Multidimensional Persistent Topology as a Metric Approach to Shape Comparison

Patrizio Frosini[1,2]

[1]Department of Mathematics, University of Bologna, Italy
[2]ARCES - Vision Mathematics Group, University of Bologna, Italy
frosini@dm.unibo.it

GETCO 2010
Geometric and Topological Methods in Computer Science
Aalborg University, January 11-15, 2010

The point of this talk

In brief, the main message of this talk:

TECHNIQUES FOR THE STABLE COMPUTATION AND THE COMPARISON OF PERSISTENT TOPOLOGY IN THE MULTIDIMENSIONAL SETTING (I.E., FOR FILTERING FUNCTIONS TAKING VALUES IN $\mathbb{R}^k$) ARE AVAILABLE.

Outline

1. A Metric Approach to Shape Comparison
2. Lower Bounds for the Natural Pseudodistance
3. New Results in the Multidimensional Setting
4. Experiments

The results I am going to present refer to a collective work of the Vision Mathematics Group (Niccolò Cavazza, Andrea Cerri, Barbara Di Fabio, Massimo Ferri, Patrizio Frosini, Claudia Landi).

The experimental results I shall show at the end of this talk have been obtained in a joint work with the C.N.R. IMATI Group (Silvia Biasotti, Daniela Giorgi).

Shape depends on persistent perceptions

Massimo and Claudia have already presented some motivations to study Persistent Topology.
Just a few words to recall our approach to shape comparison:

- "Science is nothing but perception." *Plato*
- "Reality is merely an illusion, albeit a very persistent one." *Albert Einstein*

Our formal setting

As shown by Massimo and Claudia, we propose that

- Each perception is formalized by a pair $(X, \vec{\varphi})$, where X is a topological space and $\vec{\varphi}$ is a continuous function.
- X represents the set of observations made by the observer, while $\vec{\varphi}$ describes how each observation is interpreted by the observer.

Our formal setting

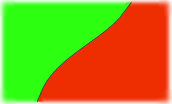
- Persistence is quite important. Without persistence (in space, time, with respect to the analysis level...) perception could have little sense. This remark compels us to require that
  - X is a topological space and $\vec{\varphi}$ is a continuous function; this function $\vec{\varphi}$ describes X from the point of view of the observer. It is called a measuring function.
  - Persistent Topology is used to study the stable properties of the pair $(X, \vec{\varphi})$.

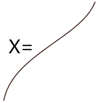
Our formal setting

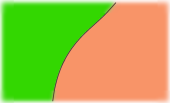
- A possible objection: sometimes we have to manage discontinuous functions (e.g., color).

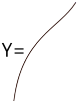
Our formal setting

- A possible objection: sometimes we have to manage discontinuous functions (e.g., color).
- An answer: in that case the topological space X can describe the discontinuity set, and persistence can concern the properties of this topological space with respect to a suitable measuring function.









As measuring functions we can take $\vec{\varphi} : X \to \mathbb{R}^2$ and $\vec{\psi} : Y \to \mathbb{R}^2$, where the components φ_1, φ_2 and ψ_1, ψ_2 represent the colors on each side of the considered discontinuity set.

Our formal setting

A categorical way to formalize our approach

Let us consider a category $\mathcal{C}$ such that

- The objects of $\mathcal{C}$ are the pairs $(X, \vec{\varphi})$ where X is a compact topological space and $\vec{\varphi} : X \to \mathbb{R}^k$ is a continuous function.
- The set $Hom\left((X, \vec{\varphi}), (Y, \vec{\psi})\right)$ of all morphisms between the objects $(X, \vec{\varphi})$, $(Y, \vec{\psi})$ is a subset of the set of all homeomorphisms between X and Y (possibly empty).

If $Hom\left((X, \vec{\varphi}), (Y, \vec{\psi})\right)$ is not empty we say that the objects $(X, \vec{\varphi})$, $(Y, \vec{\psi})$ are comparable.

Our formal setting

Do not compare apples and oranges...

Remark: $Hom\left((X,\vec{\varphi}),(Y,\vec{\psi})\right)$ can be empty also in case X and Y are homeomorphic.

Example:

- Consider a segment $X = Y$ embedded into $\mathbb{R}^3$ and consider the set of observations given by measuring the color $\vec{\varphi}(x)$ and the triple of coordinates $\vec{\psi}(x)$ of each point x of the segment.

Our formal setting

Do not compare apples and oranges...

Remark: $Hom\left((X,\vec{\varphi}),(Y,\vec{\psi})\right)$ can be empty also in case X and Y are homeomorphic.

Example:

- Consider a segment $X = Y$ embedded into $\mathbb{R}^3$ and consider the set of observations given by measuring the color $\vec{\varphi}(x)$ and the triple of coordinates $\vec{\psi}(x)$ of each point x of the segment.
- It does not make sense to compare the perceptions $\vec{\varphi}$ and $\vec{\psi}$. In other words the pairs $(X,\vec{\varphi})$ and $(Y,\vec{\psi})$ are not comparable, even if $X = Y$.

Our formal setting

Do not compare apples and oranges...

Remark: $Hom\left((X,\vec{\varphi}),(Y,\vec{\psi})\right)$ can be empty also in case X and Y are homeomorphic.

