GROUPS007 CIRM MARSEILLE
Weak 2 – Outer space and Teichmuller Space

A Distance on Outer Space

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Joint work with Armando Martino

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Abstract. *We introduce a metric on Outer space defined via stretching factors, and we discuss its basic properties. Such a metric is the analogous of the Thurston metric on Teichmuller space. We refer the reader to the beautiful survey* [Vog02] *for detailed introduction to Outer space.*

1. Introduction

This is a LaTeXversion of the talk I gave at CIRM of Luminy during the workshop *Outer space and Teichmuller space*. The style of exposition is informal and many details are missing. I refer the reader to the forthcoming joint work with A. Martino [FM07] for a full detailed discussion.

Sections 2-7 contain more or less the original talk, while Section 8 is an addendum wrote after the problems session of the workshop.

Many thanks to my collaborator A. Martino, to the organizers A. Hilion and P. Hubert, and to M. Lustig (also organizer) and J. Los because their comments and suggestions are never useless.

2. Notation

- F is a free group of finite rank.
- The Outer space of F is denoted by CV. Elements of CV are triples $A = (A, d_A, \alpha)$ where A is a graph, with all vertices of valence at least three, whose fundamental group is F; d_A is a simplicial metric on A (i.e. the assignment of a positive length for each edge), and α is an homotopy equivalence from the standard rose with rank(F) equal petals R to A. Two triples

(A, d_A, α) and (B, d_B, β) are equivalent whenever there is an homothety $h : A \to B$ such that $\beta \sim h \circ \alpha$. We will often use the short notation A for (A, d_A, α).
- The actions of F and $\mathrm{Aut}(F)$ on CV are by composition with α, that is to say, $\Phi(A, d_A, \alpha) = (A, d_A, \alpha \circ \Phi)$.

3. Curves, Currents and Lengths

Geodesic currents are a generalization of reduced, simplicial curves. Namely, one can think of the space $\mathrm{Curr}(F)$ of the geodesic currents as the positive cone of the closure of the vector space generated by reduced closed curves on the standard rose R, with the weak-$*$ topology. We do not want to be more precise here because in what follows currents are not strictly necessary. In fact, everything we do here can be done using only reduced closed simplicial curves on graphs. We refer to [Kap06] for a good introduction on the theory of geodesic currents on free groups (see also [Fra06].)

Introducing currents has two main advantages:

- We can use its linear structure for studying curves.
- $\mathrm{Curr}(F)$ is a compact convex set (in an infinite dimensional space.)

Definition 3.1 (Translation length)**.** *For any $A \in CV$ the translation length $L_A(\gamma)$ of a closed curve γ is the length of the geodesic representative of γ in A. The definition of lengths is extended by linearity to the space of geodesic currents.*

Lengths of currents can be explicitly computed by the formula

$$L_A(\eta) = \sum L_A(e)\eta(\mathrm{Cyl}(e))$$

where the sum runs over all edges e of the graph A, $L_A(e)$ is the length of e with respect to d_A, and $\mathrm{Cyl}(e)$ denotes the cylinder of e, which is roughly speaking the set of geodesics passing trough e. If η is a closed geodesic, then $\eta(\mathrm{Cyl}(e))$ is simply the number of occurrences of e in η.

Fact 3.2. *L_A determines A. In fact, CV is homeomorphic to a subset of the space of length-functions on geodesics, with the topology of pointwise convergence.*

Remark 3.3. *The length is a bilinear operator!*

That remark makes no sense if we do not specify the linear structures. Let's give more details.

Definition 3.4 (Simplices of CV). *For $A = (A, d_A, \alpha)$ in CV the simplex of A is the set*

$$\sigma_A = \{(B, d_B, \beta) : B = A, \beta = \alpha\}.$$

Let $e_1, \cdots, e_k$ be the edges of A. Then σ_A has a natural linear structure given by the following identification with the positive cone of $\mathbb{R}^k$

$$B \in \sigma_A \to (L_B(e_1), \cdots, L_B(e_k)).$$

Also, there is a natural projection of the space of currents to $\mathbb{R}^k$ given by

$$\eta \to (\eta(\mathrm{Cyl}(e_1)), \cdots, \eta(\mathrm{Cyl}(e_k))).$$

By the above formula, the length of a current only depends on its projection on $\mathbb{R}^k$, and in such cohordinates the length becomes simply the usual Euclidean scalar product, which is obviously bilinear.

Lemma 3.5. *The length is equivariant with respect to the action of $Aut(F)$. That is to say, for any $\eta \in Curr(F)$*

$$L_A(\Phi(\eta)) = L_{\Phi A}(\eta)$$

for any $\Phi \in Aut(F)$.

