds on the curvatures. We refer the reader to [6] for further details on that direction.

Moreover, even if the hypothesis that the image of $\rho$ is non-elementary is crucially used in Corollary 3.5, it is not strictly necessary. More precisely, suppose that $\rho(\Gamma)$ is elementary. Then there exists a $\rho(\Gamma)$-invariant set $A \subset \partial \mathbb{H}^n$ with either one or two points.

In the latter case, there exists a whole geodesic $c$ in $\mathbb{H}^n$ which is $\rho(\Gamma)$-invariant and it is easy to see that a natural map whose image is contained in $c$ exists. Therefore, Theorem 1.2 is true in this case.
In the former case, we can suppose that, in the half-space model \( \mathbb{H}^{n-1} \times \mathbb{R}^+ \) of \( \mathbb{H}^n \), the point \( \infty \) is fixed by \( \rho(\Gamma) \). Then, one can easily construct a \( \rho \)-equivariant map \( D \) whose image is contained in the horosphere \( \{(z,1) : z \in \mathbb{R}^{n-1}\} \). Thus, if \( \text{Jac}_D(x) \) is bounded, then Theorem 1.2 is proved by raising \( D \) to a sufficiently high horosphere. It follows that, if \( \mathbb{H}^k/\Gamma \) is compact, or simply if \( \mathbb{H}^k/\Gamma \) retracts to a compact set (and this is the case if for example \( \Gamma \) is geometrically finite), then Theorem 1.2 is true even if \( \rho(\Gamma) \) has a fixed point.

3. Construction of B-C-G natural maps

In this section we prove the following Theorem, which, together with Theorem 1.1 (proved in Section 4), directly implies Theorem 1.2. For this section we keep the notation fixed in Section 2.

**Theorem 3.1.** — Let \( \Gamma < \text{Isom}(\mathbb{H}^k) \) be an infinite discrete group. Let \( \rho : \Gamma \to \text{Isom}(\mathbb{H}^n) \) be a representation whose image is non-elementary. If there exists a family \( \{\lambda_z\} \) of developing measures for \( \rho \), then a natural map exists.

**Proof.** — Fix a family
\[
\{\lambda_z\}_{z \in \partial \mathbb{H}^k}
\]
of developing measures for \( \rho \). The idea of the proof is to use the developing measures to push-forward the Patterson-Sullivan measures \( \mu_x \)'s to measures \( \beta_x \)'s on \( \partial \mathbb{H}^n \), and define the natural map by
\[
x \mapsto \mu_x \mapsto \beta_x \mapsto \text{bar}(\beta_x).
\]
Then, the properties of the natural map will follow as in [6].

The push-forward the measures \( \mu_x \)'s is defined as follows. For each \( x \in \mathbb{H}^k \), define \( \beta_x \) as the positive Borel measure on \( \partial \mathbb{H}^n \) given by \( \beta_x = \mu_x \times \{\lambda_z\} \). Namely, for all \( \varphi \in C(\partial \mathbb{H}^n) \)
\[
\int_{\partial \mathbb{H}^n} \varphi(y) \, d\beta_x(y) = \int_{\partial \mathbb{H}^k} \left( \int_{\partial \mathbb{H}^n} \varphi(y) \, d\lambda_z(y) \right) \, d\mu_x(z)
\]
\[
= \int_{\partial \mathbb{H}^k} \left( \int_{\partial \mathbb{H}^n} \varphi(y) \, d\lambda_z(y) \right) e^{-\delta(\Gamma) B_K(x,z)} \, d\mu_O(z).
\]

Note that the measure \( \beta_x \) is well-defined and has finite mass because of conditions (2) and (3) of Definition 2.10. Moreover, since \( \beta_x = \pi_N \ast (\mu_x \times \{\lambda_z\}) \), if the family \( \{\lambda_z\} \) is of the form \( \{\delta_{D(z)}\} \) for a \( \mu_O \)-measurable function \( D : \partial \mathbb{H}^k \to \partial \mathbb{H}^n \), then \( \beta_x = \text{bar}(\beta_x) \).

□
Lemma 3.2. — The family of measures \( \{ \beta_x \}_{x \in \mathbb{H}^k} \) is \( \rho \)-equivariant, that is to say, for all \( x \in \mathbb{H}^k \) and \( \gamma \in \Gamma \)

\[ \beta_{\gamma x} = \rho(\gamma)_* \beta_x. \]

Proof. — For any \( \varphi \in C(\partial \mathbb{H}^n) \)

\[
\int_{\partial \mathbb{H}^n} \varphi(y) d\beta_{\gamma x}(y) = \int_{\partial \mathbb{H}^k} \left( \int_{\partial \mathbb{H}^n} \varphi(y) d\lambda_z(y) \right) d\mu_{\gamma x}(z)
\]

\[
= \int_{\partial \mathbb{H}^k} \left( \int_{\partial \mathbb{H}^n} \varphi(y) d\lambda_{\gamma z}(y) \right) d\mu_x(z)
\]

\[
= \int_{\partial \mathbb{H}^k} \left( \int_{\partial \mathbb{H}^n} \varphi(\rho(\gamma)y) d\lambda_z(y) \right) d\mu_x(z) = \int_{\partial \mathbb{H}^n} \varphi(y) d(\rho(\gamma)_* \beta_x)(y).
\]

\[]

Lemma 3.3. — For all \( x, y \in \mathbb{H}^k \), the measures \( \beta_x \) and \( \beta_y \) are in the same density class of measures.

Proof. — We have to show that for all positive functions \( \varphi \in C(\partial \mathbb{H}^n) \)

we have

\[
\int_{\partial \mathbb{H}^n} \varphi d\beta_x = 0 \iff \int_{\partial \mathbb{H}^n} \varphi d\beta_y = 0.
\]

This follows from the fact that \( \mu_x \) and \( \mu_y \) are in the same density class. Indeed, if \( \Phi \) denotes the function \( z \mapsto \int_{\partial \mathbb{H}^n} \varphi d\lambda_z \), since the developing measures are positive, \( \Phi \) is positive, and

\[
\int_{\partial \mathbb{H}^n} \varphi d\beta_x = \int_{\partial \mathbb{H}^n} \Phi d\mu_x = 0 \iff 0 = \int_{\partial \mathbb{H}^n} \Phi d\mu_y = \int_{\partial \mathbb{H}^n} \varphi d\beta_y.
\]

\[]

Lemma 3.4. — For all \( x \in \mathbb{H}^k \), \( ||\beta_x|| > 0 \).

Proof. — It follows from condition (4) of Definition 2.10 and from the fact that \( ||\mu_x|| > 0 \).

\]

Corollary 3.5. — For all \( x \in \mathbb{H}^k \), the measure \( \beta_x \) is not concentrated on two points.

Proof. — By Lemma 3.4, \( \beta_x \) is not the zero-measure. Suppose that \( \beta_x \) has an atom of positive weight at \( y_0 \in \partial \mathbb{H}^n \). By Lemmas 3.2 and 3.3, \( \beta_x \) has an atom of positive weight at each point of the \( \rho(\Gamma) \)-orbit of \( y_0 \), which contains most than two points because \( \rho(\Gamma) \) is non-elementary.

It follows that for all \( x \in \mathbb{H}^k \), the barycentre of the measure \( \beta_x \) is well-defined and belongs to \( \mathbb{H}^n \). We define the natural map \( F : \mathbb{H}^k \to \mathbb{H}^n \) by

\[ F(x) = \text{bar}(\beta_x). \]
By condition (1) of Lemma 2.8, the natural map is characterised by the implicit equation

\[(3.1) \quad G(x, F(x)) = 0\]

where

\[G(x, \xi) = \int_{\partial \mathbb{H}^n} dB_N(\xi, \theta)(\cdot) d\beta_x(\theta)\]

\[= \int_{\partial \mathbb{H}^k} \int_{\partial \mathbb{H}^n} dB_N(\xi, \theta)(\cdot) d\lambda_z(\theta) e^{-\delta(\Gamma)} B_K(x, z) d\mu_O(z).\]

The function \(G\) is smooth because the Busemann functions \(B_K\) and \(B_N\) are smooth. Then, by the implicit function theorem, we get that \(F\) is smooth. Moreover, by Lemma 3.2 and claim (2) of Lemma 2.8, it follows that \(F\) is \(\rho\)-equivariant.

By differentiating equation (3.1) we get that for all \(u \in T_x \mathbb{H}^k\) and \(v \in T_{F(x)} \mathbb{H}^n\)

\[(3.2) \quad \int_{\partial \mathbb{H}^k} \int_{\partial \mathbb{H}^n} DdB_N|_{(F(x), \theta)}(dF_x(u), v) d\lambda_z d\mu_x\]

\[= \delta(\Gamma) \int_{\partial \mathbb{H}^k} \int_{\partial \mathbb{H}^n} dB_N(F(x), \theta)(v) dB_K(x, z)(u) d\lambda_z d\mu_x.\]

Equation (3.2) is the analogous of equation (2.3) of [6]. The proof of the properties of the natural map now goes exactly as in [6, p. 152-154], and the proof of Theorem 3.1 is complete. \(\Box\)

An immediate corollary of Theorem 1.2 is the following fact.

**Corollary 3.6.** — Let \(X, Y\) be two compact hyperbolic manifold of (possibly different) dimension at least two. Then in each homotopy class of maps \(f : Y \to X\) there exists a smooth map \(F : Y \to X\) such that \(|\text{Jac}(F)| \leq 1\). Moreover, if \(|\text{Jac} F(y)| = 1\), then \(d_y F\) is an isometry.