Example:

- Consider a segment $X = Y$ embedded into $\mathbb{R}^3$ and consider the set of observations given by measuring the color $\vec{\varphi}(x)$ and the triple of coordinates $\vec{\psi}(x)$ of each point x of the segment.
- It does not make sense to compare the perceptions $\vec{\varphi}$ and $\vec{\psi}$. In other words the pairs $(X,\vec{\varphi})$ and $(Y,\vec{\psi})$ are not comparable, even if $X = Y$.
- We express this fact by setting $Hom\left((X,\vec{\varphi}),(Y,\vec{\psi})\right) = \emptyset$.

Our formal setting

We can now define the following (extended) pseudometric:

$$\delta\left((X,\vec{\varphi}),(Y,\vec{\psi})\right) = \inf_{h\in Hom\left((X,\vec{\varphi}),(Y,\vec{\psi})\right)} \max_i \max_{x\in X} |\varphi_i(x) - \psi_i \circ h(x)|$$

if $Hom\left((X,\vec{\varphi}),(Y,\vec{\psi})\right) \neq \emptyset$, and $+\infty$ otherwise.

We shall call $\delta\left((X,\vec{\varphi}),(Y,\vec{\psi})\right)$ the natural pseudodistance between $(X,\vec{\varphi})$ and $(Y,\vec{\psi})$.

The functional $\Theta(h) = \max_i \max_{x\in X} |\varphi_i(x) - \psi_i \circ h(x)|$ represents the "cost" of the matching between observations induced by h. The lower this cost, the better the matching between the two observations is.

Our formal setting

- The natural pseudodistance δ measures the dissimimilarity between the perceptions expressed by the pairs $(X, \vec{\varphi})$, $(Y, \vec{\psi})$.

Our formal setting

- The natural pseudodistance δ measures the dissimimilarity between the perceptions expressed by the pairs $(X, \vec{\varphi})$, $(Y, \vec{\psi})$.
- The value δ is small if and only if we can find a homeomorphism between X and Y that induces a small change of the measuring function (i.e., of the shape property we are interested to study).

Our formal setting

- The natural pseudodistance δ measures the dissimimilarity between the perceptions expressed by the pairs $(X, \vec{\varphi})$, $(Y, \vec{\psi})$.
- The value δ is small if and only if we can find a homeomorphism between X and Y that induces a small change of the measuring function (i.e., of the shape property we are interested to study).

- For more information:

Our formal setting

- The natural pseudodistance δ measures the dissimimilarity between the perceptions expressed by the pairs $(X, \vec{\varphi})$, $(Y, \vec{\psi})$.
- The value δ is small if and only if we can find a homeomorphism between X and Y that induces a small change of the measuring function (i.e., of the shape property we are interested to study).

- For more information:
- P. Donatini, P. Frosini, *Natural pseudodistances between closed manifolds*, Forum Mathematicum, 16 (2004), n. 5, 695-715.

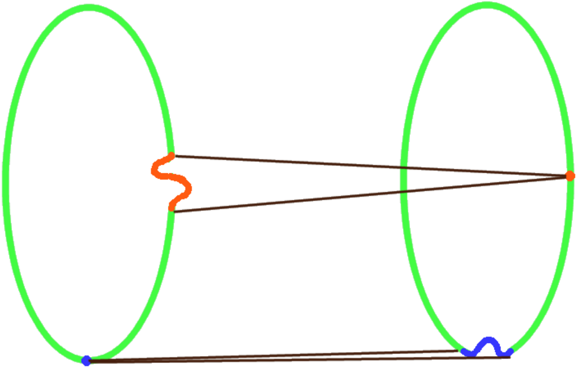
Our formal setting

- The natural pseudodistance δ measures the dissimimilarity between the perceptions expressed by the pairs $(X, \vec{\varphi})$, $(Y, \vec{\psi})$.
- The value δ is small if and only if we can find a homeomorphism between X and Y that induces a small change of the measuring function (i.e., of the shape property we are interested to study).

- For more information:
- P. Donatini, P. Frosini, *Natural pseudodistances between closed manifolds*, Forum Mathematicum, 16 (2004), n. 5, 695-715.
- P. Donatini, P. Frosini, *Natural pseudodistances between closed surfaces*, Journal of the European Mathematical Society, 9 (2007), 331-353.

Our formal setting

Why do we just consider homeomorphisms between X and Y?
Why couldn't we use, e.g., relations between X and Y?



Our formal setting

The following result suggests not to do that:

Non-existence Theorem

Let $\mathcal{M}$ be a closed Riemannian manifold. Call H the set of all homeomorphisms from $\mathcal{M}$ to $\mathcal{M}$. Let us endow H with the uniform convergence metric d_{UC}: $d_{UC}(f,g) = \max_{x\in\mathcal{M}} d_{\mathcal{M}}(f(x), g(x))$ for every $f, g \in H$, where $d_{\mathcal{M}}$ is the geodesic distance on $\mathcal{M}$.
Then (H, d_{UC}) cannot be embedded in any compact metric space (K, d) endowed with an internal binary operation $\bullet$ that extends the usual composition $\circ$ between homeomorphisms in H and commutes with the passage to the limit in K.

In particular, we cannot embed H into the set of binary relations on $\mathcal{M}$.

