AN EASY (HORIZONTAL) WALK THROUGH FAKE OCTAGONS

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Abstract. A fake octagon is a genus two translation surface with only one singular point and the same periods as the octagon. Existence of infinitely many fake octagons was established first by McMullen [12] in 2007, and more generally follows from dynamical properties of so called isoperiodic foliation.

The purpose of this note is to describe an infinite family of fakes constructed by means of elementary methods. We describe an easy cut-and-paste surgery and show that the n^{th} iterate of that surgery is a fake octagon Oct_n. Moreover we shows that $\mathrm{Oct}_n \neq \mathrm{Oct}_m$ for $n \neq m$, and that any Oct_n can be approximated arbitrarily well by some other Oct_m. This note is intended to be elementary and fully accessible to non-expert readers.

1. Introduction

Bibliography on translation surfaces is immense, we cite here only the celebrated handbooks of dynamical systems (see for instance [5, 6, 8, 10, 11]), the nice survey [14], as well as [16] and [17], and references therein. Also, we refer to Section 2 for precise definitions, staying colloquial in this introduction.

The translation surface obtained by gluing parallel sides of a regular octagon is commonly known as "the octagon". A fake octagon is a translation surface with one singular point and the same periods as the octagon.

It is well known that periods are local coordinates for the moduli space of translations surfaces of fixed genus and singular divisor. Periods comes in two flavours: absolute and relative: former ones are translation vectors associated to closed loops, the latter are those associated to saddle connections (i.e. path connecting singular points). So-called isoperiodic deformations consist in changing relative periods without touching absolute ones. Isoperiodic loci are leaves of the isoperiodic foliation (also known as absolute period foliation or kernel foliation). Local coordinates on isoperiodic leaves are given by positions of singular points with respect to a fixed singular point, chosen as origin. As a consequence, translation surfaces of the minimal stratum (that is, with a unique singular point) cannot be continuously and isoperiodically deformed in that stratum (all periods are absolute).

A priori, it is not therefore clear whether or not, given X in the minimal stratum, there is a translation surface, still in the minimal stratum, with same periods as X. If any, such surfaces are called "fake X". In fact, the question of finding fakes of famous translation surfaces, as for instance the octagon, was a nice coffee-break problem in dynamical system conferences some years ago. Nowadays, this is literature.

Fakes where introduced and studied by McMullen in [12, 13] — who gave a complete and detailed description of isoperiodic leaves in genus two — and dynamical properties of isoperiodic foliation where established in [3, 7] in general (in particular ergodicity and classification of leaf-closures).

From [12, 13, 3, 7] it follows in particular that if periods of X are not discrete (e.g. the octagon), then X has infinitely many fakes. More precisely, the isoperiodic leaf through

X intersects the minimal stratum $\mathcal{H}_{2g-2}$ in a set whose closure has positive dimension. In particular, any such X can be approximated by fakes.

The purpose of this note is to give easy proves of such results for the particular case of the octagon by using elementary methods; where "easy" means "explicable in a conference coffee-break". The "elementary methods" we use are surgeries that are the topological viewpoint of the so-called Schiffer variations. Given the octagon, we describe a surgery (that we call "left-surgery") that produces a fake octagon and that can be iterated. We will then prove that all fakes produced by iterating left-surgeries are in fact different form each other, exhibiting therefore an explicit infinite family of fakes octagons. Also, we will show that any fake of the family can be arbitrarily approximated by iterations. Finally, we note that all our fakes are along an "horizontal" line of the isoperiodic leaf of the octagon: the Schiffer variations are always in the horizontal direction. Finally, we discuss ingredients needed for possible generalisations. Our main result is summarised as follows:

Theorem (Theorem 4.3, Remark 4.4)**.** *Fake octagons obtained by iterated left-surgeries on the octagon are different from each other, and any such fake can be arbitrarily approximated by iterates.*

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2. Isoperiodic foliation and fakes

Translation structures on closed, connected, oriented surfaces can be defined in many different ways, for instance:

- They can be viewed as Euclidean structures with cone-singularities of cone-angles multiple of 2π, up to isometries that reads as translations in local charts. Equivalently, they are branched $\mathbb{C}$-structures whose holonomy consits of translations, where "branched" means that the developing map is not just a local homeomorphism but can also be a local branched covering;
- or as pairs (X, ω) where X is a Riemann surface and ω a holomorphic 1-form, up to biolomorphisms;
- or quotients of poligons in $\mathbb{C}$ via gluings that identify pairs of parallel edges via translations, up to suitable "tangram" relations.

Third construction clearly produces a Euclidean structure with cone-singularities, which, by pulling back the structure of $(\mathbb{C}, dz)$ produces a complex structure together with a 1-form (whose zeroes correspond to cone-singularities). In fact, it turns out that all viewpoints are equivalent (we refer to [14] for more details). Any singular point as an order: if viewed as a cone-point, then it has order d if the total angle is $2\pi + 2\pi d$; if viewed as a zero of ω, then it has order d if locally $\omega = z^d dz$.