Proof. We have to say what exactly $L_A(\eta)$ means. If η is a geodesic on R, we can push it forward to a curve in A using α and then compute the length of its geodesic representative in A. The same holds for currents, and the claim is now tautological. □

4. The distance

We have now all the ingredients for cocking a distance.

Definition 4.1 (Distance). *For any $A, B \in$ CV we define*

$$\Lambda(A, B) = \sup_{x \in Curr(F)} \frac{L_A(x)}{L_B(x)} \cdot \sup_{x \in Curr(F)} \frac{L_B(x)}{L_A(x)}$$

$$d(A, B) = \log(\Lambda(A, B)).$$

Remark 4.2. *Since in general $\sup 1/f(x) = 1/\inf f(x)$, we have*

$$\Lambda(A, B) = \sup \frac{L_B(x)}{L_A(x)} \Big/ \inf \frac{L_B(x)}{L_A(x)}.$$

In particular, $\Lambda(A, B) \geq 1$. Moreover, the space of currents is compact (note that sup and inf can be taken over currents or simply over all curves because curves are dense in currents) so sup and inf are attained. Thus

$$\Lambda(A, B) = \max \frac{L_B(x)}{L_A(x)} \Big/ \min \frac{L_B(x)}{L_A(x)}$$

In particular, $\Lambda(A,B) < \infty$.

The basic properties of $d(A,B)$ are readily checked:

(1) $d(A,B)$ is scale-invariant, so it is well-defined CV.
(2) By Remark 4.2 we get $0 \leq d(A,B) < \infty$.
(3) $d(A,B) = 0$ if and only if $A = B$ as classes of graphs in CV. Indeed, $d(A,B) = 0$ if and only if the function L_B/L_A is constant (as its max coincide with its min.) By Fact 3.2 this implies that $A = \lambda B$ for some λ, and therefore A and B are equivalent in CV.
(4) Obviously $d(A,B) = d(B,A)$.
(5) The triangular inequality holds. Indeed, let η_0 a current that realizes $\max L_B/L_A$ and let C a third point in CV. We have

$$\max \frac{L_B}{L_A} = \frac{L_B(\eta_0)}{L_A(\eta_0)} = \frac{L_B(\eta_0)}{L_C(\eta_0)} \cdot \frac{L_C(\eta_0)}{L_A(\eta_0)} \leq \max \frac{L_B}{L_C} \max \frac{L_C}{L_A}$$

and the same holds for $\min L_B/L_A$.

We therefore get that d is a well-defined distance on CV.

Remark 4.3. *Suppose that there is a k-Lipschitz map* $f : A \to B$ *which is homotopic to* $\beta \circ \alpha^{-1}$. *Then* $\sup L_B/L_A \leq k$.

Fact 4.4 (The topology)**.** *The topology induced by d on CV is the usual one. Namely, it is the one of pointwise convergence on the space of translation lengths.*

Proof. We show that the two topologies have the same converging sequences, that being enough since both topologies have countable basis. Let $A_n = (A_n, d_{A_n}, \alpha_n)$ and (A, d_A, α) be in CV.

If $d(A_n, A) \to 0$, then $\Lambda(A_n, A) \to 1$, i.e.

$$\frac{\sup(L_{A_n}/L_A)}{\inf(L_{A_n}/L_A)} \to 1$$

whence we deduce $\forall x\, [L_{A_n}(x) \to L_A(x)]$.

Vice versa, if $L_{A_n} \to L_A$, then there exist a sequence $k_n \to 1$ and k_n-Lipschitz maps $\varphi_n : A_n \to A$ and $\psi_n : A \to A_n$ such that φ_n is homotopic to $\alpha \circ \alpha_n^{-1}$ and ψ_n to $\alpha_n \circ \alpha^{-1}$. By Remark 4.3 we get $d(A_n, A) \to 0$. □

Remark 4.5 (Compact balls, completeness)**.** *The proof of Fact 4.4 proves more. Indeed, its argument shows that for any radius r and for any* $A \in CV$ *the d-ball of centre A and radius r is compact. This in particular implies that the metric d is complete.*

Fact 4.6 (Iterations of automorphisms)**.** *Let* $\Phi \in \mathrm{Aut(F)}$ *be an automorphism of exponential growth. Then for any* $A \in \mathrm{CV}$ *the sequence* $\Phi^n A$ *is a quasi-geodesic (as a map of* $\mathbb{Z} \to \mathrm{CV}$*.)*

Proof. If Φ has exponential growth so does Φ^{-1}. That means that $\sup_x L(\Phi(x))/L(x) > kc^n$ for some $k > 0$ and $c > 1$, where the length L is calculated in your favourite, fixed, free basis (and the same holds for Φ^{-1}.)