Lower bounds for δ, via Persistent Topology

Size homotopy groups

In the following we shall set $X\langle\vec{\varphi} \preceq \vec{u}\rangle = \{x \in X : \vec{\varphi}(x) \preceq \vec{u}\}$ and $\Delta^+ = \{(\vec{u}, \vec{v}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{u} \prec \vec{v}\}$.

The concept of size homotopy group:

Frosini&Mulazzani 1999

Assume that $\mathcal{M}$ is a C^1-submanifold of the Euclidean space and $\vec{\varphi} : \mathcal{M} \to \mathbb{R}^k$ is a C^1 function. For each pair $(\vec{u}, \vec{v}) \in \Delta^+$ and every $x \in X\langle\vec{\varphi} \preceq \vec{u}\rangle$ let us consider the j-th homotopy groups $\pi_j(X\langle\vec{\varphi} \preceq \vec{u}\rangle)$ and $\pi_j(X\langle\vec{\varphi} \preceq \vec{v}\rangle)$ based at x. Let us consider also the homomorphism $i_{(\vec{u},\vec{v})_*} : \pi_j(X\langle\vec{\varphi} \preceq \vec{u}\rangle) \to \pi_j(X\langle\vec{\varphi} \preceq \vec{v}\rangle)$ induced by the embedding $i_{(\vec{u},\vec{v})}$ of the set $X\langle\vec{\varphi} \preceq \vec{u}\rangle$ into the set $X\langle\vec{\varphi} \preceq \vec{v}\rangle$. The j-th size homotopy group of $(\mathcal{M}, \vec{\varphi})$ based at x and associated to $(\vec{u}, \vec{v})$ is the group $i_{(\vec{u},\vec{v})_*}(\pi_j(X\langle\vec{\varphi} \preceq \vec{u}\rangle))$.

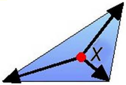
Lower bounds for δ, via Persistent Topology

Pareto-critical points

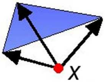
Let us recall the concept of Pareto-critical point:
Assume that $\mathcal{M}$ is a C^1 closed manifold and $\vec{\varphi} : \mathcal{M} \to \mathbb{R}^k$ is a C^1 function. We shall say that $x \in \mathcal{M}$ is a Pareto-critical (or pseudocritical) point if the convex hull of the vectors $\nabla\varphi_i(x)$ contains the null vector. If x is a Pareto-critical point, then its image $\vec{\varphi}(x)$ is called a Pareto-critical (or pseudocritical) value.

Example:

$$\vec{\varphi} : \mathcal{M} = \mathbb{R}^2 \to \mathbb{R}^3$$



x is Pareto-critical



x is not Pareto-critical

Lower bounds for δ, via Persistent Topology

Pareto-critical points

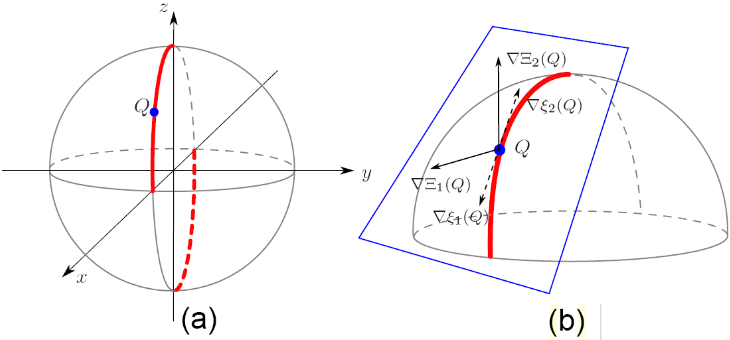


Figure: (a) The sphere $S^2 \subseteq \mathbb{R}^3$ endowed with the measuring function $\vec{\xi} = (\xi_1, \xi_2) : S^2 \to \mathbb{R}^2$, defined as $\vec{\xi}(x, y, z) = (x, z)$ for each $(x, y, z) \in S^2$. The Pareto-critical points of $\vec{\xi}$ are depicted in bold red. (b) The point Q is a Pareto-critical point for $\vec{\xi}$, since the vectors $\nabla\xi_1(Q)$ and $\nabla\xi_2(Q)$ are parallel with opposite verse.

Lower bounds for δ, via Persistent Topology

Natural pseudodistance and size homotopy groups

- The natural pseudodistance is usually difficult to compute.
- The following result allows us to get a lower bound for the natural pseudodistance δ, by computing the size homotopy groups.

Lower bounds for δ, via Persistent Topology.

Natural pseudodistance and size homotopy groups.

Frosini&Mulazzani 1999

Assume that

- $\mathcal{M}, \mathcal{N}$ are C^1-submanifolds of the Euclidean space
- $\vec{\varphi} : \mathcal{M} \to \mathbb{R}^k, \vec{\psi} : \mathcal{N} \to \mathbb{R}^k$ are C^1 functions.