**Proof.** — By Theorem 1.2, any \(f : Y \to X\) is homotopic to the natural map corresponding to the representation \(f_*\). Such a map has the requested properties because, by Lemma 2.5, we have \(\delta(\pi_1(Y)) \leq \dim(Y)\). \(\Box\)

As Theorem 1.2, Corollary 3.6 was proved by Besson, Courtois and Gallot in the special case in which \(\rho\) is discrete and faithful and both \(\Gamma\) and \(\rho(\Gamma)\) are convex co-compact ([6].)
4. Weak extension of equivariant maps: existence of developing measures

In this section we prove Theorem 1.1. In particular, to obtain a family of developing measures, we show that any equivariant map weakly extend to the boundary in the setting of Borel measures. We keep here the notation fixed in Section 2, included the one of Theorem 2.6 for the Patterson-Sullivan measures.

The rough idea is the following. Consider the graphic $G(D)$ of $D$ as a subspace of $\mathbb{H}^k \times \mathbb{H}^n$. Then, for each measure $\mu$ on $\mathbb{H}^k$ we can consider the graphic measure $\eta$ on $\mathbb{H}^k \times \mathbb{H}^n$, that is, the only measure which is concentrated on $G(D)$ and whose push-forward $\pi_K \ast \eta$ is $\mu$. The sequence $\{\eta_s^x\}$ of the graphic measures corresponding to the measures $\{\mu_s^x\}$, weakly converges to a measure $\eta_x$ concentrated on $\partial \mathbb{H}^k \times \mathbb{H}^n$. Then, by disintegrating the measure $\eta_x$ we obtain a family $\{\alpha_z\}_{z \in \partial \mathbb{H}^k}$ of measures on $\mathbb{H}^n$.

By making the convolution with the family of visual measures of $\partial \mathbb{H}^n$, we get a family of developing measures.

For each $x \in \mathbb{H}^k$ and $s > \delta(\Gamma)$ we define a positive Borel measure of finite mass on $\mathbb{H}^k \times \mathbb{H}^n$ as follows:

$$\eta_s^x = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma O)} \delta_{(\gamma O,D(\gamma O))} = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma O)} \delta_{(\gamma O,\rho(\gamma)D(O))}$$

where $\delta_{(x,y)}$ denotes the Dirac measure concentrated on $(x,y) \in \mathbb{H}^k \times \mathbb{H}^n$.

The measures $\{\eta_s^x\}$ are the graphic measures associated to $\{\mu_s^x\}$ and $D$, and they are concentrated on $\mathbb{H}^k \times \mathbb{H}^n$. Note that $\eta_s^x = \mu_s^x \times \{\delta_D(z)\}$ (see Section 2.)

The group $\text{Isom}(\mathbb{H}^k) \times \text{Isom}(\mathbb{H}^n)$ acts on $\mathbb{H}^k \times \mathbb{H}^n$ by

$$(g_1,g_2)(x,y) = (g_1x, g_2y).$$

**Lemma 4.1.** — For all $s > \delta(\Gamma)$ the family $\{\eta_s^x\}$ is $\rho$-equivariant, that is, for all $x \in \mathbb{H}^k$ and $\psi \in \Gamma$

$$\eta_{\psi \ast x}^s = (\psi, \rho(\psi)) \ast \eta_s^x.$$
Proof.

$$\eta^s_{\psi,x} = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(\psi x, \gamma O)} \delta_{(\gamma O, D(\gamma O))} = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x, \psi^{-1} \gamma O)} \delta_{(\gamma O, D(\gamma O))}$$

$$= \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma O)} \delta_{(\psi \gamma O, D(\psi \gamma O))}$$

$$= \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma O)} \delta_{(\psi \gamma O, \rho(\psi)(D(\gamma O)))} = (\psi, \rho(\psi)) \ast \eta^s_{\psi,x}.$$
function (defined for \( \mu \)-almost all \( z \))
\[
\psi_{j,\eta}^z = \frac{\chi_{U_j(z)}}{\mu(U_j(z))}
\]
where \( \chi_A \) denotes the characteristic function of the set \( A \) (note that \( \psi_{j,\eta}^z \to \delta_z \) for \( \mu \)-almost all \( z \)). Then, for \( \mu \)-almost all \( z \) and for all \( \varphi \in C(\mathbb{H}^n) \)
\[
\int_{\mathbb{H}^n} \varphi(y) d\alpha^z_n(y) = \lim_{j \to \infty} \int_{\mathbb{H}^k \times \mathbb{H}^n} \psi_{j,\eta}^z(x)\varphi(y) d\eta(x,y).
\]

From now on, the set \( \{\alpha_z\}_{z \in \mathbb{H}^k} \) will denote the family of measures that disintegrates \( \eta_0 \), where we set \( \alpha_z = 0 \) for \( z \in \mathbb{H}^k \). It seems worth mentioning that the choice of \( \alpha_z \) for \( z \) in the interior of \( \mathbb{H}^k \) is not relevant for our purposes. Indeed, since any \( \mu_x \) is concentrated on the limit set of \( \Gamma \), which is contained in the boundary at infinity of \( \mathbb{H}^k \), we have \( \mu_x(\mathbb{H}^k) = 0 \). In particular, the value of \( \alpha_z \) for \( z \) in the interior of \( \mathbb{H}^k \) does not affect the product measure \( \mu_x \times \{\alpha_z\} \).

**Definition 4.2.** — For each \( x \in \mathbb{H}^k \) we set \( \eta_x = \mu_x \times \{\alpha_z\} \).

Now, we have two problems. First, while it is true that any \( \{\eta^s_x\} \) has a converging sub-sequence, it is not clear a priori that the same sub-sequence works for all \( x \). Second, by definition, the family \( \{\alpha_z\} \) disintegrates the limit measure \( \eta_0 \), and there is no reasons for such a family to disintegrate the limits of \( \eta^s_x \) for all \( x \). In other words, it is not clear a priori that \( \eta^s_x \) converges to \( \eta_x \). The following theorem settles both questions. This is the core of the proof of Theorem 1.1, as it implies that the family \( \{\alpha_z\} \) is equivariant (see Lemma 4.4 below.)

**Theorem 4.3.** — For all \( x \in \mathbb{H}^k \), the sequence \( \{\eta^s_x\} \) weakly converges to \( \eta_x \). Where \( \{s_i\} \) is the sequence of indices such that \( \eta^s_0 \to \eta_0 \).

**Proof.** — We fix \( x \in \mathbb{H}^k \). Since the measures \( \eta^s_x \) are bounded in norm, up to pass to sub-sequences, we can suppose that \( \{\eta^s_x\} \) weakly converges to a measure \( \tilde{\eta}_x \). We prove now that any such possible limit \( \tilde{\eta}_x \) coincides with \( \eta_x \), and this will prove the thesis. Let \( \{\tilde{\alpha}_z\} \) be the family that disintegrates \( \tilde{\eta}_x \). By weak continuity of push forward
\[
\pi_{K*}(\tilde{\eta}_x) = \mu_x = \pi_{K*}(\eta_x).
\]

Therefore, it is sufficient to show that \( \tilde{\alpha}_z = \alpha_z \) for \( \mu_O \)-almost all \( z \) (recall that since \( \mu_x \) and \( \mu_O \) are in the same density-class, the notions of \( \mu_O \)-negligible set and \( \mu_x \)-negligible set coincide.) For any positive function \( \varphi \in C(\mathbb{H}^n) \)
\[ C(\mathbb{H}^n) \text{ and } z \in \partial \mathbb{H}^k \]

\[
\int_{\mathbb{H}^n} \varphi(y) \, d\tilde{\alpha}_z(y) = \lim_{j \to \infty} \int_{\mathbb{H}^k \times \mathbb{H}^n} \psi_j^z(\tilde{\eta}_z) \varphi(y) \, d\tilde{\eta}_z(\xi, y)
\]

\[
= \lim_{j \to \infty} \lim_{s_i \to \delta(\Gamma)^+} \int_{\mathbb{H}^k \times \mathbb{H}^n} \psi_j^z(\tilde{\eta}_z) \varphi(y) \, d\eta_z(s_i, \xi, y)
\]

\[
= \lim_{j \to \infty} \lim_{s_i \to \delta(\Gamma)^+} \int_{\mathbb{H}^k} \psi_j^z(\tilde{\eta}_z) \varphi(D(\xi)) \, d\mu_z^s(\xi)
\]

\[
= \lim_{j \to \infty} \lim_{s_i \to \delta(\Gamma)^+} \int_{U_j(z)} \varphi(D(\xi)) \frac{\mu_O(U_j(z)) e^{-\delta(\Gamma)B_K(x, z)}}{\mu_x(U_j(z))} \, d\mu_z^s(\xi)
\]

whence, using the definition of \( \mu_z^s \), and setting

\[
A_j(z) = \frac{\mu_O(U_j(z)) e^{-\delta(\Gamma)B_K(x, z)}}{\mu_x(U_j(z))},
\]

we get

\[
\int_{\mathbb{H}^n} \varphi(y) \, d\tilde{\alpha}_z(y)
\]

\[
(4.1) \quad = \lim_{j \to \infty} A_j(z) \lim_{s_i \to \delta(\Gamma)^+} \int_{U_j(z)} \frac{\varphi(D(\xi))}{\mu_O(U_j(z)) e^{-\delta(\Gamma)B_K(x, z)}} \, d\mu_z^s(\xi)
\]

\[
= \lim_{j \to \infty} A_j(z) \lim_{s_i \to \delta(\Gamma)^+} \frac{1}{c(s_i)} \sum_{g \in \Gamma \atop gO \in U_j} \frac{\varphi(D(gO)) e^{-s_i d(x, gO)}}{\mu_O(U_j(z)) e^{-\delta(\Gamma)B_K(x, z)}}.
\]