As usual, we will refer to a surface endowed with a translation structure as a *translation surface*. Singular points are also referred to as *saddles*.

If a translation surface has genus g, then by Gauss-Bonnet (or by a characteristic count) the sum of the orders of singular points is $2g - 2$.

The moduli space of translation surface of genus g — that we denote simply by $\mathcal{H}$ if there is no ambiguity on the genus — is naturally stratified by the singular divisor: if κ is a partition of $2g-2$ (more precisely a list of non increasing positive integers summing up to $2g-2$) then the stratum $\mathcal{H}(\kappa)$ consists of all translation surfaces whose singular points have orders as prescribed by κ. For example, in genus $g = 2$ there are only two strata: the principal, or generic, stratum $\mathcal{H}_{1,1}$ — consisting of translation surfaces with two simple singular points (with cone-angles 4π each) — and the minimal stratum $\mathcal{H}_2$ — consisting of translation surfaces having only one singular point of cone-angle 6π. It turns out that any stratum is a complex orbifold of dimension $2g+s-1$ where $s=|\kappa|$ is the number of singular points.

A part obvious issues due to orbifold structure, periods give coordinates on any stratum. More precisely, if S is a translation surface with singular locus $\Sigma=\{x_1,\dots,x_s\}$, then we consider the relative homology $H_1(S,\Sigma;\mathbb{Z})$. If $\gamma_1,\dots,\gamma_{2g}$ is a basis of $H_1(S;\mathbb{Z})$ and $\eta_2,\dots,\eta_s$ are arcs connecting x_1 to $x_2,\dots,x_s$, then the family $\gamma_1,\dots,\gamma_{2g},\eta_2,\dots,\eta_s$ is a basis of $H_1(S,\Sigma;\mathbb{Z})$. By using the (X,ω) viewpoint of translation surface, the period map

$$(X,\omega)\mapsto (\int_{\gamma_1}\omega,\dots,\int_{\gamma_{2g}}\omega,\int_{\eta_2}\omega,\dots,\int_{\eta_s}\omega)$$

is a local chart $\mathcal{H}(\kappa)\to\mathbb{C}^{2g+s-1}$. These are the so called **period coordinates**. In other words, we consider $[\omega]\in H^1(S,\Sigma;\mathbb{C})$. Periods of curves γ_i's are usually called **absolute periods**, while those of η_i's are **relative periods**.

There is a natural period map $Per:\mathcal{H}\to\mathbb{C}^{2g}=H^1(S;\mathbb{C})$ that associates to any translation surface its absolute periods

$$Per:(X,\omega)\mapsto (\int_{\gamma_1}\omega,\dots,\int_{\gamma_{2g}}\omega)$$

The so-called **isoperiodic foliation** $\mathcal{F}$ (also known as *kernel foliation* or *absolute period foliation*) is the foliation locally defined by the fibers of Per. Namely, two translation surfaces are in the same leaf of $\mathcal{F}$ if they can be continuously deformed one into the other without changing absolute periods. The isoperiodic foliation is globally defined in $\mathcal{H}=\cup_\kappa\mathcal{H}(\kappa)$, and its leaves have dimension $2g-3$. Isoperiodic foliation has been extensively studied, for instance in [12, 13, 3, 7, 1, 9, 15].

One of the problems in studying isoperiodic foliation, is to determine the foliation induced by $\mathcal{F}$ on each stratum. For instance, in the minimal stratum $\mathcal{H}_{2g-2}$ there is no room for deformations: locally, any leaf of $\mathcal{F}$ intersects transversely such stratum in a single point. Given $X\in\mathcal{H}_{2g-2}$, a "**fake** X" is a translation surface, different from X, but with same absolute periods as X and only one singular point, that is to say, if F_X is the leaf of $\mathcal{F}$ through X, then a "fake X" is a point in $F_X\cap\mathcal{H}_{2g-2}$.

Example 2.1. The so-called *octagon* is the translation surfaces obtained by gluing parallel sides of a regular octagon sitting in $\mathbb{C}$ with an edge in the segment $[0,1]$. It is a genus two surface with a single singular point. A *fake octagon* is an intersection point of the isoperiodic leaf of the octagon with the minimal stratum $\mathcal{H}_2$, i.e. any translation surface with the same (absolute) periods as the octagon (the same area) and only one singular point.

3. Traveling isoperiodic leaves by moving singular points

If X has s singular points, then there are $s-1$ degrees of freedom for perturbing X without changing its absolute periods (we can change the relative periods of $\eta_2,\dots,\eta_s$). It turns out that local parameters are exactly the positions of singular points; more precisely, the relative positions of $x_2,\dots,x_s$ with respect to x_1. So we can travel the isoperiodic leaf through X by "moving" singular points. From an analytic viewpoint such moves are known as Schiffer variations. We adopt here a more topological cut-and-paste viewpoint. We briefly recall the basic construction, referring to [3, 2] for a more detailed discussion.