Recall that by Lemma 3.5 we have $L_{\Phi A}(\eta) = L_A(\Phi_*\eta)$. Moreover,

$$\frac{L_A(\Phi_*^{n+m}\eta)}{L_A(\Phi_*^m\eta)} = \frac{L_A(\Phi_*^n(\Phi_*^m\eta))}{L_A(\Phi_*^m\eta)}$$

so

$$\sup_\eta \frac{L_A(\Phi_*^{n+m}\eta)}{L_A(\Phi_*^m\eta)} = \sup_\eta \frac{L_A(\Phi_*^n\eta)}{L_A(\eta)} = \sup_\eta \frac{L_A(\Phi_*^n\eta)}{L(\Phi_*^n\eta)} \cdot \frac{L(\Phi_*^n\eta)}{L(\eta)} \cdot \frac{L(\eta)}{L_A(\eta)}$$

In the last term of above inequality, the first and the last factor are bounded below by constants because A lies at finite distance from the rose used for calculating L. The middle term is bounded below by kc^n by our hypothesis of exponential growth. Similarly, using that also Φ^{-1} has exponential growth, we can show that

$$\Lambda(\Phi^{n+m}A, \Phi^m A) > kc^n$$

for some constants $k > 0$ and $c > 1$, this giving

$$d(\Phi^{n+m}A, \Phi^m A) > \log k + n \log c$$

The other inequality is even easier, and does not need any assumption on Φ:

$$\sup_\eta \frac{L_A(\Phi_*^{n+m}\eta)}{L_A(\Phi_*^m\eta)} = \sup_\eta \frac{L_A(\Phi_*^{n+m}\eta)}{L_A(\Phi_*^{n+m-1}\eta)} \cdot \frac{L_A(\Phi_*^{n+m-1}\eta)}{L_A(\Phi_*^{n+m-2}\eta)} \cdots \frac{L_A(\Phi_*^{1+m}\eta)}{L_A(\Phi_*^m\eta)}$$

which is bounded above by

$$\left(\sup_\eta \frac{L_A(\Phi_*\eta)}{L_A(\Phi_*\eta)}\right)^n$$

whence

$$\Lambda(\Phi^{n+m}A, \Phi^m A) \le \Lambda(\Phi A, A)^n$$

and

$$d(\Phi^{n+m}A, \Phi^m A) \le nd(\Phi A, A).$$

□

5. Best Lipschitz constants

The following lemma, originally proved by T. White, is contained in a never published preprint. Thanks to M. Bestvina (White's advisor) who gave the preprint to M. Lustig, who gave it to A. Martino, we now know that result.

Lemma 5.1 (White map)**.** *Let $A, B \in CV$. Let k the infimum of Lipschitz constants of maps from A to B in the homotopy-class of $\beta \circ \alpha^{-1}$. Then, there is a map $f : A \to B$ in the class of $\beta \circ \alpha^{-1}$ such that:*

- *f is k-Lipschitz.*
- *f has constant speed on edges.*
- *The union of edged stretched exactly by a factor k contains a closed geodesic whose image is a geodesic.*

Corollary 5.2. *Let A, B, k as in Lemma 5.1. Then*

$$\sup \frac{L_B}{L_A} = k.$$

Moreover, such a supremum is realized by a closed curve.

Corollary 5.3. *Let $A, B \in CV$ let K_{AB} be the best Lipschitz constants of maps $A \to B$. Then*

$$d(A, B) = \log K_{AB} + \log K_{BA}.$$

So we have that the distance d is the analogous on CV of the symmetrized version of Thurston and Teichmuller metrics on Teichmuller space.

Remark 5.4. *The construction of the White map tells us more. Indeed, one can show that the curves realizing $\sup L_B/L_A$ has one of the following shapes. Either an O-curve (i.e. an embedded circle), or an 8-curve (i.e. an embedded bouquet of two circles) or an O-O-curve (i.e. two circles joined by an arc.)*

Corollary 5.5. *The distance $d(A, B)$ is computable in a finite number of steps.*

Proof. Just compute the stretching factor of all (finitely many) possible O-, 8-, and O-O-curves. ☐

6. Geodesics

The following are the leading observations we used to study geodesics for the metric d.

- Suppose one wants to find a geodesic between two point $A, B \in CV$. It is enough to show that there is a path A_t such that the triangular inequality $d(A, A_t) \leq d(A_t, B)$ is an equality at each time t.
- Let $A, C, B \in CV$. Suppose that one knows that there are currents η_0 and η_1 such that
  - η_0 realizes both $\sup L_B/L_C$ and $\sup L_C/L_A$
  - η_1 realizes both $\sup L_A/L_C$ and $\sup L_C/L_B$.

  Then, the triangular inequality becomes equality.