Moreover,

\[
\frac{\varphi(D(gO)) e^{-s_i d(x, gO)}}{\mu_O(U_j(z)) e^{-\delta(\Gamma)B_K(x, z)}}
\]

\[
= \frac{\varphi(D(gO))}{\mu_O(U_j(z))} \cdot \frac{e^{-s_i (d(x, gO) - d(O, gO))}}{e^{-\delta(\Gamma)B_K(x, z)}} \cdot e^{-s_i d(O, gO)}
\]

\[
= \frac{\varphi(D(gO))}{\mu_O(U_j(z))} \cdot \frac{e^{-\delta(\Gamma)(d(x, gO) - d(O, gO))}}{e^{-\delta(\Gamma)B_K(x, z)}} \cdot e^{-s_i d(O, gO)} \cdot e^{-s_i d(O, gO)}.
\]

From the definition of the Busemann function \( B_K \), it follows that for all \( z \in \partial \mathbb{H}^k \) and \( p \in \mathbb{H}^k \)

\[
(4.2) \quad \lim_{\xi \to z} (d(p, \xi) - d(O, \xi)) = B_K(p, z).
\]
Therefore, there exist two sequences \( \{ E_j^+ \} \) and \( \{ E_j^- \} \), converging to 1 as \( j \to \infty \), and such that for all \( g \in \Gamma \) with \( gO \subseteq U_j(z) \)

\[
E_j^- \leq \frac{e^{-\delta(\Gamma)(d(x,gO) - d(O,gO))}}{e^{-\delta(\Gamma)B_K(x,z)}} \leq E_j^+.
\]

Moreover, equation (4.2) implies that the term \( (d(x,gO) - d(O,gO)) \) is bounded, so for each \( j \)

\[
\lim_{s_i \to \delta(\Gamma)^+} \frac{e^{-s_i(d(x,gO) - d(O,gO))}}{e^{-\delta(\Gamma)(d(x,gO) - d(O,gO))}} = 1
\]

uniformly on \( U_j \). Whence, since \( \varphi \) is positive,

\[
\lim_{j \to \infty} A_j(z) \lim_{s_i \to \delta(\Gamma)^+} \frac{1}{c(s_i)} \sum_{g \in \Gamma, gO \subseteq U_j} \frac{\varphi(D(gO)) e^{-s_i d(x,gO)}}{\mu_O(U_j(z)) e^{-\delta(\Gamma)B_K(x,z)}}
\]

\[
\leq \lim_{j \to \infty} A_j(z) E_j^+ \lim_{s_i \to \delta(\Gamma)^+} \frac{1}{c(s_i)} \sum_{g \in \Gamma, gO \subseteq U_j} \frac{\varphi(D(gO))}{\mu_O(U_j(z))} e^{-s_i d(O,gO)}
\]

\[
= \lim_{j \to \infty} A_j(z) E_j^+ \lim_{s_i \to \delta(\Gamma)^+} \int_{U_j(z)} \varphi(D(x)) \frac{1}{\mu_O(U_j(z))} d\mu_{O}(x)
\]

and similarly for \( E_j^- \). Since for \( \mu_O \)-almost all \( z \) we have \( \lim_j A_j(z) = 1 \), and since \( E_j^\pm \to 1 \), we get that \( \mu_O \)-almost everywhere

\[
\lim_{j \to \infty} A_j(z) E_j^\pm \lim_{s_i \to \delta(\Gamma)^+} \int_{U_j(z)} \varphi(D(\xi)) \frac{1}{\mu_O(U_j(z))} d\mu_{O}(\xi)
\]

\[
= \lim_{j \to \infty} \lim_{s_i \to \delta(\Gamma)^+} \int_{U_j(z)} \varphi(D(\xi)) \frac{1}{\mu_O(U_j(z))} d\mu_{O}(\xi)
\]

\[
= \lim_{j \to \infty} \lim_{s_i \to \delta(\Gamma)^+} \int_{\mathbb{R}^k} \psi_j^{z,nO}(\xi) \varphi(D(\xi)) d\mu_{O}(\xi)
\]

\[
= \lim_{j \to \infty} \lim_{s_i \to \delta(\Gamma)^+} \int_{\mathbb{R}^k \times \mathbb{R}^n} \psi_j^{z,nO}(\xi, y) d\eta_{O}(\xi, y)
\]

\[
= \int_{\mathbb{R}^n} \varphi(y) d\alpha_z(y).
\]

Finally, from equations (4.1) – (4.4), \( \mu_O \)-almost everywhere we get

\[
\int_{\mathbb{R}^n} \varphi(y) d\alpha_z(y) \leq \int_{\mathbb{R}^n} \varphi(y) d\tilde{\alpha}_z(y) \leq \int_{\mathbb{R}^n} \varphi(y) d\alpha_z(y)
\]

and the claim follows. \( \square \)
In particular, Theorem 4.3 implies that the measures $\eta_x$’s are $\rho$-equivariant. Indeed, since the measures $\{\eta_x\}$ are $\rho$-equivariant, and since the push-forward is continuous for the weak convergence, for all $x \in \mathbb{H}^k$ and $\gamma \in \Gamma$

$$\eta^\gamma_x = (\gamma, \rho(\gamma))_* \eta^\gamma_x \rightarrow (\gamma, \rho(\gamma))_* \eta_x.$$  

Moreover, since $\pi_K(\eta_x) = \mu_x$, each measure $\eta_x$ is concentrated in $\partial \mathbb{H}^k \times \mathbb{H}^n$.

**Lemma 4.4.** — The family $\{\alpha_z\}$ is $\rho$-equivariant, that is, for all $\gamma \in \Gamma$ and $\mu_O$-almost all $z \in \partial \mathbb{H}^k$

$$\alpha^\gamma_z = \rho(\gamma)_* \alpha_z.$$

**Proof.** — From point (3) of Theorems 2.6 and the $\rho$-equivariance of the $\eta_x$’s, it follows that for all $\varphi \in C(\mathbb{H}^k \times \mathbb{H}^n)$

$$\int \int_{\partial \mathbb{H}^k \times \mathbb{H}^n} \varphi(\gamma z, y) \, d\alpha^\gamma_z(y) \, d\mu_O(z) = \int \int_{\partial \mathbb{H}^k \times \mathbb{H}^n} \varphi(z, y) \, d\alpha_z(y) \, d\mu_O(z) = \int \int_{\mathbb{H}^k \times \mathbb{H}^n} \varphi(z, y) \, d\eta_O = \int \int_{\mathbb{H}^k \times \mathbb{H}^n} \varphi(\gamma z, \rho(\gamma)y) \, d\eta_O = \int \int_{\partial \mathbb{H}^k \times \mathbb{H}^n} \varphi(\gamma z, \rho(\gamma)y) \, d\alpha_z(y) \, d\mu_O(z) = \int \int_{\partial \mathbb{H}^k \times \mathbb{H}^n} \varphi(\gamma z, y) \, d(\rho(\gamma)_* \alpha_z)(y) \, d\mu_O(z).$$

Whence, the measures $\alpha^\gamma_z$ and $\rho(\gamma)_* \alpha_z$ equal $\mu_O$-almost everywhere.  \(\square\)

Now, for each $y \in \mathbb{H}^n$ let $\nu_y$ be the visual measure on $\partial \mathbb{H}^n$ centred at $y$. More precisely, choose the disc model of $\mathbb{H}^n$ whose centre is $O$ and let $\nu_O$ be the standard probability measure on $S^{n-1} \simeq \partial \mathbb{H}^n$. Then, for all $g \in \text{Isom}(\mathbb{H}^n)$ define

$$\nu_g O = g_* \nu_O.$$  

This definition is not ambiguous because $\nu_O$ is $\text{Stab}(O)$-invariant, where $\text{Stab}(O) = \{g \in \text{Isom}(\mathbb{H}^n) : g(O) = O\}$. For $y \in \partial \mathbb{H}^n$ simply define $\nu_y = \delta_y$.

For all $z \in \partial \mathbb{H}^k$ define $\lambda_z$ a measure on $\partial \mathbb{H}^n$ by $\lambda_z = \alpha_z * \{\nu_y\}$. That is, for all $\varphi \in C(\partial \mathbb{H}^n)$

$$\int_{\partial \mathbb{H}^n} \varphi \, d\lambda_z = \int_{\mathbb{H}^n} \left( \int_{\partial \mathbb{H}^n} \varphi(\theta) \, d\nu_y(\theta) \right) \, d\alpha_z(y).$$
Note that such an integral is well-defined because, for any \( \varphi \), the function \( y \mapsto \int_{\partial H^n} \varphi \, d\nu_y \) is continuous in \( y \).

**Remark 4.5.** — Since the measure \( \nu_y \) depends continuously on \( y \), by Lemma 2.11, the convolution with the family of visual measures is weakly continuous.

We show now that \( \{ \lambda_z \}_{z \in \partial H^k} \) is a family of developing measures for \( \rho \).

**Lemma 4.6.** — The family \( \{ \lambda_z \} \) is \( \rho \)-equivariant, that is, for \( \mu_O \)-almost all \( z \in \partial H^k \) and all \( \gamma \in \Gamma \), we have \( \lambda_{\gamma z} = \rho(\gamma)^* \lambda_z \).