Let x be a singular point and let γ be a segment, or more generally a path, starting at x. If x has degree d, then γ has d **twins**, that is to say, paths starting at x with the same developed image as γ (by simplicity we assume here that none of such twin contains a saddle in its interior). Explicitly, if γ is a segment, its twins are segments forming angles $2\pi, 4\pi, \dots, d2\pi$ with γ. For any twin of γ we can perform a cut-and-paste surgery as follows: We cut along γ and the chosen twin, and then we glue in the unique other way coherent with orientations. This is better described in Figure 1.

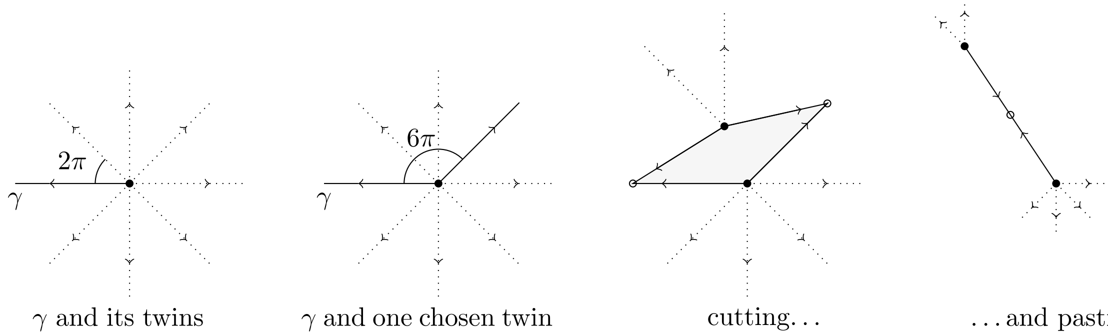


FIGURE 1. Moving singular points via cut-and-paste surgeries

A first remark on that surgery, is that endpoints of γ and the twin can be both regular, both singular, or one regular and the other singular point. Given the angles at endpoints, and the angle between γ and its twin, we can easily recover angles after the surgery (see Figure 2):

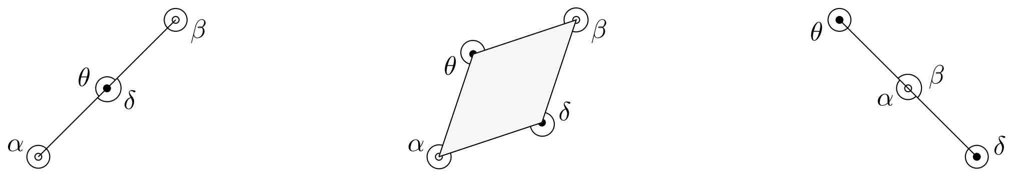


FIGURE 2. Angles before and after surgery

In Figure 2, before the surgery the full-dotted singular point has total angle $\theta + \delta$, and after it splits in two points. The two empty-dotted points paste together to form a point of total angle $\alpha + \beta$. All $\alpha, \beta, \theta, \delta$ are multiple of 2π (they are 2π precisely when the corresponding point is regular).

Note that our surgeries take place locally, near a singular point. It follows that they do not affect absolute periods (wile clearly they affect relative periods). It turns out that these moves are the only way to isoperiodically deform a translation surface. (See [3, 2]).

It maybe useful to remark at this point that such surgeries may or may not preserve strata. With notations as in Figure 2, if $\alpha, \beta, \theta, \delta$ are all 2π, then what we are doing is to move a singular point from the starting point of γ to its endpoint (in this case the stratum does not change).

If $\delta, \theta > 2\pi$, and $\alpha, \beta = 2\pi$, then we are splitting a singular point in two separate singular points and creating a singular point of angle 4π. (The sum of resulting degrees equals that of initial ones). So in this case we are changing stratum.

Similarly, if for instance $\alpha = 4\pi$, and $\theta, \delta, \beta = 2\pi$, the surgery collapses together two singular points, hence again changing stratum. There are more possibilities, and other kind of surgeries are possible (for instance by cut and pasting along many twins simultaneously). We refer the interested reader to [2, 3] for further details.

The last needed remark, is that it may happen that γ is a loop, starting and ending at the same point. In this case twins of γ may or may be not loops, and conversely. Also, it can even happen that γ is embedded, but the twin is not. In such cases some topological disaster may happen (the surgery could for instance disconnect the surface) and one has to check what happens carefully.

We will use surgeries where γ is a closed **saddle connection**, that is to say a straight segment starting and ending at the same singular point, but we will always require that twins of γ are embedded segments. It is readily checked that in this case no disasters occur. We refer to such a cut-and-paste as **saddle connection surgery**. See Figure 3.