Let $\mathcal{P}_{\vec{\psi}}$ be the set of Pareto-critical points of the function $\vec{\psi}$. Assume also that $(\vec{u}', \vec{v}'), (\vec{u}'', \vec{v}'') \in \Delta^+$ and that a point $x \in \mathcal{M}\langle\vec{\varphi} \preceq \vec{u}'\rangle$ exists for which the following statement holds:

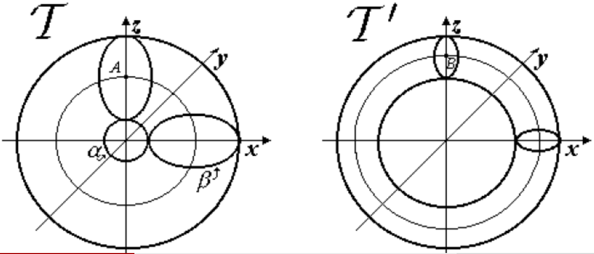
- For each $y \in \mathcal{P}_{\vec{\psi}}$ with $\vec{\psi}(y) \preceq \vec{u}''$, the first size homotopy group of $(\mathcal{M}, \vec{\varphi})$ based at x and associated to $(\vec{u}', \vec{v}')$ is not isomorphic to a subgroup of any quotient of the first size homotopy group of $(\mathcal{N}, \vec{\varphi})$ based at y and associated to $(\vec{u}'', \vec{v}'')$.

Then $\min_i \min\{u_i'' - u_i', v_i' - v_i''\} \leq \delta\left((X, \vec{\varphi}), (Y, \vec{\psi})\right)$.

Lower bounds for δ, via Persistent Topology.

Natural pseudodistance and size homotopy groups.

Example: Consider the two tori $\mathcal{T}, \mathcal{T}' \subset \mathbb{R}^3$ generated by the rotation around the y-axis of the circles lying in the plane yz and with centers $A = (0, 0, 3)$ and $B = (0, 0, 4)$, and radii 2 and 1, respectively. As measuring function φ (resp. φ') on $\mathcal{T}$ (resp. on $\mathcal{T}'$) we take the restriction to $\mathcal{T}$ (resp. to $\mathcal{T}'$) of the function $\zeta : \mathbb{R}^3 \to \mathbb{R}$, $\zeta(x, y, z) = z$. We point out that, for both $\mathcal{T}$ and $\mathcal{T}'$, the image of the measuring function is the closed interval $[-5, 5]$.



Lower bounds for δ, via Persistent Topology.

Natural pseudodistance and size homotopy groups.

We want to prove that the natural pseudodistance between $(\mathcal{T}, \varphi)$ and $(\mathcal{T}', \varphi')$ is 2. In order to do that, let us consider the homeomorphism f, that takes each point of the former torus to the point having the same toroidal coordinates in the latter. We can easily verify that $\Theta(f) = 2$. So we have only to prove that $\delta((\mathcal{T}, \varphi), (\mathcal{T}', \varphi')) \geq 2$. This inequality follows from the previous theorem by choosing $x = (0, 0, -5)$, $u' = 1$, $v' = 5 - \epsilon$, $u'', v'' = 3 - \epsilon$ and observing that if ϵ is any small enough positive number, then the first size homotopy group of $(\mathcal{T}, \varphi)$ based at x and associated to $(1, 5 - \epsilon)$ is $\mathbb{Z} * \mathbb{Z}$ while the first size homotopy group of $(\mathcal{T}', \varphi')$ based at y and associated to $(3 - \epsilon, 3 - \epsilon)$ is $\mathbb{Z}$.
From previous theorem we obtain that
$\delta((\mathcal{T}, \varphi), (\mathcal{T}', \varphi')) \geq \min\{(3 - \epsilon) - 1, (5 - \epsilon) - (3 - \epsilon)\} = 2 - \epsilon$.
This implies the wanted inequality.

Lower bounds for δ, via Persistent Topology

Natural pseudodistance and persistent homology groups

Let us recall the foliation method, illustrated in previous talks by Massimo and Claudia:

$$\vec{l} = (l_1, \ldots, l_k),\ \vec{b} = (b_1, \ldots, b_k), \text{ with } \|\vec{l}\| = 1,\ l_i > 0,\ \sum_i b_i = 0$$

- $\Delta^+ = \{(\vec{u}, \vec{v}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{u} \prec \vec{v}\}$ is foliated by the 2D half-planes with parametric equations:

$$\pi_{(\vec{l},\vec{b})} : \begin{cases} \vec{u} = s\vec{l} + \vec{b} \\ \vec{v} = t\vec{l} + \vec{b} \end{cases} \qquad s, t \in \mathbb{R}, s < t$$

- For every $(\vec{l}, \vec{b})$, define $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} : X \to \mathbb{R}$ by

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(x) = \max_{i=1,\ldots,k} \left\{ \frac{\varphi_i(x) - b_i}{l_i} \right\}.$$

Lower bounds for δ, via Persistent Topology

Reduction of the multidimensional rank invariant to the 1-dimensional case

Reduction Theorem

For every $(\vec{u},\vec{v}) = (s\vec{l}+\vec{b}, t\vec{l}+\vec{b}) \in \pi_{(\vec{l},\vec{b})}$ it holds that

$$\check{\rho}_{(X,\vec{\varphi}),q}(\vec{u},\vec{v}) = \check{\rho}_{(X,F^{\vec{\varphi}}_{(\vec{l},\vec{b})}),q}(s,t).$$

On each leaf of the foliation size functions can be represented as persistence diagrams.