**Proof.** — By Lemma 4.4, for \( \mu_O \)-almost all \( z \in \partial H^k \) and all \( \varphi \in C(\partial H^n) \)

\[
\int_{\partial H^n} \varphi \, d\lambda_{\gamma z} = \int_{\partial H^n} \int_{\partial H^n} \varphi(\theta) \, d\nu_y(\theta) \, d\alpha_{\gamma z}(y) = \int_{\partial H^n} \int_{\partial H^n} \varphi(\theta) \, d\nu_{\rho(\gamma)y}(\theta) \, d\alpha_z(y)
\]

\[
= \int_{\partial H^n} \int_{\partial H^n} \varphi(\rho(\gamma)\theta) \, d\nu_y(\theta) \, d\alpha_z(y)
\]

\[
= \int_{\partial H^n} \varphi \circ \rho(\gamma) \, d\lambda_z = \int_{\partial H^n} \varphi \, d(\rho(\gamma)^* \lambda_z).
\]

\( \square \)

**Lemma 4.7.** — For any \( \varphi \in C(\partial H^n) \), the function \( z \mapsto \int_{\partial H^n} \varphi(\theta) \, d\lambda_z(\theta) \) is \( \mu_O \)-integrable.

**Proof.** — This follows directly from the definition of \( \lambda_z \), because the family \( \{ \alpha_z \} \) disintegrates \( \eta_O \), and \( \mu_O = \pi_K^* \eta_O \).

**Lemma 4.8.** — For \( \mu_O \)-almost all \( z \in \partial H^k \), we have \( ||\lambda_z|| = 1 \).

**Proof.** — For all \( x \in H^k \) and \( y \in \partial H^n \), the measures \( \eta_x \) and \( \nu_y \) are positive. Then the measures \( \alpha_z \)'s are positive, and this implies that the measures \( \lambda_z \)'s are positive. Thus

\[
||\lambda_z|| = \int_{\partial H^n} 1 \, d\lambda_z = \int_{\partial H^n} \int_{\partial H^n} 1 \, d\nu_y(\theta) \, d\alpha_z(y) = \int_{\partial H^n} 1 \, d\alpha_z = ||\alpha_z|| = 1
\]

\( \mu_O \)-almost everywhere because \( \{ \alpha_z \} \) disintegrates \( \eta_O \).

Therefore, the family \( \{ \lambda_z \} \) satisfies properties (1) – (4) of Definition 2.10. So it is a family of developing measure for \( \rho \), and the proof of Theorem 1.1 is complete.

\( \square \)

### 5. Sequence of \( \varepsilon \)-natural maps

In this section we prove Theorem 1.3, showing that the natural map constructed in Sections 3 and 4 is the limit of a sequence of \( \varepsilon \)-natural maps.
We keep the notation of Sections 2-4. Through this section we suppose that \( \Gamma \) diverges at \( \delta(\Gamma) \); by Lemma 2.5, this is the case if \( \mathbb{H}^k/\Gamma \) has finite volume.

Let \( A \subset \mathbb{H}^k \) be the Dirichlet domain of \( O \). The set \( A \) is a fundamental domain for \( \Gamma \) containing \( O \). Let \( \sigma \) be any Borel probability measure on \( A \).

**Definition 5.1.** For each \( s > \delta(\Gamma) \) and \( x \in \mathbb{H}^k \) we define \( m^s_x \) a positive Borel measure on \( \mathbb{H}^k \) by

\[
m^s_x = \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma O)} \gamma \ast \sigma.
\]

Note that if \( \sigma = \delta_O \), then \( m^s_x = \mu^s_x \).

**Lemma 5.2.** Let \( \{\gamma_n\} \) be any sequence of elements of \( \Gamma \). If \( \gamma_n(O) \) converges to a point \( \theta \in \partial \mathbb{H}^k \), then for any sequence \( \{x_n\} \subset A \), the sequence \( \{\gamma_n(x_n)\} \) converges to \( \theta \).

**Proof.** Suppose the contrary. Since \( \mathbb{H}^k \) is compact, up to passing to a sub-sequence, we can suppose that \( \{\gamma_n(x_n)\} \) converges to a point \( \zeta \neq \theta \) in \( \mathbb{H}^k \). Then, the geodesics joining \( \gamma_n(x_n) \) to \( \gamma_n(O) \) accumulate near the geodesic between \( \zeta \) and \( \theta \). This can not happen because \( \Gamma \) is discrete. \( \square \)

**Theorem 5.3.** For each \( x \in \mathbb{H}^k \), if \( \mu_x \) denotes the Patterson-Sullivan measure constructed as in Theorem 2.6, then

\[
m^s_x \rightharpoonup \mu_x
\]

in \( \mathbb{H}^k \), when \( s \to \delta(\Gamma)^+ \).

**Proof.** We have to show that for each \( \varphi \in C(\mathbb{H}^k) \), \( \int \varphi \, dm^s_x \to \int \varphi \, d\mu_x \). Let \( \varphi \in C(\mathbb{H}^k) \). Since \( \mu^s_x \to \mu_x \), we will have finished by proving that

\[
\lim_{s \to \delta(\Gamma)^+} \left| \int_{\mathbb{H}^k} \varphi \, dm^s_x - \int_{\mathbb{H}^k} \varphi \, \mu^s_x \right| = 0.
\]

Let \( C > 0 \) be a small constant and let \( A_1 \subset A \) be a compact set such that \( O \in A_1 \) and \( \sigma(A \setminus A_1) < C \). Since the supports of the measures \( m^s_x \) and \( \mu^s_x \) are contained in \( \mathbb{H}^k \), and since \( O \in A \), we have

\[
(5.1) \quad \int_{\mathbb{H}^k} \varphi \, d(m^s_x - \mu^s_x) = \sum_{\gamma \in \Gamma} \int_{\gamma(A_1)} \varphi \, d(m^s_x - \mu^s_x) + \sum_{\gamma \in \Gamma} \int_{\gamma(A \setminus A_1)} \varphi \, dm^s_x.
\]
Looking at the second summand,

\[
\left| \sum_{\gamma \in \Gamma} \int_{\gamma(A \setminus A_1)} \varphi \, dm^s_x \right| \\
\leq \max(\|\varphi\|) \frac{1}{c(s)} \sum_{\gamma \in \Gamma} \int_{\gamma(A \setminus A_1)} e^{-sd(x,\gamma O)} \, d\gamma \sigma(\xi) \\
= \max(\|\varphi\|) \frac{1}{c(s)} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma O)} \sigma(A \setminus A_1) \leq C \cdot \max(\varphi) \cdot ||\mu^s_x||. 
\]

We estimate now the first summand. Since \(c(s) \to \infty\), for any finite subset \(\Gamma_1\) of \(\Gamma\)

\[
\lim_{s \to \delta(\Gamma)^+} \left| \sum_{\gamma \in \Gamma_1} \int_{\gamma(A_1)} \varphi \, dm^s_x \right| = \lim_{s \to \delta(\Gamma)^+} \left| \sum_{\gamma \in \Gamma_1} \int_{\gamma(A_1)} \varphi \, d\mu^s_x \right| = 0. 
\]

Moreover, since \(\varphi\) is continuous, it is uniformly continuous on \(H_k\) for any metric that induces the usual topology on \(H_k\) (recall that the hyperbolic metric of \(H_k\) is not a metric on \(H_k\)). Therefore, Lemma 5.2 implies that, except for a finite number of elements of \(\Gamma\), we have

\[
|\varphi(\gamma O) - \varphi(\gamma \xi)| < C 
\]

independently on \(\gamma\) and on \(\xi \in A_1\). Let \(\Gamma_1\) be a finite subset of \(\Gamma\) such that

\[
\left| \sum_{\gamma \in \Gamma \setminus \Gamma_1} \int_{\gamma(A_1)} \varphi \, d(m^s_x - \mu^s_x) \right| \\
= \frac{1}{c(s)} \sum_{\gamma \in \Gamma \setminus \Gamma_1} e^{-sd(x,\gamma O)} \left( \int_{\gamma(A_1)} \varphi(\xi) \, d\gamma \sigma(\xi) \\
- \varphi(\gamma O) \left( \sigma(A_1) + \sigma(A \setminus A_1) \right) \right) \leq C \int_{H_k^+} |\varphi| \, d\mu^s_x + \frac{1}{c(s)} \sum_{\gamma \in \Gamma \setminus \Gamma_1} e^{-sd(x,\gamma O)} \int_{A_1} |\varphi(\gamma \xi) - \varphi(\gamma O)| \, d\sigma(\xi) \\
\leq C \int_{H_k^+} |\varphi| \, d\mu^s_x + \frac{C}{c(s)} \sum_{\gamma \in \Gamma \setminus \Gamma_1} e^{-sd(x,\gamma O)} \cdot \sigma(A_1) \leq C \int_{H_k^+} (1 + |\varphi|) \, d\mu^s_x.
\]

Whence the claim follows, combining with (5.1), (5.2) and (5.3), since \(C\) can be chosen arbitrarily small. \(\square\)

Now we proceed as in Sections 3 and 4. Namely, we fix a \(\rho\)-equivariant map \(D\), we define measures \(n^s_x = m^s_x \times \{\delta_{D(z)}\}\), and we chose a sequence \(s_i \to \delta(\Gamma)^+\) such that \(n^s_{\gamma_i} O\) converges to a measure \(n_O\). We disintegrate \(n_O\) as \(n_O = \mu_O \times \{a_z\}\), and we define \(n_x = \mu_x \times \{a_z\}\). As in Theorem 4.3 one can
show that \( n^s_i \to n_x \). We define then \( b^*_x = (D_* m^*_x) \ast \{\nu_y\} = \pi_{N_*} (n^s_x) \ast \{\nu_f\} \), and \( b_x = \pi_{N_*} (n_x) \ast \{\nu_f\} \).

