Necessary conditions for discontinuities of multidimensional persistent Betti numbers

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Topological persistence has proven to be a promising framework for dealing with problems concerning the analysis of data. In this context, it was originally introduced by taking into account 1-dimensional properties of data, modeled by real-valued functions. More recently, topological persistence has been generalized to consider multidimensional properties of data, coded by vector-valued functions. This extension enables the study of *multidimensional persistent Betti numbers*, which provide a representation of data based on the properties under examination. In this contribution we establish a new link between multidimensional topological persistence and Pareto optimality, proving that discontinuities of multidimensional persistent Betti numbers are necessarily pseudocritical or special values of the considered functions.

Keywords: topological persistence, foliation method, Pareto optimality.

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Introduction

Topological data analysis aims at studying stable properties of data, in order to infer their global structure and capture meaningful information about the phenomena they represent. Topological persistence (hereafter simply persistence) is a possible approach to topological data analysis, based on the assumption that meaningful information may be described as (sets of) functions defined on data. In the classical persistence setting, data are represented by a topological space X, and each considered function f is real-valued, that is, $f : X \to \mathbb{R}$. The collection of sub-level sets $X_u = f^{-1}((-\infty, u])$, $u \in \mathbb{R}$, forms a nested sequence and induces a filtration of X. Homology allows for tracking the occurrence of meaningful topological events along

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the filtration, such as the birth or the death of connected components, tunnels and void, and to rank these events by importance, that is, according to their life length. In this way, it is possible to provide a multi-scale description of data: long-lived features provide a coarse, global information, whereas short-lived ones stand for noise and details. All the information can then be encoded in a parameterized version of Betti numbers, known in the literature as *persistent Betti numbers* [29], a *rank invariant* [11] and, for the 0*th* homology, a *size function* [30].

The constantly growing interest for the persistence approach to data analysis is due to the fact that persistent Betti numbers (PBNs) can be stored in compact, yet informative descriptors called *persistent diagrams*, which are robust to functions' perturbations [13,18,22]. Roughly, this means that global topological features are stable under small changes in the considered functions, whereas variations may happen at a local scale.

From the application viewpoint, stability implies resistance to noise, thus persistence emerged as a viable option for concretely analyzing and comparing data from the topological perspective. The first results in this sense were reached using size functions for shape analysis [41,42]. After that, persistence has revealed to be useful for a wide range of applications, including point cloud data analysis [19,33], holes detection in sensor networks [23] segmentation [37,39] and image analysis [3,20], beyond confirming its potential for shape description, comparison and retrieval [6,15,18,26].

Multidimensional persistence

However, a common scenario in applications is to deal with multi-parameter information. This is usually the case, for example, in the analysis and comparison of time-varying CT scans in medical imaging. Moreover, sometime the properties of data to be studied are intrinsically multidimensional, such as the coordinates of an object in a 2D or 3D image (e.g. for tracking applications), or photometric properties, which are usually taken into account in digital image segmentation. These considerations drove the attention to the concept of *size homotopy group* [34] and, later on, to the theory of *multidimensional persistence* [11]. Here the term multidimensional, or equivalently k-dimensional, refers to the fact that data properties are described through functions taking values in $\mathbb{R}^k$ [9,28,35].

While scalar-valued functions induce 1-parameter filtrations, the use of vector-valued functions give rise to multi-parameter filtrations, which in turn enables the study of *multidimensional* persistent Betti numbers. An approach to this research is the one proposed in [5,7], which is based on the *foliation method*: The authors show that, when $k > 1$, a dimensionality reduction can be used to decompose k-dimensional PBNs into a family of 1-dimensional PBNs. This allows for the definition of a proven stable distance between k-dimensional PBNs [13], which can be effectively evaluated through suitable approximation techniques [4,12]. Beyond stability, the foliation method has led to prove that multidimensional PBNs allow for the reconstruction of planar curves, thus providing the first advancement towards the solution of the inverse problem in persistence [32].

Motivations and contribution of the paper

Previous work says not so much about the intrinsic structure of multidimensional persistent Betti numbers, which is therefore still not clear. This is actually in contrast with the 1-dimensional situation, since the basic properties of PBNs associated with scalar functions are well-known.

Increasing the knowledge about the structure of multidimensional PBNs would be interesting not only from the theoretical point of view, but also from the application perspective, since it may open the way toward new techniques for their evaluation and comparison, possibly giving insights on how to improve the existing methods.

In this paper we start to fill this gap by proving some new result result about the discontinuities of multidimensional PBNs. By making use of the foliation method, we establish a connection between multidimensional persistence and Pareto Optimality, which is a central notion in the field of Multi-Objective Optimization. More precisely, we show that the discontinuities of a multidimensional PBNs can be located just at points with at least one *pseudocritical* (a.k.a. *Pareto critical*) coordinate (Theorem 2.4).