Multidimensional Matching Distance

$$D_{match}\left(\check{\rho}_{(X,\vec{\varphi}),q}, \check{\rho}_{(Y,\vec{\psi}),q}\right) = \sup_{(\vec{l},\vec{b})} \min_{i=1,\dots,k} l_i \cdot d_{match}(\check{\rho}_{(X,F^{\vec{\varphi}}_{(\vec{l},\vec{b})}),q}, \check{\rho}_{(Y,F^{\vec{\psi}}_{(\vec{l},\vec{b})}),q})$$

Lower bounds for δ, via Persistent Topology
Size functions and persistent homology groups

Claudia has shown that the following result holds for the matching distance D_{match}:

Multidimensional Stability Theorem

If X is a compact and locally contractible space and $\vec{\varphi}, \vec{\psi} : X \to \mathbb{R}^k$ are continuous functions, then

$$D_{match}\left(\check{\rho}_{(X,\vec{\varphi}),q}, \check{\rho}_{(X,\vec{\psi}),q}\right) \leq \max_{x \in X} \|\vec{\varphi}(x) - \vec{\psi}(x)\|_\infty .$$

Lower bounds for δ, via Persistent Topology
Size functions and persistent homology groups

The previous result can be reformulated in this way:

A Lower Bound for the Natural Pseudodistance

If X, Y are compact and locally contractible topological spaces, and $\vec{\varphi}: X \to \mathbb{R}^k, \vec{\psi}: X \to \mathbb{R}^k$ are continuous functions then

$$D_{match}\left(\check{\rho}_{(X,\vec{\varphi}),q}, \check{\rho}_{(Y,\vec{\psi}),q}\right) \le \delta\left((X,\vec{\varphi}),(Y,\vec{\psi})\right).$$

This result allows us to get a lower bound for the natural pseudodistance δ, by computing the rank invariants.

Localizing discontinuities of the rank invariants

Our main result about discontinuities

A theorem localizing the discontinuities of the rank invariants

Assume that $\mathcal{M}$ is a C^1 closed manifold and $\vec{\varphi}:\mathcal{M}\to\mathbb{R}^k$ is a C^1 function. Let $(\vec{u},\vec{v})\in\Delta^+$ be a discontinuity point for $\check{\rho}_{(\mathcal{M},\vec{\varphi})}$. Then at least one of the following statements holds:

(i) $\vec{u}$ is a discontinuity point for $\check{\rho}_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{v})$;

(ii) $\vec{v}$ is a discontinuity point for $\check{\rho}_{(\mathcal{M},\vec{\varphi})}(\vec{u},\cdot)$.

Moreover,

- If (i) holds, then a projection p exists such that $p(\vec{u})$ is a Pareto-critical value for $p\circ\vec{\varphi}$;
- If (ii) holds, then a projection p exists such that $p(\vec{v})$ is a Pareto-critical value for $p\circ\vec{\varphi}$.

Localizing discontinuities of the rank invariants

Why is the previous result important?

The previous result allows us to divide Δ^+ in connected components where the rank invariant is constant. As a consequence, it implies a new procedure to compute the multidimensional rank invariant, requiring to compute it just at one point for each connected component.

Evaluating the matching distance between rank invariants

Reformulating the foliation method

In order to proceed, let us reformulate the foliation method. We need to use a different parametrization of the planes in our foliation. The question is: does a change of the parametrization produce a different matching distance?

Fortunately, we can prove the following statement:

The multidimensional matching distance is invariant with respect to reparametrizations of the half-planes foliating Δ^+.

Evaluating the matching distance between rank invariants

Reformulating the foliation method

More precisely, the following result can be proven:

Invariance with respect to reparametrization (I)

For each pair $(\vec{\lambda}, \vec{\beta}) \in \mathbb{R}^k \times \mathbb{R}^k$ let us consider the half-plane $\pi_{(\vec{\lambda},\vec{\beta})}$ defined by the following parametric equation:

$$\pi_{(\vec{\lambda},\vec{\beta})} : \begin{cases} \vec{u} = s\vec{\lambda} + \vec{\beta} \\ \vec{v} = t\vec{\lambda} + \vec{\beta} \end{cases} \qquad s, t \in \mathbb{R}, s < t$$

Assume $\Lambda \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^k$ are two sets such that the collection of half-planes $\left\{\pi_{(\vec{\lambda},\vec{\beta})}\right\}_{(\vec{\lambda},\vec{\beta})\in\Lambda\times B}$ is a foliation of Δ^+.

$(\longrightarrow)$

Evaluating the matching distance between rank invariants

Reformulating the foliation method

Invariance with respect to reparametrization (II)

Let $\vec{\varphi} : X \to \mathbb{R}^k$, $\vec{\psi} : Y \to \mathbb{R}^k$ be two continuous functions. For every $(\vec{\lambda}, \vec{\beta}) \in \Lambda \times B$, define $F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})} : X \to \mathbb{R}$ and $F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})} : Y \to \mathbb{R}$ by

$$F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}(x) = \max_{i=1,\dots,k} \left\{ \frac{\varphi_i(x) - \beta_i}{\lambda_i} \right\}, \quad F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}(y) = \max_{i=1,\dots,k} \left\{ \frac{\psi_i(y) - \beta_i}{\lambda_i} \right\}.$$

Then

$$D_{match}\left(\check{\rho}_{(X,\vec{\varphi}),q}, \check{\rho}_{(Y,\vec{\psi}),q}\right) =$$

$$\sup_{(\vec{\lambda},\vec{\beta}) \in \Lambda \times B} \min_{i=1,\dots,k} \lambda_i \cdot d_{match}(\check{\rho}_{(X,F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}),q}, \check{\rho}_{(Y,F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}),q}).$$

Evaluating the matching distance between rank invariants

Reformulating the foliation method

Because of the previous theorem, the following parametrization of the planes in our foliation produces the same matching distance we have presented previously.