Finally, for each \( s > \delta(\Gamma) \) we set \( s = (1 + \varepsilon)\delta(\Gamma) \) and we define maps \( F(x) = \text{bar}(b_x) \) and \( F_\varepsilon(x) = \text{bar}(b^*_x) \). The map \( F \) is a natural map, in the sense that it has the properties (1) – (4) of Theorem 1.2. The maps \( F_\varepsilon \) the \( \varepsilon \)-natural maps constructed in [6], so they are smooth, \( \rho \)-equivariant, and for all \( p \geq 3 \) and \( \varepsilon > 0 \), \( \text{Jac}_p F_\varepsilon(x) \leq \left( \frac{(1 + \varepsilon)\delta(\Gamma)}{p - 1} \right)^p \).

**Proposition 5.4.** — The maps \( F_\varepsilon \) punctually converge to the map \( F \).

**Proof.** — From the weak continuity of the push-forward and from Lemma 2.11, we get
\[
b^*_x \rightharpoonup b_x.
\]
Then, the claim follows from Point 3 of Lemma 2.8. \( \square \)

**Remark 5.5.** — The weak convergence of \( b^*_x \) to \( b_x \) is enough to prove stronger convergences. For example, it can be shown that the derivatives of \( F_\varepsilon \) converges the ones of \( F \), whence one gets that the convergence of the \( \varepsilon \)-natural maps is locally uniform (see [14] for details).

**6. Rigidity of representations**

In this section we give a proof of Theorem 1.4, referring the reader to [14] for a fully detailed discussion on the matter.

**Proof of Theorem 1.4.** — Let \( M \) be a complete hyperbolic \( k \)-manifold of finite volume and let \( \rho : \pi_1(M) \to \text{Isom}(\mathbb{H}^n) \) be a representation. We consider \( \mathbb{H}^k \) as the universal cover of \( M \), and we identify \( \pi_1(M) < \text{Isom}(\mathbb{H}^k) \) with the group of deck transformations of \( \mathbb{H}^k \to M \).

We denote by a pseudo-developing map any piecewise smooth, \( \rho \)-equivariant map \( D : \mathbb{H}^k \to \mathbb{H}^n \). The volume of a pseudo-developing map \( D \) is defined by integrating the pull-back of the volume form of \( \mathbb{H}^n \) on a fundamental domain for \( M \). Equivalently, if \( \omega_N \) denotes the volume form of \( \mathbb{H}^n \), by equivariance, the form \( D^* \omega_N \) descends to \( M \) and we set \( \text{vol}(D) = \int_M D^* \omega_N \).

Now, let us suppose that \( M \) is compact. In this case we define the volume of \( \rho \) as the infimum of the volumes of all the pseudo-developing maps for \( \rho \):
\[
\text{vol}(\rho) = \inf_D \text{vol}(D).
\]
Note that the volume of any elementary representation vanishes, so that we can suppose $\rho$ to be non-elementary.

The existence of a natural map immediately gives the inequality; indeed Theorem 1.2 point 3, together with the fact that $\delta(\pi_1(M)) = k - 1$, tells us that the volume of a natural map is less than $\text{vol}(M)$. Moreover, if $\rho$ has maximum volume, then from Theorem 1.2 point 4 it follows that in each point the differential of a natural map is an isometry. Since $\text{vol}(\rho)$ is maximal, we deduce that the image of a natural map is a locally minimal sub-manifold of $\mathbb{H}^n$. The claim now follows because a locally isometric and locally minimal immersion from a Riemannian manifold to another is totally geodesic.

In the non-compact case, some problem arises. From now on we suppose that $M$ is non-compact. If we keep the above definition of volume of $\rho$ we get that the volume of any representation vanishes. This is because any non-compact manifold collapses to a spine, which is a codimension-one object. To avoid such a pathology we need to require that a pseudo-developing map has a nice behaviour on the cusps. For $G$ a group of isometries, we denote by $\text{Fix}(G)$ the set of fixed points of $G$ (including the points at infinity). If $C$ is a cusp of $M$, it is readily checked that $\pi_1(C)$ is Abelian and that $\text{Fix}(\pi_1(C))$ consists of a unique point of $\partial \mathbb{H}^k$.

We say that a pseudo-developing map $D$ properly ends if for any cusp $C$ of $M$, if $\xi = \text{Fix}(\pi_1(C))$ and $\alpha(t)$ is a geodesic ray ending at $\xi$, then all limit points of $D(\alpha(t))$ lie either in $\text{Fix}(\rho(\pi_1(C))) \subset \mathbb{H}^n$ or in a finite union of $\rho(\pi_1(C))$-invariant geodesics. It easy to see that properly ending pseudo-developing maps always exist.

Now we can define the volume of $\rho$ by taking the infimum of the volumes of properly ending pseudo-developing maps.

$$\text{vol}(\rho) = \inf_{D \text{ properly ending}} \text{vol}(D).$$

We would like to use a natural map as in the compact case, but the problem now is that a natural map given by Theorem 1.2 in general does not end properly.

Nevertheless, we can use the $\varepsilon$-natural maps. Indeed, it can be shown (see [14]) that any $\varepsilon$-natural map constructed as in Section 5 properly ends, and this gives immediately the inequality. Moreover, by Proposition 5.4 and Remark 5.5 we have that the volumes of the $\varepsilon$-natural maps converge to the volume of a natural map. Thus, if the volume of $\rho$ is maximal, then the volume of a natural map is maximal and we conclude as in the compact case. \qed
7. Measurable extension of natural maps

This section is devoted to proving Theorem 1.5. We keep here the notation of previous sections. In particular we recall that \( \{\mu_x\}_{x \in \mathbb{H}^k} \) is the family of Patterson-Sullivan measures, and that \( \{\lambda_z\}_{z \in \partial \mathbb{H}^k} \) is a family of developing measures.

**Definition 7.1.** — Let \( \{\gamma_i O\} \) be a sequence in the \( \Gamma \)-orbit of \( O \). We say that \( \gamma_i O \) conically converges to \( \omega \in \partial \mathbb{H}^k \) if \( \gamma_i O \to \omega \) and there exists a geodesic \( \sigma \), ending at \( \omega \), such that the distance of \( \gamma_i O \) from \( \sigma \) is bounded.

The conical limit set of \( \Gamma \), denoted by \( \Lambda_c \), is the set of the limits of conically converging sequences in the \( \Gamma \)-orbit of \( O \).

Clearly, the conical limit set is a sub-set of the limit set of \( \Gamma \). In order to prove that the natural maps extend to the boundary we need the following result.

**Theorem 7.2.** — For each \( f \in L^1(\partial \mathbb{H}^k, \mu_O) \) there exists a set \( Z \) with \( \mu_O(Z) = 0 \) such that for all \( \omega \in \Lambda_c \setminus Z \), and for any sequence \( \{\gamma_i\} \subset \Gamma \) such that \( \gamma_i O \) conically converges to \( \omega \)

\[
\int_{\partial \mathbb{H}^k} f(\theta) d\mu_{\gamma_i O}(\theta) \to f(\omega).
\]

Before proving Theorem 7.2 we show how it implies Theorem 1.5.

**Proof of Theorem 1.5.** — First, we prove the existence part. By definition (see Section 2) we have

\[
B_{\beta_x}(y) = \int_{\partial \mathbb{H}^k} B_N(y, \theta) d\beta_x(\theta) = \int_{\partial \mathbb{H}^n} \int_{\partial \mathbb{H}^k} B_N(y, \theta) d\lambda_z(\theta) d\mu_x(z)
\]

and a similar formula holds for the derivatives of \( B_{\beta_x} \). Therefore, for each \( y \) there exists a \( \mu_O \)-negligible set \( Z \subset \partial \mathbb{H}^k \) such that for all \( \omega \in \partial \mathbb{H}^k \setminus Z \) we have

\[
\lim_{x \to \omega} B_{\beta_x}(y) = B_{\lambda_x}(y)
\]

Where “\( \lim_{x \to \omega} \)” means “for any sequence \( \{x_i\} \) in the \( \Gamma \)-orbit of \( O \), conically converging to \( \omega \)...”. The same statement holds for the derivatives of \( B_{\beta_x} \). Now, let \( Y \) be a countable dense subset of \( \mathbb{H}^n \). Then (since countable unions of negligible sets are negligible), there exists a \( \mu_O \)-negligible set \( W \subset \partial \mathbb{H}^k \) such that the above limit holds for all \( \omega \in \partial \mathbb{H}^k \setminus W \), all \( y \in Y \) and all the derivatives of \( B_{\beta_x} \). It follows that the barycentre of \( \beta_x \), that is the unique point of minimum of \( B_{\beta_x} \), converges to the barycentre...
of \( \lambda_\omega \), which is well-defined because \( \lambda_\omega \) is not the sum of two Dirac deltas with equal weights. Therefore, if \( F \) denotes the natural map constructed using the family \( \{ \lambda_z \} \), setting \( \overline{F}(\omega) = \text{bar}(\lambda_\omega) \), we have that for \( \mu_O \)-almost all \( \omega \), for any sequence \( \{ \gamma_i O \} \) conically converging to \( \omega \)

\[
\lim_{\gamma_i O \rightarrow \omega} F(\gamma_i O) = \overline{F}(\omega) = \text{bar}(\lambda_\omega) \tag{7.1}
\]

The map \( \overline{F} \) is measurable because it can be viewed as a limit of continuous functions. Finally, it is readily checked that \( F(\gamma_i O) \) and \( F(\gamma_i x) \) have the same limit, and this completes the proof of the existence part. It remains now to prove the last part of Theorem 1.5.