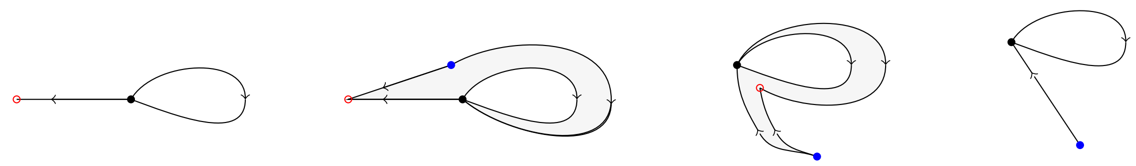


FIGURE 3. A saddle connection surgery

Remark 3.1. If X is in $\mathcal{H}_{2g-2}$, then a saddle connection surgery produces a translation with the same absolute period of X. If in addiction the angle between the closed saddle connection and the chosen twin is exactly 2π, then the resulting surface is in $\mathcal{H}_{2g-2}$ (the full-dotted blue point in Figure 3 is a regular point). So, if different from X, it is a fake X. Moreover, the closed saddle connection used by the surgery, remains a closed saddle connection of the same length and direction after the surgery.

4. ITERATED SURGERIES ON THE OCTAGON

In this section we describe a sequence of fake octagons Oct_n obtained from the octagon $\mathrm{Oct} = \mathrm{Oct}_0$ via a sequence of saddle connection surgeries. In particular, each surgery will be a saddle connection surgery along a fixed closed saddle connection. We will then prove that all fakes Oct_n are in fact different from each other.

We parameterise our octagon by gluing parallel sides of two polygons as in Figure 4. Edges have length one, all vertices are identified to each other and form the unique singular point.

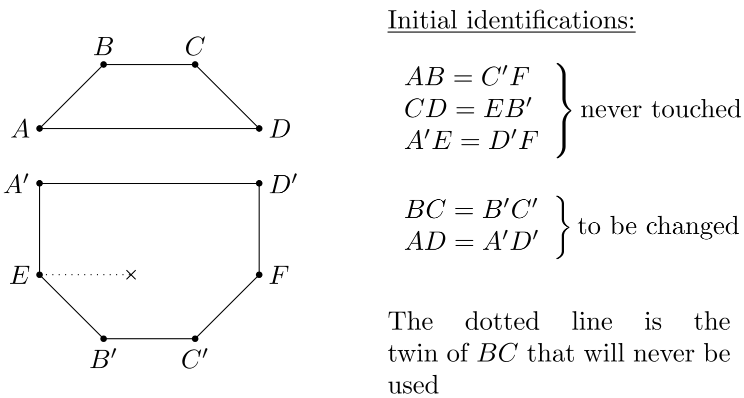


FIGURE 4. The octagon

The octagon has three horizontal (closed) saddle connections. Only one, which in the picture is BC, has length 1, and the other two AD, EF have length $1+\sqrt{2}$. This property will be preserved by all saddle connection surgeries. We therefore describe our surgeries from an intrinsic viewpoint, exploiting this property.

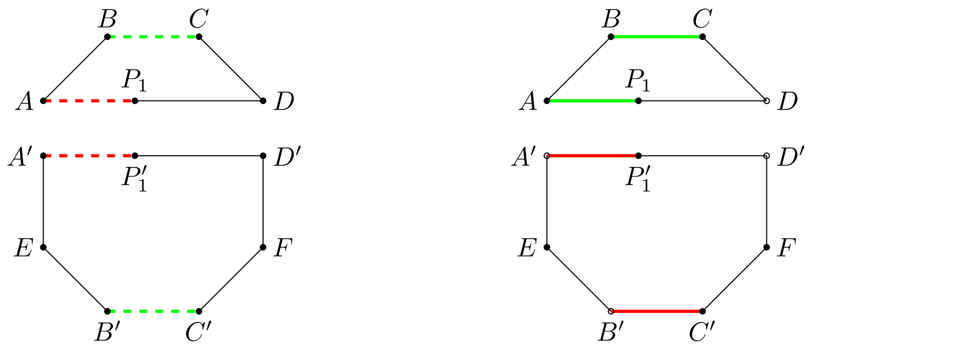
Let γ be the unique unitary horizontal closed saddle connection, being the other two of length $1+\sqrt{2}$. By definition of twin, the two twins of γ are sub-segments of those longer saddle connections. Since γ is horizontal, the end of γ forms with the start of γ an angle which is an odd multiple of π. In fact for the octagon that angle is 3π. Since the total angle around the singular point is 6π, then the twins of γ form angles $\pm\pi$ with respect to the end of γ. We orient γ from left to right, and name γ_L be the twin on the "left side", that is to say, the angle measured clockwise from the end of γ to γ_L is π. Let γ_R be the other twin. We define **left surgery** the saddle connection surgery along γ and γ_L, and **right surgery** that along γ and γ_R. (See also Figure 5). The angle between γ_L (or γ_R) and γ is exactly 2π, so left and right surgeries produce elements of $\mathcal{H}_2$ (see Remark 3.1) It is immediate to check that the inverse of a left surgery is a right surgery along γ^{-1}.