To establish the correlation between PBNs and Pareto Optimality, we first need to consider multi-parameter filtrations given by C^1 functions. However, many interesting situations arise when the properties under study are modeled by less regular functions: for example, the distance function from a point cloud. Therefore, as a further contribution we refine the previous result to the case of continuous functions, by using the notions of *special point* and *special value* (Theorem 2.5).

This paper is organized in two sections. In Section 1 the basic results about multidimensional size functions are recalled, while in Section 2 our main theorems are proved.

1. Preliminary Results on Persistence

In this section we recall some basic definitions and results about multidimensional persistence, by focusing on those that will be useful for what follows. According to the topic of this paper, the main reference here is [13]. For further details about persistence in the multidimensional setting, the reader is also referred to [11,34].

In the classical persistence framework, the main object of study is the so-called *filtration*, that is, a nested sequence of subspaces of a given space X. In [24], it has been shown that every complete, compact and stable filtration consists in the sublevel sets of a suitable continuous function defined on X.

Unless clearly stated, hereafter X is a compact, triangulable topological space. Any function from X to $\mathbb{R}^k$ is called *a filtering function*, and is supposed to be continuous. The relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^k$ as follows: for $u = (u_1, \ldots, u_k)$

and $v = (v_1, \ldots, v_k)$, we write $u \preceq v$ (resp. $u \prec v$) if and only if $u_i \leq v_i$ (resp. $v_i < v_i$) for every $i = 1, \ldots, k$. Moreover, $\mathbb{R}^k$ is equipped with the usual max-norm, that is, $\|u\|_\infty = \max_i |u_i|$.

For every function $f : X \to \mathbb{R}^k$, the sublevel set $\{x \in X : f(x) \preceq u\}$ is denoted by $X\langle f \preceq u\rangle$. We also introduce the following notations: the open set $\{(u,v) \in \mathbb{R}^k \times \mathbb{R}^k : u \prec v\}$ will be referred to as Δ^+, while $\Delta = \partial\Delta^+$; Δ^* will denote the set $\Delta^+ \cup \{(u,\infty) : u \in \mathbb{R}^k\}$. Finally, $\overline{\Delta^*} = \Delta^* \cup \Delta$.

Whenever $u \prec v$, we consider the inclusion of $X\langle f \preceq u\rangle$ into $X\langle f \preceq v\rangle$. Such inclusion induces in turn a homomorphism of homology groups $\iota_j^{u,v} : H_j(X\langle f \preceq u\rangle) \to H_j(X\langle f \preceq v\rangle)$. Following [13], we assume here to make use of Čech homology, and refer the reader to that paper for a detailed explanation about preferring this homology theory to others. The image of $\iota_j^{u,v}$ consists of the j-homology classes of cycles which "are born" no later than u and are "still alive" at v, and is called the *multidimensional jth persistent homology group of* (X, f) *at* (u, v).

Our requirements on X imply that multidimensional persistent homology groups are finitely generated [8,13]. Moreover, by further assuming to work with coefficients in a field $\mathbb{K}$, we have that homology groups are vector spaces, and homomorphisms induced in homology by continuous maps are linear maps. Thus the rank of $\iota_j^{u,v}$, i.e. the dimension of its image, completely determines persistent homology groups, leading to the notion of *persistent Betti numbers*.

Definition 1.1 (Persistent Betti Numbers). The *persistent Betti numbers function (briefly PBNs function) of* $f : X \to \mathbb{R}^k$ is the function $\beta_f : \Delta^+ \to \mathbb{N}$ taking each $(u,v) \in \Delta^+$ to

$$\beta_f(u,v) = \operatorname{rk} \iota_j^{u,v}.$$

Note that, for each $j \in \mathbb{Z}$, we have different PBNs for f (which should be denoted $\beta_{f,j}$, say). However, for the sake of notational simplicity, we omit any reference to j. This will also apply to the notations used for other concepts in this paper, such as multiplicities. In what follows, we will also refer to the case of filtering functions taking values in $\mathbb{R}^k$ by the term "k-dimensional".

Among the properties of PBNs, it is worth mentioning monotonicity, which will be useful in the rest of the paper. For $v \in \mathbb{R}^k$, we denote by $\beta_f(\cdot, v)$ the function taking each $u \prec v$ to the value $\beta_f(u,v)$. A similar meaning will be given to $\beta_f(u, \cdot)$.

Proposition 1.1 (Monotonicity). *The function $\beta_f(\cdot, v)$ is non-decreasing in u, while the function $\beta_f(u, \cdot)$ is non-increasing in v (with respect to $\preceq$).*

1.1. *The case $k = 1$*

We now briefly review the particular case when f is real-valued. We start with an example. Figure 1(a) shows a curve depicted by a solid line, i.e. the space X, filtered by the ordinate function f. In Figure 1(b) the associated PBNs function β_f for the 0th homology is given.