Fact 6.1. *In each simplex of CV, segments are geodesics.*

Proof. . We proof the following: for any line in a simplex of CV there are currents η_0, η_1 such that for any $X < Y$ in that line we have

$$\sup \frac{L_X}{L_Y} = \frac{L_X(\eta_0)}{L_Y(\eta_0)} \qquad \sup \frac{L_Y}{L_X} = \frac{L_Y(\eta_1)}{L_X(\eta_1)}.$$

This follows from the fact that L_X/L_Y is the ratio of two linear function and hence has no critical points. □

Fact 6.2. *Geodesics are not unique.*

Proof. Let $A, B \in CV$ be two points in the same simplex, and let C be the middle point of the segment $\overline{AB}$. It is easy to check that, generically, for C' sufficiently close to C the curve consisting of the two segments $\overline{AC'}$ and $\overline{C'B}$ is a geodesic. □

So, the metric d looks like the path metric in the Eixample of Barcelona.

One can also consider the two asymmetric factors of the Λ function, and define the left and right part of the distance as follows

$$d_R(A, B) = \log(\sup \frac{L_B}{L_A}) \qquad d_L(A, B) = \log(\sup \frac{L_A}{L_B}).$$

The notation left and right is because the right part is the logarithm of the minimal Lipschitz constant of maps $A \to B$, so "we go to right."

We say that a path A_t from A to B is a right-geodesic if the ordered triangular inequality is an equality at each t

$$d_R(A, B) = d_R(A, A_t) + d_R(A_t, B).$$

A folding argument starting from a White map provides right- and left-geodesics.

Fact 6.3. *For any $A, B \in CV$ there is a right- and a left-geodesic between them.*

Usually, the asymmetric geodesic that one founds are different. This makes more complicated the problem to find a geodesic for the symmetric metric. At the moment we do not know in general whether (CV, d) is or not a geodesic space (see also the discussion if Section 8.)

7. Boundaries

Once one has a metric space, one can start to search for compactifications and boundaries at infinity. In particular one can give the following definition

Definition 7.1. *The boundary at infinity $\partial_\infty CV$ of Outer space consist of equivalent classes of quasi-geodesics sequences in CV, where*

$$(X_n) \sim (Y_n) \Leftrightarrow d(X_n, Y_n) < c$$

for a positive constant c uniform in n.

This gives a nice filtration of the usual boundary of CV consisting of length functions. Namely, if $X, Y \in \overline{CV}$ and $X_n \to X$ and $Y_n \to Y$ are quasi-geodesic in the same class, then the quantity

$$\frac{\sup L_X/L_Y}{\inf L_X/L_Y}$$

is well defined, and its logarithm has the properties of a metric. So we can say that a boundary point of $\partial_\infty CV$ is a metric space, which has a boundary at infinity, etc...

8. Symmetric Vs Asymmetric

Following M. Lustig, a good metric on Outer space should, at least:

- be a possibly asymmetric metric on CV (non-negative, distinguishes points, and satisfies ordered triangular inequality;)
- have a natural action of $\mathrm{Aut}(F)$ (i.e. by isometries;)
- induce the usual topology on CV;
- characterize the north-south dynamic of iwip automorphisms.

The choice of a symmetric complete metric is of course more elegant, but there are reasons for thinking that the *nature of Outer space is asymmetric* because, as in real life, *going from A to B is not the same that going from B to A.*

We summarize here the difference of the symmetric and asymmetric metrics described so far, namely d and d_R.

8.1. **Du côté de chez d.**
ADVANTAGES: The metric d is a metric. That is is a well-defined function on CV which is symmetric, non-negative and $d(A, B)$ if and only if $A = B$; it satisfies the triangular inequality. The metric is complete and balls are compact. The topology induced on CV is the usual one and $Aut(F)$ acts by isometries on (CV, d). An iwip automorphism has quasi-geodesic orbits.

PROBLEMS: We still do not know whether (CV, d) is a geodesic space, and whether a geodesic axis (if any) of an automorphism with north-south dynamic is related to other constructions like train-tracks and folding paths.

8.2. **Du côté de chez d_R.**
ADVANTAGES: It can be proved that the sub-level sets of d_R induces the usual topology on CV (namely, $A_n \to A$ in CV if and only if $d_R(A_n, A) \to 0$. We notice that here the assumption of total volume 1 is crucial.) Ordered triangular inequality holds, and $Aut(F)$ acts by isometries. (CV, d_R) is geodesic, and geodesics can be described in terms of folding procedures. In particular, the folding procedure from a train-track for an iwip gives a geodesic axis.

PROBLEMS: The metric d_R is not a metric. It is not scale-invariant, but once we choose, for any element of CV, the representative with total volume 1, then d_R is well-defined. It is not complete (there are points in the boundary that are at finite distance from the standard rose.)

As a final remak we just notice that a geodesic for d is also a geodesic for both d_R and d_L.

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