It will be clear from what follows that left and right surgeries preserve the two properties of having one unitary horizontal saddle connection (and two of length $1+\sqrt{2}$), and that the angle between the start and the end of γ is 3π. Therefore, we can iterate left and right surgeries.

Definition 4.1. For $n \in \mathbb{Z}$ we define Oct_n as the translation surface obtained from the octagon Oct_0 by n left surgeries (for negative n we apply right surgeries).

Before giving a global description of Oct_n, we start by looking in details at first steps. Coming back to pictures, left surgeries will always affect the horizontal saddle connection $\gamma = BC$ and its twin on the line AD. Specifically, the twin of BC along EF will never come in play. Also, we never change diagonal identifications $AB = C'F, CD = EB'$, nor the vertical one $A'E = D'F$.

Let's start. We cut and paste along BC and its twin on the line AD. See Figure 5.



First cut along BC and its twin AP_1

New identifications: $BC = AP_1, B'C' = A'P_1', P_1D = P_1'D'$

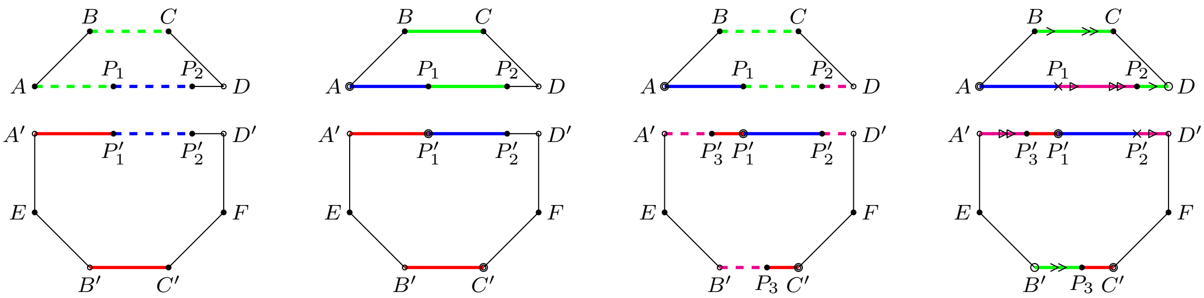
FIGURE 5. First left surgery: first fake Oct_1.

In that picture, dashed lines mean cuts, i.e. segments that where previously identified and are no longer identified. Colours visualise new identifications. Note that after the surgery, not all vertices are identified to each other. In particular, $A' = B' = D = D'$ is a regular point. All other vertices are identified, give rise to the unique singular point, and the result is indeed a fake octagon: it is our Oct_1. We will label with a full dot the singular point, and with other symbols those other vertices that are regular points (we use same label for vertices that are identified). Also, we will use the "dot" notation for concatenation of segments, e.g. "$XY \cdot ZT$" denotes the concatenation of segments ZT after XY, clearly this makes sense only if Y is identified with Z.

When we cut the twin of BC (oriented as BC) we see two avatars of it in the picture: one with the surface on its left, and one on its right. We denote by P_1 the endpoint of the cut having the surface on its left side, and P_1' the other.

After the surgery, the saddle connection BC has again two twins, one emanating from P_1 along the line P_1D and another emanating from E.

We then obtain Oct_2 via a second left surgery, cutting and pasting along BC and its twin on the line P_1D. See Figure 6 (left side). As above, when cutting along that twin, we denote by P_2 the endpoint of the cut having the surface in its left side, and P_2' the other.



Second surgery:

On the left the cut along BC and its twin P_1P_2.

On the right the new identifications:
$(B'C' = A'P_1', P_2D = P_2'D'$, and)
$BC = P_1P_2, AP_1 = P_1'P_2'$.

Third surgery:

On the left the cut along BC and its twin $P_2D{\cdot}B'P_3$.

On the right the new identifications:
$(AP_1 = P_1'P_2', P_3'P_1' = P_3C'$, and)
$BC = P_2D \cdot B'P_3, P_1P_2 = P_2'D' \cdot A'P_3'$.

FIGURE 6. Second and third fakes Oct_2 and Oct_3.

One more left surgery, along BC and its twin emanating from P_2, will produce Oct_3. See Figure 6 (right side). Again, P_3 and P_3' are the endpoints of the cut of the twin having the surface on the left and right side respectively.

We are now ready to describe the gluing pattern of Oct_n. For this purpose it is more convenient to pass to a simpler — even if less "octagonal" — viewpoint. Namely, we glue the upper quadrilateral to the bottom one, by identifying sides AB and $C'F$. See Figure 7.