Given the maps \( \overline{F}_1 \) and \( \overline{F}_2 \), we construct the corresponding natural maps \( F_1, F_2 \). For \( i = 1, 2 \) and for almost all \( \omega \) in the conical limit set of \( \Gamma \), if \( \{ \gamma_n O \} \) is a sequence conically converging to \( \omega \), then by \( \tag{7.1} \), \( F_i(\gamma_n O) \rightarrow \text{bar}(\overline{F}_i(\omega)) \). By equivariance we have

\[
F_i(\gamma_n O) = \rho(\gamma_n)F_i(O) \quad i = 1, 2.
\]

Up to pass to a sub-sequence, either \( \rho(\gamma_n) \) converges to an isometry \( \psi \) of \( \mathbb{H}^n \), or the limit of \( \rho(\gamma_n)y \) belongs to \( \partial \mathbb{H}^n \) and does not depend on \( y \in \mathbb{H}^n \) (see for example [15]). In the former case, we get \( \psi(F_1(O)) = \text{bar}(\overline{F}_1(\omega)) \); but also, for all \( \gamma \in \Gamma \) we have

\[
\psi(\rho(\gamma)F_i(O)) = \psi(F_i(\gamma O)) = \lim \rho(\gamma_n)F_i(\gamma O).
\]

The sequence \( \gamma_n \gamma O \) conically converges to \( \omega \), since \( \gamma_n O \) does. Then, by equivariance and by \( \tag{7.1} \), we get

\[
\lim \rho(\gamma_n)F_i(\gamma O) = \lim F_i(\gamma_n(\gamma O)) = \text{bar}(\overline{F}_i(\omega))
\]

whence \( \psi(\rho(\gamma)F_i(O)) = \psi(F_i(O)) \) and thus \( \rho(\gamma)F_i(O) = F_i(O) \), contradicting the fact that the image of \( \rho \) is non-elementary. Therefore, we are in the latter case and in particular

\[
\text{bar}(\overline{F}_1(\omega)) = \lim \rho(\gamma_n)F_i(O) = \lim \rho(\gamma_n)F_2(O) = \text{bar}(\overline{F}_2(\omega)) \in \partial \mathbb{H}^n
\]

but this is possible if and only if \( \overline{F}_i(\omega) = \delta_{\text{bar}(F_i(\omega))} \). So \( \overline{F}_1 \) and \( \overline{F}_2 \) are ordinary functions with values in \( \partial \mathbb{H}^n \), and they coincide almost everywhere. \( \square \)

**Proof of Theorem 7.2.** — The map \( x \mapsto \int_{\partial \mathbb{H}^k} f(\theta) \, d\mu_x(\theta) \) can be viewed as the harmonic extension of \( f \), and one can prove its convergence to \( f \) along cones and almost everywhere by using standard techniques of harmonic analysis. We give a proof for completeness.

For the whole proof, we work in the half space model \( \mathbb{H}^k = \mathbb{R}^{k-1} \times \mathbb{R}^+ \), using the following notation. For a point \( x \in \mathbb{H}^k \), we denote by \( (x', x_k) \in \mathbb{R}^{k-1} \times \mathbb{R}^+ \)
$\mathbb{R}^{k-1} \times \mathbb{R}^+$ its coordinates in the half-space model, by the symbol $|x|$ we denote the Euclidean norm of $x$ in the model, and $B(\omega, r)$ will denote the ball of centre $\omega$ and Euclidean radius $r$. In the half-space model, setting $\delta = \delta(\Gamma)$ we have

$$e^{-\delta B_k(x, \xi)} = \left( \frac{x_k(1 + |\xi|^2)}{|\xi - x'|^2 + x_k^2} \right)^\delta$$

and for all $\gamma \in \Gamma$, if $x = \gamma O$

$$1 = ||\mu_O|| = ||\gamma_* \mu_O|| = ||\mu_{\gamma O}|| = \int_{\partial \mathbb{H}^k} \left( \frac{x_k(1 + |\xi|^2)}{|\xi - x'|^2 + x_k^2} \right)^\delta \mu_{\gamma O}(\xi) \cdot$$

We will work on a fixed ball of centre 0 and radius $R$ of $\mathbb{R}^{k-1}$. This is not restrictive because proving convergence almost everywhere for all balls is equivalent to prove convergence almost everywhere.

For all $\omega \in \mathbb{R}^{k-1}$, we denote by $C_\omega(\alpha)$ the vertical cone in $\mathbb{H}^{k-1} \times \mathbb{R}^+$ of vertex $\omega$ and emi-angle $\alpha$. For any non-negative $g \in L^1(\partial \mathbb{H}^k, \mu_O)$, $\omega \in \partial \mathbb{H}^k$, $\alpha \in (0, \pi/2)$ we define the maximal operator $M_\alpha g(\omega)$ by

$$M_\alpha g(\omega) = \sup_{\gamma \in \Gamma, \gamma O \in C_\omega(\alpha)} \int_{\partial \mathbb{H}^k} g(\xi) \, d\mu_{\gamma O}(\xi)$$

and the so-called Hardy Littlewood operator $N g(\omega)$ by

$$N g(\omega) = \sup_{r > 0} \frac{1}{\mu_O(B(\omega, r))} \int_{B(\omega, r)} g(\xi) \, d\mu_O(\xi).$$

From now on, the symbol $c$ will denote a generic constant, and different occurrences may denote different constants. If not specified, the constants do not depend on the other quantities we are considering.

**Lemma 7.3.** — There exists a constant $c$ such that for every point $\omega$ of the limit set of $\Gamma$ and any $r > 0$

$$\mu_O(B(\omega, r)) \leq cr^\delta.$$

**Proof.** — For all $x \in \mathbb{H}^k$

$$||\mu_x|| = \int_{\partial \mathbb{H}^k} \left( \frac{x_k(1 + |\xi|^2)}{|\xi - x'|^2 + x_k^2} \right)^\delta \geq \int_{\partial \mathbb{H}^k} \left( \frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^\delta.$$

Suppose now that $x$ is of the form $x = \gamma O$, with

$$1 = ||\mu_x|| = \int_{\partial \mathbb{H}^k} \left( \frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^\delta \geq c_1 r \mu_O(B(\omega, r)).$$

Then $|\xi - x'| \leq |\xi - \omega| + |x' - \omega| \leq (1 + c_2) r$ for any $\xi \in B(\omega, r)$. Then

$$1 = ||\mu_x|| = \int_{\partial \mathbb{H}^k} \left( \frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^\delta \geq c_1 r^\delta \mu_O(B(\omega, r)).$$
and the claim holds for such points. Let $a > 1$ and for $j \in \mathbb{Z}$, consider the set $A_j = \{x \in \mathbb{H}^k : |x - \omega| \in [a^j, a^{j+1})\}$. If for a certain $j \in \mathbb{Z}$ the set $A_j$ contains a point of the $\Gamma$-orbit of $O$, then for $r \in [a^j, a^{j+1}]$

$$\frac{r}{a} \leq \frac{a^{j+1}}{a} = a^j \leq a^{j+1} = a^j a \leq ra$$

and inequalities (7.2) hold with $c_2 = a = 1/c_1$. Let now $r > 0$. Since $\omega$ lies on the limit set, there exists $j$ with $a^j \leq r$ and such that $A_j$ intersects the $\Gamma$-orbit of $O$. Let $j_0$ be the maximum of such $j$’s. If $a^{j_0+1} \geq r$, we have finished. Otherwise, since $\mu_O$ is concentrated on the limit set

$$\mu_0(B(\omega, r)) = \mu(B(\omega, a^{j_0+1})) \leq c(a^{j_0+1})^\delta \leq cr^\delta.$$  

This completes the proof of Lemma 7.3.  

\[\square\]

**Lemma 7.4.** — There exists a constant $c$, depending only on $\alpha$, such that for any non-negative $g \in L^1(\partial \mathbb{H}^k, \mu_O)$, $\omega \in \partial \mathbb{H}^k$, $\alpha \in (0, \pi/2)$, we have

$$M_\alpha g(\omega) \leq cN g(\omega).$$
Proof. — Let \( x = \gamma O \) with \( \gamma \in \Gamma \) and \( \gamma O \in C_\omega(\alpha) \). As noticed above, it is not restrictive to work in the ball \( B(0,R) \).

\[
\int_{\partial \mathbb{H}^k} g(\xi) d\mu_x(\xi) \leq (1 + R^2)^{\delta} \int_{\partial \mathbb{H}^k} g(\xi) \left( \frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^{\delta} d\mu_O(\xi)
\]

\[
= c \left[ \int_{B(\omega,x_k)} g(\xi) \left( \frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^{\delta} d\mu_O(\xi) + \right.
\]

\[
+ \sum_{j \geq 0} \int_{B(\omega,2^{j+1}x_k) \setminus B(\omega,2^jx_k)} g(\xi) \left( \frac{x_k}{|\xi - x'|^2 + x_k^2} \right)^{\delta} d\mu_O(\xi) \right]
\]

\[
\leq c \left[ \frac{1}{c} \int_{B(\omega,x_k)} g(\xi) d\mu_O(\xi) + \right.
\]

\[
+ \sum_{j \geq 0} \frac{1}{(c2^j x_k^2)^{\delta}} \int_{B(\omega,2^{j+1}x_k) \setminus B(\omega,2^jx_k)} g(\xi) d\mu_O(\xi) \right]
\]

\[
\leq c \left[ \frac{1}{\mu_O(B(\omega,x_k))} \int_{B(\omega,x_k)} g(\xi) d\mu_O(\xi) + \right.
\]

\[
+ \sum_{j \geq 0} \frac{2^{-j}}{\mu_O(B(\omega,2^{j+1}x_k))} \int_{B(\omega,2^{j+1}x_k)} g(\xi) d\mu_O(\xi) \right]
\]

\[
\leq cNg(\omega) \left( 1 + \sum_{j \geq 0} 2^{-j} \right) \leq cNg(\omega).
\]

The constant \( c \) actually depends on \( \alpha \) because we used that for \( x \in C_\alpha(\omega) \) and \( \xi \in B(\omega,2^{j+1}x_k) \setminus B(\omega,2^jx_k) \) we have \( |\xi - x'|^2 + x_k^2 \geq c2^j x_k^2 \). It can be shown that \( c \) is bounded by \((\tan \alpha)^{2\delta}\).