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\vec{\beta} = (\beta_1, \ldots, \beta_k)$, with $\sum_i \lambda_i = 1$, $\lambda_i > 0$, $\sum_i \beta_i = 0$

- $\Delta^+ = \{(\vec{u}, \vec{v}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{u} \prec \vec{v}\}$ is foliated by the 2D half-planes with parametric equations:

$$\pi_{(\vec{\lambda},\vec{\beta})} : \begin{cases} \vec{u} = s\vec{\lambda} + \vec{\beta} \\ \vec{v} = t\vec{\lambda} + \vec{\beta} \end{cases} \qquad s, t \in \mathbb{R}, s < t.$$

- For every $(\vec{\lambda}, \vec{\beta})$, define $F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})} : X \to \mathbb{R}$ by

$$F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}(x) = \max_{i=1,\ldots,k} \left\{ \frac{\varphi_i(x) - \beta_i}{\lambda_i} \right\}.$$

Evaluating the matching distance between rank invariants

2-dimensional case

Let us consider the previously defined foliation.
We shall denote by *Ladm* the set of all admissible pairs.
We recall the definition of the matching distance in the case $k = 2$:

$$D_{match}\left(\check{\rho}_{(X,\vec{\varphi}),q}, \check{\rho}_{(Y,\vec{\psi}),q}\right)$$

$$= sup_{(\vec{\lambda},\vec{\beta})\in Ladm}\mu(\vec{\lambda}) \cdot d_{match}\left(\check{\rho}_{\left(X,F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}\right)}, \check{\rho}_{\left(Y,F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}\right)}\right)$$

$$= sup_{(\vec{\lambda},\vec{\beta})\in Ladm} d_{match}\left(\check{\rho}_{\left(X,\mu(\vec{\lambda})\cdot F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}\right)}, \check{\rho}_{\left(Y,\mu(\vec{\lambda})\cdot F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}\right)}\right)$$

where $\mu(\vec{\lambda}) = \min\{\lambda_1, \lambda_2\}$, $F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}(x) = \max\left\{\frac{\varphi_1(x)-\beta_1}{\lambda_1}, \frac{\varphi_2(x)-\beta_2}{\lambda_2}\right\}$,
$F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}(x) = \max\left\{\frac{\psi_1(x)-\beta_1}{\lambda_1}, \frac{\psi_2(x)-\beta_2}{\lambda_2}\right\}$.

Evaluating the matching distance between rank invariants

Our main result about the perturbation of the leaf in the foliation

The following statement holds:

Change of leaves and matching distance

Let us set $C = \max\{\|\vec{\varphi}\|_\infty, \|\vec{\psi}\|_\infty\}$ and $d(\vec{\lambda},\vec{\beta}) = d_{match}\left(\check{\rho}_{\left(X,\mu\cdot F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}\right)}, \check{\rho}_{\left(Y,\mu\cdot F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}\right)}\right)$. Let us assume that $\|(\vec{\lambda},\vec{\beta}) - (\vec{\lambda}',\vec{\beta}')\|_\infty \leq \epsilon$, with $\epsilon \leq \frac{1}{4}$. Then

$$\left|d(\vec{\lambda},\vec{\beta}) - d(\vec{\lambda}',\vec{\beta}')\right| \leq \epsilon \cdot (32C + 2)$$

Evaluating the matching distance between rank invariants

Let us simplify our notations

The strip $(0,1) \times \mathbb{R}$

In order to simplify the study of $d(\vec{\lambda},\vec{\beta})$, we observe that $(\vec{\lambda},\vec{\beta})$ is identified by the pair (λ_1,β_1) (since $\lambda_2 = 1-\lambda_1$ and $\beta_2 = -\beta_1$). In the following we shall speak of the value of $d(\vec{\lambda},\vec{\beta})$ at the point $(\lambda_1,\beta_1) \in (0,1) \times \mathbb{R}$: we shall mean the value of $d(\vec{\lambda},\vec{\beta})$ at the point $((\lambda_1,\lambda_2),(\beta_1,\beta_2))$.

Evaluating the matching distance between rank invariants

Relationship between $d(\vec{\lambda}, \vec{\beta})$ and the 1-dimensional matching distance

The knowledge of the function $d(\vec{\lambda}, \vec{\beta})$ implies the knowledge of the 1-dimensional matching distance:

Theorem

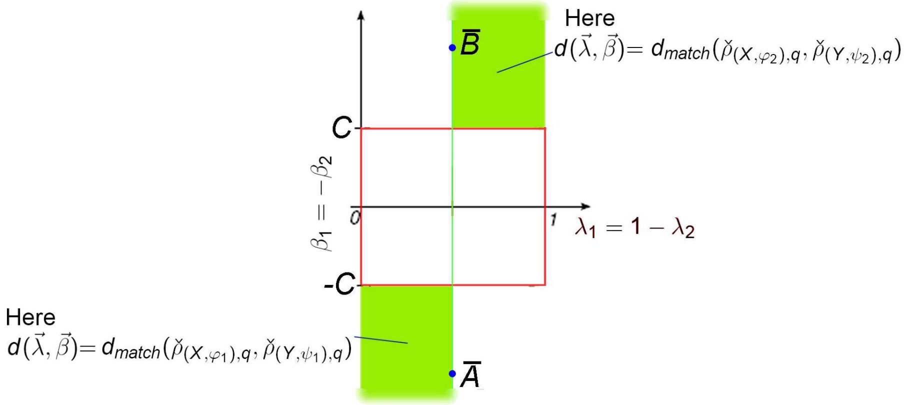
$$d(\vec{\lambda}, \vec{\beta}) = \begin{cases} \frac{\min(\lambda_1, 1-\lambda_1)}{\lambda_1} \cdot d_{match}\left(\check{\rho}_{(X,\varphi_1),q}, \check{\rho}_{(Y,\psi_1),q}\right), & \text{if } \beta_1 \leq -C \\ \frac{\min(\lambda_1, 1-\lambda_1)}{1-\lambda_1} \cdot d_{match}\left(\check{\rho}_{(X,\varphi_2),q}, \check{\rho}_{(Y,\psi_2),q}\right), & \text{if } \beta_1 \geq C \end{cases}$$

where $C = \max\{\|\vec{\varphi}\|_\infty, \|\vec{\psi}\|_\infty\}$.

Evaluating the matching distance between rank invariants

Let us simplify our notations

In plain words, considering the strip $(0,1) \times \mathbb{R}$, we have the situation represented in this figure:



An Algorithm to Compute the Multidimensional Matching Distance

Previous results open the way to the approximation of the matching distance between 2-dimensional rank invariants.

Indeed, if we take a finite grid of points G in $(0,1) \times \mathbb{R}$ in such the way that each point of $(0,1) \times \mathbb{R}$ has distance from G less than ϵ then the matching distance

$$D_{match}\left(\check{\rho}_{(X,\vec{\varphi}),q}, \check{\rho}_{(Y,\vec{\psi}),q}\right) = \sup_{(\lambda_1,\beta_1)\in(0,1)\times\mathbb{R}} d(\vec{\lambda},\vec{\beta})$$

is approximated with an error less than $\epsilon \cdot (32C + 2)$ by the pseudodistance

$$\widetilde{D}_{match}\left(\check{\rho}_{(X,\vec{\varphi}),q}, \check{\rho}_{(Y,\vec{\psi}),q}\right) = \sup_{(\lambda_1,\beta_1)\in G} d(\vec{\lambda},\vec{\beta})$$

where $C = \max\{\|\vec{\varphi}\|_\infty, \|\vec{\psi}\|_\infty\}$.

An Algorithm to Compute the Multidimensional Matching Distance

Therefore, in order to compute the matching distance between rank invariants we can proceed this way:

- We fix an error tolerance $\eta > 0$ and set $\epsilon = \frac{1}{8}$;
- We choose a finite grid whose ϵ dilation includes the set $(0,1) \times [-C, C]$;
- We consider two further points $\bar{A} = (\frac{1}{2}, -(C + \frac{1}{2}))$ and $\bar{B} = (\frac{1}{2}, C + \frac{1}{2})$;
- We compute $d(\vec{\lambda}, \vec{\beta})$ for each point of $G \cup \{\bar{A}, \bar{B}\}$ and call D the maximum of these values;
- If $\epsilon \cdot (32C + 2) \leq \eta$, D is the wanted approximation of the 2-dimensional matching distance and the algorithm ends; otherwise we refine the grid in the neighborhood of radius ϵ (w.r.t. the L_∞ norm) of each points of G at whose center $d(\vec{\lambda}, \vec{\beta})$ takes a value having a distance from D less than $\epsilon \cdot (32C + 2)$. Then we go again to the previous point, after replacing ϵ with $\frac{\epsilon}{2}$.

The Multidimensional Matching Distance in Action

- The following figures A, B, C, D, E show some examples of the computation of the 2-dimensional matching distance between 3D models taken from the SHREC 2007 database.
- The 2-dimensional measuring function is $\vec{\varphi} = (\varphi_1, \varphi_2)$, with φ_1 the integral geodesic distance and φ_2 the distance from the vector $\vec{w} = \frac{\int_S (x-B)\|x-B\| \ d\sigma}{\int_S \|x-B\|^2 \ d\sigma}$, where S is the surface of the 3D object that we are studying and B is its barycenter. The functions φ_1, φ_2 are normalized so that they range in the interval $[0, 1]$.
- An analogous procedure is used for the measuring function $\vec{\psi}$. This implies that the constant $C = \max(\|\vec{\varphi}\|_\infty, \|\vec{\psi}\|_\infty)$ is equal to 1.

The Multidimensional Matching Distance in Action

- We fix an error tolerance η equal to 5% of the constant C, that is, $\eta = 0.05$. Six iterations are required for the threshold $t = \epsilon \cdot (32C + 2)$ to become less than η.
- Each plot in Figures A, B, C, D, E shows the values of $d(\vec{\lambda}, \vec{\beta})$. In the color coding, red corresponds to higher values, whereas blue corresponds to lower values.