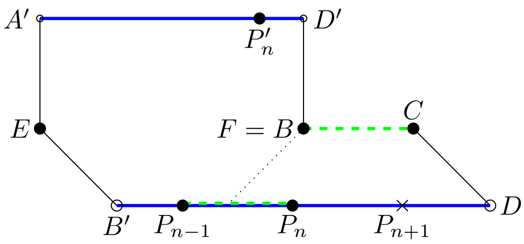


FIGURE 7. A less "octagonal" viewpoint. P_n is identified with P_n'. $P_{n-1}P_n$ is where BC is glued at step n, while P_nP_{n+1} is the next twin we cut at step $n+1$. Segment $P_nD \cdot B'P_{n-1}$ is identified with $P_n'D' \cdot A'P_n'$. This is the n^{th} fake Oct_n.

Horizontal gluings are determined, once we know positions of points P_n and P_n', as follows. Since B' is identified with D, segment $B'D$ can be parameterised by a circle of length $2+\sqrt{2}$. Points P_{n-1} and P_{n+1} are the points of the circle $B'D$ at distance 1 from

P_n, respectively on the left and right side of P_n. At step n, segment BC is identified with $P_{n-1}P_n$ — this is the unique unitary horizontal saddle connection — and segment $P'_nD'\cdot A'P_n$ is identified with P_nP_{n-1} (which, in Figure 7, is the concatenation of segments $P_nD \cdot B'P_{n-1}$), the latter being a horizontal saddle connection of length $1+\sqrt{2}$. The third horizontal saddle connection, namely EF, is never involved and always has length $1+\sqrt{2}$. The unique singular point is $P_{n-1} = P_n = P'_n = B = C = E$, and a quick check shows that the angle between the start and the end of the unitary closed horizontal saddle connection is 3π.

The twin of BC that will be used in next surgery is P_nP_{n+1} (which is identified with the corresponding segment starting from P'_n), and it is readily checked that a left surgery along BC and its twin P_nP_{n+1} produces again a configuration of the same type, with different positions of P_n and P'_n:

If we parameterise $B'D$ with $[0, 2+\sqrt{2}]$ and $A'D'$ with $[0, 1+\sqrt{2}]$, then we have

$$P_n \equiv n+1 \mod (2+\sqrt{2}) \qquad\qquad P'_n \equiv n \mod (1+\sqrt{2}).$$

Remark 4.2. Pictures only help in calculations, but left surgeries are intrinsically defined: any of our fakes has three horizontal saddle connections, and only one of them has length one. At any step we cut and paste along that saddle connection and its left twin. This receipt is "picture free".

Theorem 4.3. *If $n \neq m$, then $\mathrm{Oct}_n \neq \mathrm{Oct}_m$.*

Proof. The invariant that distinguishes fakes octagons from each other is the systole, namely the (family of) shortest saddle connection(s). As the octagon has edge of length one, the systole is always not longer than one. In fact, the shortest saddle connections for the true octagon all have length one, and because the irrationality of $\sqrt{2}$ this never happen again. Looking at Figure 7 we see that systoles are necessarily segments connecting some avatar of the singular point (i.e. $P_{n-1}, P_n, P'_n, E, B, C$). Point P'_n always has distance at least one from other singular points, so no systole starts from P'_n in Figure 7. Moreover, since the quadrilateral $P_{n-1}P_nCB$ is a parallelogram, for $n \neq 0$, we have three possible families of fakes octagons, determined by the position of P_n in $B'D = [0, 2+\sqrt{2}]$ (see Figure 8):

(1) $P_n \in (1, 1+\frac{1+\sqrt{2}}{2})$. The unique systole is the segment P_nB.
(2) $P_n \in (1+\frac{1+\sqrt{2}}{2}, 2+\frac{1+\sqrt{2}}{2})$. There are two systoles: $P_{n-1}B$ and P_nC.
(3) $P_n \in (0,1) \cup (2+\frac{1+\sqrt{2}}{2}, 2+\sqrt{2})$. In this case the unique systole is $P_{n-1}C$.

Since $2+\sqrt{2}$ is irrational and $P_n \equiv n+1 \mod (2+\sqrt{2})$, the possible positions of P_n on $B'D$ identified with $[0, 2+\sqrt{2}]$, form an infinite dense set. It follows that the set of lengths of systoles of the family $\{\mathrm{Oct}_n; n \in \mathbb{Z}\}$ is an infinite set. Hence, the family of fakes $\{\mathrm{Oct}_n : n \in \mathbb{Z}\}$ contains infinitely many different fakes.