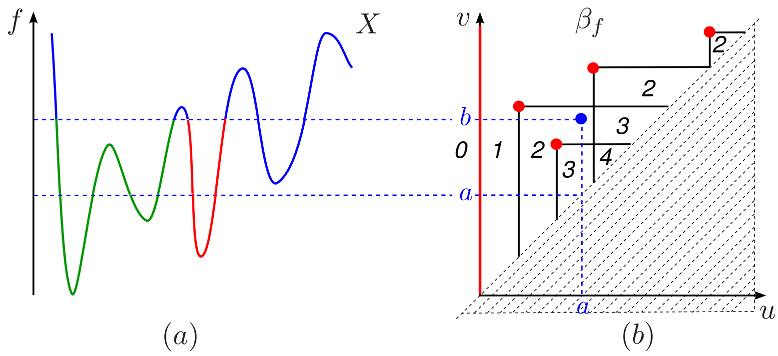

Fig. 1. (a) The topological space X and the function f. (b) The PBNs function β_f.

As can be seen, the domain $\Delta^+ = \{(u,v) \in \mathbb{R}^2 : u < v\}$ of β_f is divided into bounded and unbounded regions, in each of which the PBNs function takes a constant value: The displayed numbers coincide with the values of β_f in each region. For example, in order to compute the value of β_f at the point (a,b), it is sufficient to count how many of the three connected components in the sublevel set $X\langle f \le a\rangle$, which "are born" no later than a, are "still alive" at b. One can check that $\beta_f(a,b) = 2$: Indeed, a "death" occurs when the two green connected components merge together at some level between a and b.

Persistence diagrams

1-dimensional PBNs can be compactly described by the corresponding *persistence diagram*, a multiset of points lying on $\overline{\Delta^*}$ [22]. Persistence diagrams can be formally defined via the concept of *cornerpoint* [13,31].

Definition 1.2 (Proper cornerpoint). For every point $p = (u,v) \in \Delta^+$, the *multiplicity* $\mu(p)$ is defined as the minimum, over all the positive real numbers ε with $u + \varepsilon < v - \varepsilon$, of

$$\beta_f(u+\varepsilon, v-\varepsilon) - \beta_f(u-\varepsilon, v-\varepsilon) - \beta_f(u+\varepsilon, v+\varepsilon) + \beta_f(u-\varepsilon, v+\varepsilon).$$

When $\mu(p)$ is strictly positive, the point p is said to be a *proper cornerpoint* for β_f.

Definition 1.3 (Cornerpoint at infinity). For every point $p = (u,\infty) \in \Delta^*$, the *multiplicity* $\mu(p)$ is defined as the minimum, over all the positive real numbers ε with $u + \varepsilon < 1/\varepsilon$, of

$$\beta_f(u+\varepsilon, 1/\varepsilon) - \beta_f(u-\varepsilon, 1/\varepsilon).$$

When $\mu(p)$ is strictly positive, the point p is said to be a *cornerpoint at infinity* for β_f.

Definition 1.4 (Persistence diagram). The persistence diagram $\mathrm{Dgm}(f)$ is the multiset of all cornerpoints (both proper and at infinity) for β_f, counted with their multiplicity, union the points of Δ, counted with infinite multiplicity.

For example, the persistence diagram associated to the PBNs function in Figure $1(b)$ is given by a cornerpoint at infinity (represented by a red line) and four proper cornerpoints (still highlighted in red), together with the points of $\Delta : u = v$.

The following Representation Theorem 1.1 shows that persistence diagrams uniquely determine 1-dimensional PBNs (the inverse also holds by definition of persistence diagram). Roughly, the theorem claims that, for $(\bar{u}, \bar{v}) \in \Delta^+$, the value $\beta_f(\bar{u}, \bar{v})$ equals the number of cornerpoints lying above and on the left of $(\bar{u}, \bar{v})$.

Theorem 1.1 (Representation Theorem). *For every $(\bar{u}, \bar{v}) \in \Delta^+$, it holds that*

$$\beta_f(\bar{u}, \bar{v}) = \sum_{u \le \bar{u}\,, v > \bar{v}} \mu((u, v)) + \sum_{u \le \bar{u}} \mu((u, \infty)).$$

Matching distance

A consequence of the Representation Theorem 1.1 is that every distance between persistence diagrams naturally induces a distance between 1-dimensional PBNs. This leads to introducing the so-called *matching* (or *bottleneck*) *distance* [13,22].