We can now finish the proof of Theorem 7.2. Since \( ||\mu_{\gamma O}|| = 1 \) and \( \lim_{x \to \omega} e^{-\delta B^k(x,z)} = 0 \) for all \( z \neq \omega \), the claim is true for continuous functions. Suppose now \( f \in L^1(\partial \mathbb{H}^k, \mu_O) \), and let \( f_j \to f \) be a sequence of continuous functions converging to \( f \) \( \mu_O \)-almost everywhere and in \( L^1 \). We have

\[
\left| \int_{\partial \mathbb{H}^k} f(\xi) d\mu_{\gamma O}(\xi) - f(\omega) \right| \leq \left| \int_{\partial \mathbb{H}^k} f(\xi) - f_j(\xi) d\mu_{\gamma O}(\xi) \right|
\]

\[
\quad + \left| \int_{\partial \mathbb{H}^k} f_j(\xi) d\mu_{\gamma O}(\xi) - f_j(\omega) \right|
\]

\[
\quad + |f_j(\omega) - f(\omega)|.
\]
The second summand of the second term goes to zero as $\gamma O \to \omega$ because $f_j$ is continuous. For $\mu_O$-almost all $\omega$ the last summand can be chosen arbitrarily small because $f_j \to f$ $\mu_O$-almost everywhere. By Lemma 7.4, the first summand of the second term is bounded by $cN(f_j-f)(\omega)$ on each cone $C_\alpha(\omega)$, the constant $c$ depending on $\alpha$. The Hardy Littlewood operator is bounded from $L^1$ to $L^{1,\infty}$ (see for example [26]), that is, for all $g \in L^1$ and $\epsilon > 0$

$$\mu_O \left( \{ \omega \in \partial \mathbb{H}^k : |Ng(\omega)| \geq \epsilon \} \right) \leq \frac{c\|g\|_{L^1}}{\epsilon}.$$ 

Let $A_j^\epsilon = \{ \omega \in \partial \mathbb{H}^k : |N(f_j-f)(\omega)| \geq \epsilon \}$. Since $f_j \to f$ in $L^1$, for all $\epsilon$ the measure $\mu_O(A_j^\epsilon)$ goes to zero. This is equivalent to say that the characteristic function $\chi_{A_j^\epsilon} \to 0$ in $L^1$. Then, up to pass to sub-sequences, $\chi_{A_j^\epsilon} \to 0$ $\mu_O$-almost everywhere, that is, for $\mu_O$-almost all $\omega$ the quantity $|N(f_j-f)(\omega)| \leq \epsilon$ eventually on $j$. We have so proved that, for each $\alpha \in (0, \pi/2)$ there exist a negligible set $Z_\alpha^\epsilon$ such that, for all $\omega \in \partial \mathbb{H}^k \setminus Z_\alpha^\epsilon$, the quantity $|\int_{\partial \mathbb{H}^k} f(\xi) d\mu_O(\xi) - f(\omega)|$ is small than or equal to $\epsilon$ as $\gamma O$ converges to $\omega$ through $C_\alpha(\omega)$. The thesis now follows setting

$$Z = \bigcup_{\alpha \in \mathbb{Q} \cap (0, \pi/2)} Z_\alpha^\epsilon.$$ 

\[\square\]

8. Measurable Cannon-Thurston maps

In this sections we study existence, uniqueness and convergence of measurable Cannon-Thurston maps.

We keep here the notations of previous sections. In particular, if not specified, $\Gamma$ is a discrete group of $\text{Isom}(\mathbb{H}^k)$, $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ is a representation whose image is not elementary, $\{\mu_x\}$ is the family of Patterson-Sullivan measure, and $\{\lambda_z\}_{z \in \partial \mathbb{H}^k}$ is a family of developing measures for $\rho$.

The following lemma collects some ergodic properties of $\Gamma$ that we need in the sequel (see [31, Theorem A], [22, Theorem 6.3.6], [27, 23] for the proof).

**Lemma 8.1.** — Any non-elementary discrete group $\Gamma$ acts ergodically on $\Lambda$ w.r.t. the measure $\mu_O$. Therefore, $\Lambda_c$ has either zero or full $\mu_O$-measure. Moreover, the following are equivalent:

1. $\Lambda_c$ has full measure.
(2) $\Gamma$ diverges at $\delta(\Gamma)$.
(3) The geodesic flow is ergodic.
(4) $\Gamma$ acts ergodically on $\partial \mathbb{H}^k \times \partial \mathbb{H}^k$ w.r.t. $\mu_0 \times \mu_0$.

To begin with, we prove a couple of lemmas we need in the sequel.

**Lemma 8.2.** — The subset of $\partial \mathbb{H}^k$ consisting of the points $z$ such that $\lambda_z$ is the sum of two Dirac deltas with equal weights is $\mu_0$-measurable.

**Proof.** — Let $B(\theta, r)$ denotes the ball of centre $\theta \in \partial \mathbb{H}^n$ and radius $r$ in some metric of $\partial \mathbb{H}^n$, and let $Q$ be a countable, dense subset of $\partial \mathbb{H}^n$.

For any open set $B \subset \partial \mathbb{H}^n$, the function $z \mapsto \lambda_z(B)$ is $\mu_0$-measurable. It follows that the function

$$z \mapsto \inf_{0 < r_1, r_2 \in Q} \left( \sup_{\theta_1, \theta_2 \in Q} \lambda_z \left( B(\theta_1, r_1) \cup B(\theta_2, r_2) \right) \right)$$

is $\mu_0$-measurable. The pre-image of 1, which therefore is a $\mu_0$-measurable set, is the set of points $z$ such that the support of $\lambda_z$ contains at most two points. Similarly, the following sets are $\mu_0$-measurable

$$\{ z \in \partial \mathbb{H}^k : \lambda_z \text{ is concentrated on one point} \}$$
$$\{ z \in \partial \mathbb{H}^k : \lambda_z \text{ has an atom of weight } \frac{1}{2} \}$$

and the claim follows.

**Lemma 8.3.** — For all $\mu_0$-measurable set $A \subset \partial \mathbb{H}^k$ the set

$$O(A) = \{ (x, y) \in \partial \mathbb{H}^k \times \partial \mathbb{H}^k : \exists \gamma \in \Gamma : \gamma(x), \gamma(y) \in A \}$$

is $\mu_0 \times \mu_0$-measurable.

**Proof.** — Clearly, it is sufficient to show that the function

$$(x, y) \mapsto \# \{ \gamma \in \Gamma : \gamma(x), \gamma(y) \in A \}$$

is $\mu_0 \times \mu_0$-measurable. The pre-image of $(n, \infty]$ is the set

$$\bigcup \left( (\gamma_1(A) \times \gamma_1(A)) \cap \cdots \cap (\gamma_n(A) \times \gamma_n(A)) \right)$$

which is a countable union of countable measurable sets, and therefore it is measurable.

**Lemma 8.4.** — Suppose that $\Gamma$ diverges at $\delta(\Gamma)$. If $\mu_0(A) > 0$, then $O(A)$ has full measure.
Proof. — Since $\mu_O(A) > 0$, then $A \times A$ has positive measure. Thus, $O(A)$ has positive measure because it contains $A \times A$. Moreover, it is readily checked that $O(A)$ is $\Gamma$-invariant. By Lemma 8.1 the action of $\Gamma$ on $\partial \mathbb{H}^k \times \partial \mathbb{H}^k$ is ergodic, whence $O(A)$ has full measure. 

**Theorem 8.5.** — Let $\Gamma < \text{Isom}(\mathbb{H}^k)$ be a discrete group which diverges at $\delta(\Gamma)$. Let $\rho : \Gamma \to \text{Isom}(\mathbb{H}^n)$ be a representation whose image is not elementary and let $\{\lambda_z\}_{z \in \partial \mathbb{H}^k}$ be a family of developing measures for $\rho$. Then, for almost all $z$, the measure $\lambda_z$ is not the sum of two Dirac deltas with equal weights.

Proof. — By Lemma 8.1, the conical limit set has full measure in $\partial \mathbb{H}^k$ and $\Gamma$ acts ergodically on $\partial \mathbb{H}^k$. Let $E$ be the set of points $z \in \partial \mathbb{H}^k$ such that $\lambda_z$ is the sum of two deltas with equal weights. We have to prove that $\mu_O(E) = 0$. Clearly, the set $E$ is $\Gamma$-invariant, and by Lemma 8.2 it is measurable. Therefore it has either zero or full $\mu_O$-measure.

Suppose that $E$ has full $\mu_O$-measure. Then, a $\mu_O$-measurable map is well defined by

$$f : \partial \mathbb{H}^k \to (\partial \mathbb{H}^n \times \partial \mathbb{H}^n) / \mathcal{S}_2$$

$$z \mapsto \text{support of } \lambda_z.$$ 

By Lusin theorem (see for example [1]) for all $\varepsilon > 0$ there exists a compact set $A \subset \partial \mathbb{H}^k$, with $\mu_O(\partial \mathbb{H}^k \setminus A) < \varepsilon$ and such that the restriction of $f$ to $A$ is continuous. By Lemma 8.4, for any density-point $x$ of $A$, for $\mu_O \times \mu_O$-almost all $(z_1, z_2) \in \partial \mathbb{H}^k \times \partial \mathbb{H}^k$ there exists a sequence $\{\gamma_i\} \subset \Gamma$ such that for $j = 1, 2$

$$\gamma_i(z_j) \in A \quad \text{and} \quad \gamma_i(z_j) \to x.$$ 

Therefore, for $j = 1, 2$

$$f(\gamma_i(z_j)) \to f(x).$$ 

Up to pass to sub-sequences, $\rho(\gamma_i)$ converges either to an isometry of $\mathbb{H}^n$ or to a quasi-constant $C^b_a$ - a quasi-constant is a map such that $C^b_a(p) = b$ for all points $p \neq a$ of $\mathbb{H}^n$ - where the convergence is uniform on compact sets not containing $a$ (see for example [15]).