Suppose now that there is n, m such that $\mathrm{Oct}_n = \mathrm{Oct}_m$. Then (by Remark 4.2) in this case, also $\mathrm{Oct}_{n+i} = \mathrm{Oct}_{m+i}$ for any i, and so we would observe a $m-n$ periodic behaviour. In particular we would have only finitely many fakes among our Oct_n's. But, since we already proven that we have indeed infinitely many different fakes, this cannot happen. It follows that for any $n \neq m$ we have $\mathrm{Oct}_n \neq \mathrm{Oct}_m$. $\square$

Remark 4.4. The fact that the possible positions of P_n in $[0, 2\sqrt{2}]$ form an infinite dense set, implies in particular that all possibilities described in Theorem 4.3 actually arise. Another consequence is that we can find fakes Oct_n arbitrarily close to the octagon Oct_0, and in general that for any Oct_m there is a fake Oct_n arbitrarily close to, but

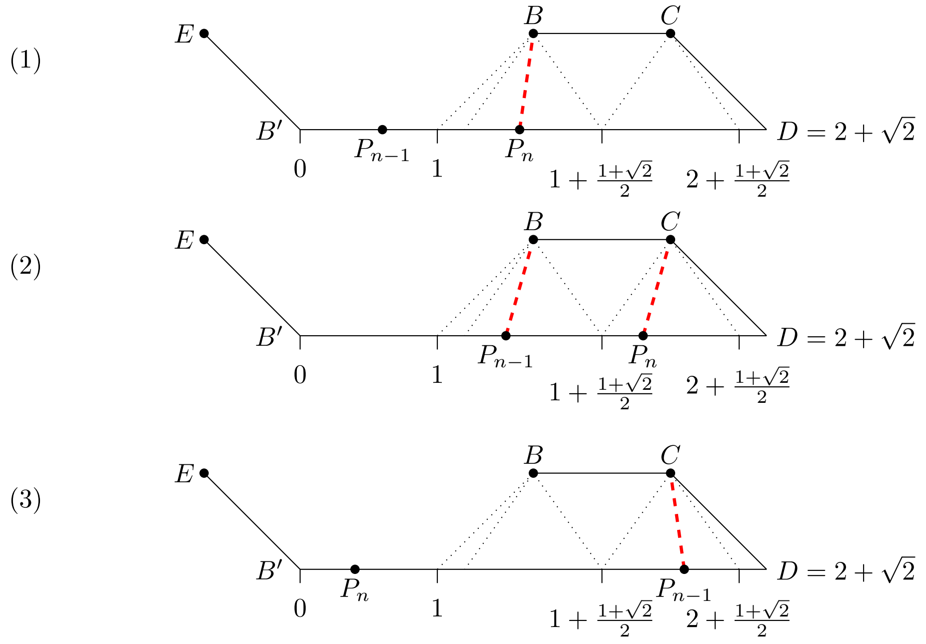


FIGURE 8. The three possible systole configurations.

different from, Oct_m. This is nothing but a manifestation of general density phenomena described in [3] and anticipated in Introduction.

Remark 4.5. Even if each any Oct_n is different from each other, the systoles may have the same length. For instance, if $1+\frac{\sqrt{2}-1}{2} < P_n < 1+\frac{\sqrt{2}+1}{2} \mod (2+\sqrt{2})$, then $\mathrm{Oct}_n, \mathrm{Oct}_{n+1}$, and Oct_{n+2} have the systole(s) of the same length (the three being in families (1), (2), (3) respectively).

This is basically all that can happens.

Proposition 4.6. *For any* Oct_m *(with* $m \neq 0$*) there is* Oct_n *with the same systole length and in family* (1)*, more precisely with* $P_n \equiv x \in (1, 1+\frac{\sqrt{2}}{2}) \mod (2+\sqrt{2})$*. Moreover,*

- *if* $P_n \in (\frac{1+\sqrt{2}}{2}, 1+\frac{\sqrt{2}}{2}) \mod (2+\sqrt{2})$*, then* Oct_m *has the same systole-length of* Oct_n *if and only if* $m = \pm n, \pm n+1, \pm n+2$*;*
- *if* $P_n \in (1, \frac{1+\sqrt{2}}{2}) \mod (2+\sqrt{2})$*, then* Oct_m *has the same systole-length of* Oct_n *if and only if* $m = n$ *or* $m = -n+2$*.*