Figure A

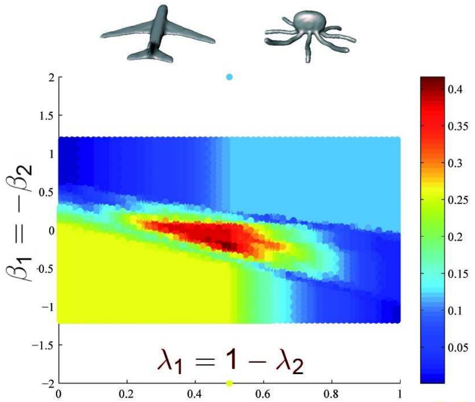


Figure: The function $d(\vec{\lambda}, \vec{\beta})$ for an airplane and an octopus models, shown on top of the plot. We fix an error tolerance η equal to 5% of the constant C, that is, $\eta = 0.05$, being $C = 1$.

Figure B

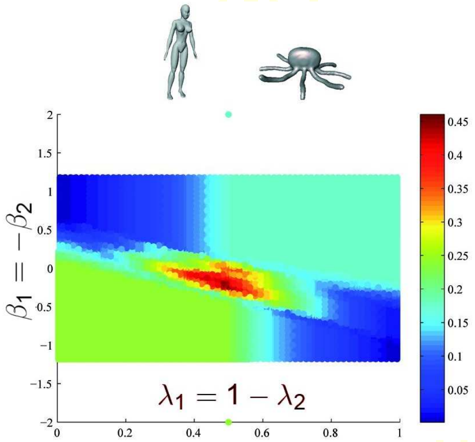


Figure: The function $d(\vec{\lambda}, \vec{\beta})$ for a human and an octopus models, shown on top of the plot. We fix an error tolerance η equal to 5% of the constant C, that is, $\eta = 0.05$, being $C = 1$.

Figure C

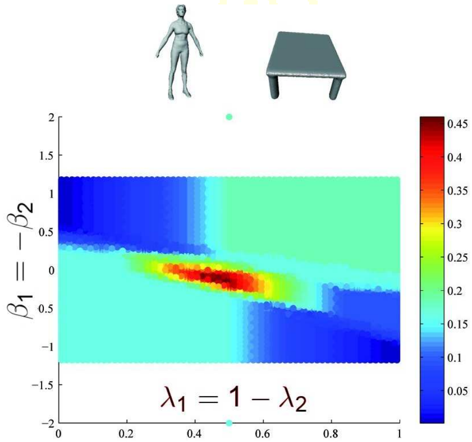


Figure: The function $d(\vec{\lambda}, \vec{\beta})$ for an airplane and a table models, shown on top of the plot. We fix an error tolerance η equal to 5% of the constant C, that is, $\eta = 0.05$, being $C = 1$.

Figure D

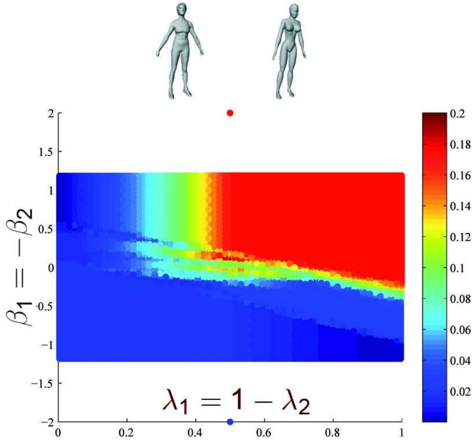


Figure: The function $d(\vec{\lambda}, \vec{\beta})$ for two human models, shown on top of the plot. We fix an error tolerance η equal to 5% of the constant C, that is, $\eta = 0.05$, being $C = 1$.

Figure E

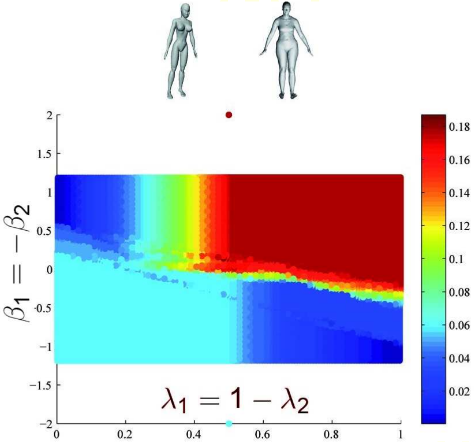


Figure: The function $d(\vec{\lambda}, \vec{\beta})$ for two human models, shown on top of the plot. We fix an error tolerance η equal to 5% of the constant C, that is, $\eta = 0.05$, being $C = 1$.

Conclusions

- We have illustrated a paradigm for shape comparison, based on a pseudometric δ between pairs $(X, \vec{\varphi})$ (named natural pseudodistance). The topological space represents the observations, while $\vec{\varphi} : X \to \mathbb{R}^k$ describes the corresponding perceptions.
- Some theorems exist, giving lower bounds for this pseudodistance. These lower bounds are based on the computation of size homotopy groups and multidimensional persistent homology groups.

Conclusions

- We have illustrated two new results about multidimensional persistent homology groups, both of them based on the foliation method:
  - A theorem allowing us to localize discontinuities of the rank invariant, based on the concept of Pareto-critical value. This result makes the computation of the rank invariant easier, since it allows us to split Δ^+ into connected components at which the rank invariant is constant.
  - A theorem bounding the change of the function $d(\vec{\lambda}, \vec{\beta})$ when we change the pair $(\vec{\lambda}, \vec{\beta})$. This result opens the way to the computation of the matching distance between rank invariants, as shown in our examples.