Let $\{\theta, \omega\}$ be the support of $\lambda_x$, that is $\lambda_x = \frac{\delta_a + \delta_\omega}{2}$, and for $j = 1, 2$ let $\{\xi_j, \zeta_j\}$ be the support of $\lambda_{z_j}$.

If $\rho(\gamma_i)$ converges to an isometry $\psi$, then for $j = 1, 2$

$$f(\gamma_i(z_j)) = \rho(\gamma_i)\{\xi_j, \zeta_j\} \to \{\psi(\xi_j), \psi(\zeta_j)\}.$$ 

Since $f(\gamma_i(z_j)) \to f(x) = \{\theta, \omega\}$, we get $f(z_1) = f(z_2)$. On the other hand, if $\rho(\gamma_i) \to C^b_a$ and if $\xi_j \neq a \neq \zeta_j$, we get

$$f(\gamma_i(z_j)) \to \{b, b\} \neq f(x).$$
We have so proved that for $\mu_O \times \mu_O$-almost all $(z_1, z_2) \in \partial \mathbb{H}^k \times \partial \mathbb{H}^k$ the supports of $\lambda z_1$ and $\lambda z_2$ share at least one point. Whence, using Fubini's theorem and Lemma 8.4, it follows that there exists $\zeta \in \partial \mathbb{H}^n$ such that for $\mu_O$-almost all $z \in \partial \mathbb{H}^k$, the support of $\lambda z$ contains $\zeta$. For $z \in \partial \mathbb{H}^k$, let $\xi(z)$ denote the other point of the support of $\lambda z$, that is

$$f(z) = \{\zeta, \xi(z)\}.$$ 

For each $\gamma \in \Gamma$, for $\mu_O$-almost all $z$ we have $f(z) = \{\zeta, \xi(z)\}$ and

$$\{\rho(\gamma)\zeta, \rho(\gamma)\xi(z)\} = \rho(\gamma)(f(z)) = f(\gamma z) = \{\zeta, \xi(\gamma z)\}.$$ 

Whence we have

Either: the set $\{\zeta\}$ is $\rho(\Gamma)$-invariant.

Or: there exists $\gamma \in \Gamma$ such that $\rho(\gamma)\zeta \neq \zeta$, which implies that $\xi(z) = \rho(\gamma)^{-1}(\zeta)$ does not depend on $z$. Therefore the set $f(z)$ does not depend on $z$ and it is $\rho(\Gamma)$-invariant.

In both cases the image of $\rho$ is elementary, which contradicts the hypotheses. It follows that the set $E$ cannot have full $\mu_O$-measure. \hfill $\square$

Theorems 1.6 and 1.7 now easily follow.

Proof of Theorem 1.6. — It is well-known (see for example [15]) that if $\mathbb{H}^k/\Gamma$ is a complete hyperbolic manifold of finite volume, then $\delta(\Gamma) = k - 1$ and $\Lambda_c$ has full-measure. Then, by Lemma 8.1, the hypotheses of Theorem 8.5 are satisfied, and the claim follows from Theorem 1.5. \hfill $\square$

Proof of Theorem 1.7. — The first claim is an immediate corollary of Theorems 1.5 and 8.5. Now, let $\varphi = F \circ G : \partial \mathbb{H}^k \to \partial \mathbb{H}^k$. The map $\varphi$ is clearly $\Gamma$-equivariant and $\mu_O$-measurable. Since the identity is also $\Gamma$-equivariant and measurable, by uniqueness it follows that $F \circ G = \text{Id}_{\partial \mathbb{H}^k}$. The same holds replacing $F \circ G$ with $G \circ F$. \hfill $\square$

In particular this extends (and provides a new proof of) the following result.

Theorem 8.6 ([9, 20, 25]). — Let $M$ be a compact hyperbolic 3-manifold fibred over the circle with fibre a surface $F$. Let $\pi_1(F) \simeq \Gamma < \text{Isom}(\mathbb{H}^2)$ and let $\rho : \Gamma \to \text{Isom}(\mathbb{H}^3)$ be the representation induced by the inclusion $\pi_1(F) \hookrightarrow \pi_1(M)$. Then, there exist measurable maps $F : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3$ and $G : \partial \mathbb{H}^3 \to \partial \mathbb{H}^2$ which are respectively $\rho$ and $\rho^{-1}$ equivariant. Moreover, almost everywhere

$$F \circ G = \text{Id}_{\mathbb{H}^2} \quad G \circ F = \text{Id}_{\mathbb{H}^3}.$$
Proof. — Clearly, $\Gamma$ satisfies the hypotheses of Theorem 1.7. Moreover, by Lemma 8.1 and [8, Corollary 2], also $\pi_1(F) < \pi_1(M) < \text{Isom}(\mathbb{H}^3)$ satisfies the hypotheses of Theorem 1.7. 

9. Convergence of Cannon-Thurston maps

We begin this section by describing an example of a converging sequence of representations whose Cannon-Thurston maps have no Cauchy subsequence with respect to the uniform convergence. Then, we prove Theorem 1.8 (point-wise convergence almost everywhere.)

Example 9.1 (Souto). — Let $\Gamma$ be the fundamental group of a compact hyperbolic surface. Then, there exists a sequence of discrete and faithful representations $\rho_n : \Gamma \to \text{Isom}(\mathbb{H}^3)$ such that, if $f_n$ and $f$ denote the corresponding Cannon-Thurston maps, then:

- $\rho_n(\Gamma)$ is quasi-Fuchsian
- $\rho_n \to \rho$
- $\rho(\Gamma)$ is geometrically finite
- There is a converging sequence of points $x_n \to x \in \partial \mathbb{H}^2$ such that $f_n(x_n) \to y \neq f(x)$. In particular, no sub-sequence of $\{f_n\}$ converges uniformly to $f$.

To see that, let $AH$ denote set of discrete and faithful representations $\rho : \Gamma \to \text{Isom}(\mathbb{H}^3)$ with the topology of algebraic convergence. Let $\rho$ be a geometrically finite representation in the closure of $AH$ (whence $\rho$ has accidental parabolics.) The domain of discontinuity of $\rho(\Gamma)$ is therefore non-empty, and we can thus pick a point $y$ in it. Since the doubly degenerate representations are dense in the closure of $AH$, there exists a sequence $\psi_n \to \rho$ with the property that the limit set of $\psi_n(\Gamma)$ is the whole sphere $\partial \mathbb{H}^3$. Let $f_n$ and $f$ be the Cannon-Thurston maps corresponding to $\psi_n$ and $\rho$. By equivariance, there exists a sequence of points $x_n \in \Lambda(\Gamma)$ such that $x_n \to x \in \Lambda(\Gamma)$ and $f_n(x_n) \to y$. Since $y \notin \Lambda(\rho(\Gamma))$ we cannot have $f_n(x_n) \to f(x)$. Finally, one can approximate the $\psi_n$’s with quasi-Fuchsian representations $\rho_n$ getting the requested properties.

Proof of Theorem 1.8. — For $z \in \Lambda(\Gamma)$ let $\delta_{f_i(z)}$ be the Dirac measure concentrated on $f_i(z)$. For $x \in \mathbb{H}^k$ consider the measures on $\partial \mathbb{H}^k \times \partial \mathbb{H}^n$ defined by

$$\eta_{x,i} = \mu_x \times \{\delta_{f_i(z)}\}.$$
Up possibly to pass to sub-sequences, the measure \( \eta_{O,i} \) have a weak limit \( \eta_O \), which we disintegrate as

\[
\eta_O = \mu_O \times \{ \beta_z \}
\]

via some family of probability measures \( \{ \beta_z \}_{z \in \Lambda(\Gamma)} \). Let now

\[
\eta_x = \mu_x \times \{ \beta_z \}.
\]

It is readily checked that \( \eta_{x,i} \) weakly converges to \( \eta_x \) (because \( d\mu_x(\theta) = e^{-\delta(\Gamma)B_O(x,\theta)} d\mu_O(\theta) \)). It follows that the measures \( \eta_x \) are \( \rho \)-equivariant. Indeed, for any fixed \( \gamma \in \Gamma \) we have \( \eta_{\gamma x,i} = (\gamma, \rho_i(\gamma)) \ast \eta_{x,i} \), and \( \rho_i(\gamma) \) converges to \( \rho(\gamma) \) uniformly on \( \partial \mathbb{H}^n \).

As in Lemma 4.4 this implies that the family \( \beta_z \) is \( \rho \)-equivariant, and uniqueness part of Theorem 1.5 implies that in fact \( \beta_z = \delta_{f(z)} \).

Now that we know that \( \eta_x = \mu_x \times \{ \delta_{f(z)} \} \), the proof is completed by the fact that the weak convergence of \( \eta_{x,i} \) to \( \eta_x \) is equivalent to the convergence of \( f_i \) to \( f \) almost everywhere (see for example [2, Lemma 2.3].) \( \square \)

**BIBLIOGRAPHY**