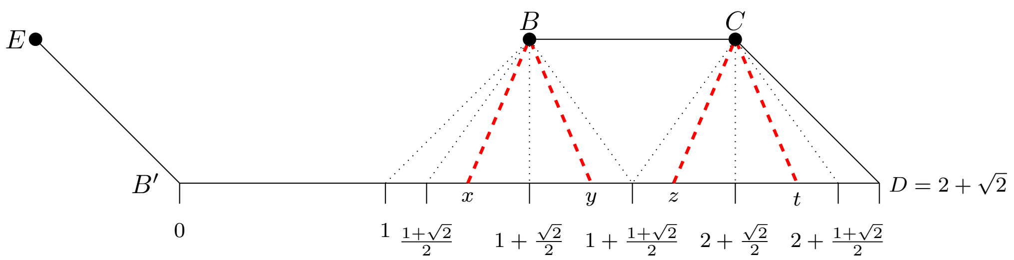


FIGURE 9. Positions having the same distance form B or C

Proof. For $x \in [0, 2+\sqrt{2}]$ let $y = y(x)$ be its symmetric with respect to $1+\sqrt{2}/2$. This is the unique other point so that $d(x, B) = d(y, B)$. Explicitly, y is determined by

$$\frac{x+y}{2} = 1 + \frac{\sqrt{2}}{2} \qquad \text{whence} \qquad x + y = 2 + \sqrt{2}.$$

Let $z = z(x) = x + 1$ and $t = t(x) = y(x) + 1$. Those are the unique points so that $d(x, B) = d(z, C) = d(t, C)$. Note that

$$x \equiv -y \equiv z - 1 \equiv -t + 1 \mod (2 + \sqrt{2}). \tag{1}$$

Such equations have integer coefficient and $2+\sqrt{2}$ is irrational. So, if we want to solve them in $\mathbb{Z}$, they reduce to genuine equalities. Namely, if $x \equiv P_n \equiv n+1 \mod (2+\sqrt{2})$ and $y \equiv P_m \equiv m+1 \mod (2+\sqrt{2})$, then $x \equiv -y \mod (2+\sqrt{2})$ if and only if $m = -n$, and similarly for points z, t.

The first consequence of this fact is that if P_m is placed in $(1+\frac{\sqrt{2}}{2}, 2+\sqrt{2})$, then there is n such that P_n is placed in $x \in (1, 1+\frac{\sqrt{2}}{2})$ (hence Oct_n is in family (1)) and P_m is either a y- or z- or t-point for x. In particular, this proves the first claim.

We may therefore assume that we have Oct_n in family (1) and search for all possible Oct_m with the same systole-length.

From the fact that congruences 1 reduces to genuine identities on $\mathbb{Z}$, we can now deduce second claims.

If $P_n \equiv x \in (\frac{1+\sqrt{2}}{2}, 1+\frac{\sqrt{2}}{2}) \mod (2+\sqrt{2})$, then the possibility for Oct_m to have the same systole-length as Oct_n are two for each family, and precisely:

- Oct_m is in family (1):
  - P_m coincides with x. This is possible only if $m = n$
  - $P_m \equiv y = -x \mod (2+\sqrt{2})$, which happens if and only if $m = -n$;
- Oct_m is in family (2):
  - $P_m \equiv z \equiv x + 1 \mod (2+\sqrt{2})$, which happens if and only if $m = n+1$. In this case $P_{m-1} \equiv x \mod (2+\sqrt{2})$;
  - $P_m \equiv t \equiv -x + 1 \mod (2+\sqrt{2})$, which happens if and only if $m = -n+1$. In this case $P_{m-1} \equiv y \equiv -x \mod (2+\sqrt{2})$;
- Oct_m is in family (3):
  - $P_{m-1} \equiv z \equiv x + 1 \mod (2+\sqrt{2})$, which happens if and only if $m = n+2$;
  - $P_{m-1} \equiv t \equiv -x+1 \mod (2+\sqrt{2})$, which happens if and only if $m = -n+2$.

If $P_n \equiv x \in (1, \frac{1+\sqrt{2}}{2}) \mod (2+\sqrt{2})$, some possibility disappears because in this case $d(y, B) > d(y, C)$ and $d(z, C) > d(z, B)$. A part the case $\mathrm{Oct}_m = \mathrm{Oct}_n$ (if and only if $m = n$), the only possibility that remains is when Oct_m belongs to family (3) and P_m is the t-point of $x \equiv P_n \mod (2+\sqrt{2})$, namely:

- $P_{m-1} \equiv t \equiv -x + 1 \mod (2+\sqrt{2})$, and this happens if and only if $m = -n+2$.

□

Remark 4.7 (Generalisations)**.** The construction of sequence $(\mathrm{Oct}_n)_{n\in\mathbb{Z}}$ used only the existence of a (horizontal) saddle connection γ having an embedded twin such that:

- The angle from the start of the twin to the start of γ is 2π. (So that the saddle connection surgery produces a point in the minimal stratum, see Remark 3.1.)
- The angle from the end of γ to the start of the twin is π.
- If the (horizontal) continuation of the twin is a saddle connection (which is longer than γ because the twin is embedded), then the angle from its start to its end is π (hence it bounds a cylinder).

Second condition implies that first one is preserved by the surgery; third condition is preserved by surgery and guarantees that the length of the twin saddle connection does not change under the surgery (to see this, just draw the twin and angles in Figure 3).

Therefore the sequence of (putative) fakes can be constructed in any such situation via left surgeries.

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