Suppose that two 1-dimensional PBNs β^1 and β^2 are given, together with the corresponding persistent diagrams Dgm^1, Dgm^2. The matching distance $d_{match}(\beta^1, \beta^2)$ is defined as

$$d_{match}\left(\beta^1, \beta^2\right) = \min_{\sigma} \max_{p \in \mathrm{Dgm}^1} \delta(p, \sigma(p)),$$

where σ varies among all the bijections between Dgm^1 and Dgm^2 and

$$\delta((u, v), (u', v')) = \min\left\{\max\left\{|u - u'|, |v - v'|\right\}, \max\left\{\frac{v - u}{2}, \frac{v' - u'}{2}\right\}\right\},$$

for every (u, v), $(u', v') \in \overline{\Delta^*}$. Here we assume the convention about ∞ that $\infty - v = v - \infty = \infty$ when $v \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{c, \infty\} = c$ and $\max\{c, \infty\} = \infty$.

In plain words, the matching distance between two PBNs functions measures the minimum cost of a correspondence between the points of the associated persistence diagrams. In doing this, the pseudometric δ is used to evaluate the pseudodistance between two points (u, v) and (u', v') as the minimum between the cost of moving one point onto the other and the cost of moving both points onto the diagonal, with respect to the max-norm and under the assumption that two points on the diagonal have vanishing pseudodistance.

1-dimensional PBNs functions are stable with respect to d_{match}: Small changes in the considered functions induces only small changes in the position of cornerpoints. This is formally stated in the following Matching Stability Theorem [13,22]:

Theorem 1.2 (Matching Stability Theorem). *If $f, g : X \to \mathbb{R}$ are continuous functions with $\max_{x\in X} |f(x) - g(x)| \leq \varepsilon$, then $d_{match}(\beta_f, \beta_g) \leq \varepsilon$.*

1.2. *Reduction to the case $k = 1$: The foliation method*

We now review the approach to multidimensional persistence proposed in [5,7]. In those works, the authors prove that the case $k > 1$ can be reduced to the framework of scalar functions by a change of variable and the use of a suitable foliation. Roughly, a foliation allows for a well-behaved decomposition of an n-dimensional manifold as a disjoint union of sub-manifolds having smaller dimension. In particular, it has been showed that a parameterized family of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ exists, such that the restriction of a k-dimensional PBNs function to each of these half-planes can be seen as a particular 1-dimensional PBNs function. The foliation method will play a central role in proving the main results of the present paper.

In what follows, elements of $\mathbb{R}^k$ will be denoted by using overarrows in case they are explicitly used as direction vectors.

Definition 1.5 (Admissible pairs). For every unit vector $\vec{m} = (m_1, \dots, m_k)$ of $\mathbb{R}^k$ such that $m_i > 0$ for $i = 1, \dots, k$, and for every point $b = (b_1, \dots, b_k)$ of $\mathbb{R}^k$ such that $\sum_{i=1}^k b_i = 0$, we shall say that the pair $(\vec{m}, b)$ is *admissible*. We shall denote the set of all admissible pairs in $\mathbb{R}^k \times \mathbb{R}^k$ by Adm_k. Given an admissible pair $(\vec{m}, b)$, we define the half-plane $\pi_{(\vec{m},b)}$ of $\mathbb{R}^k \times \mathbb{R}^k$ by the following parametric equations:

$$\begin{cases} u = s\vec{m} + b \\ v = t\vec{m} + b \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

The following proposition implies that the collection of half-planes given in Definition 1.5 is actually a foliation of Δ^+.

Proposition 1.2. *For every $(u, v) \in \Delta^+$ there exists one and only one admissible pair $(\vec{m}, b)$ such that $(u, v) \in \pi_{(\vec{m},b)}$.*

Now we can show the reduction to the case $k = 1$. Intuitively, on each half-plane $\pi_{(\vec{m},b)}$ one can find the PBNs of a certain scalar function, encoding the birth and the death of topological events that occur in the filtration induced on X by sweeping the line through u and v parameterized by $\gamma_{(\vec{m},b)} : \mathbb{R} \to \mathbb{R}^k$, with $\gamma_{(\vec{m},b)}(\tau) = \tau\vec{m} + b$. In doing this, each point of $\gamma_{(\vec{m},b)}$ is associated with a sublevel set of a certain real-valued continuous function depending on both f and $(\vec{m}, b)$.

These facts are formally stated in the following Reduction Theorem 1.3, and are illustrated in Figure 2.

Theorem 1.3 (Reduction Theorem). *Let $(\vec{m}, b)$ be an admissible pair, and let*

$f_{(\vec{m},b)} : X \to \mathbb{R}$ be defined by setting

$$f_{(\vec{m},b)}(x) = \max_{i=1,\ldots,k} \left\{ \frac{f_i(x) - b_i}{m_i} \right\} .$$

Then, for every $(u, v) = (s\vec{m} + b, t\vec{m} + b) \in \pi_{(\vec{m},b)}$ the following equality holds:

$$\beta_f(u, v) = \beta_{f_{(\vec{m},b)}}(s, t).$$

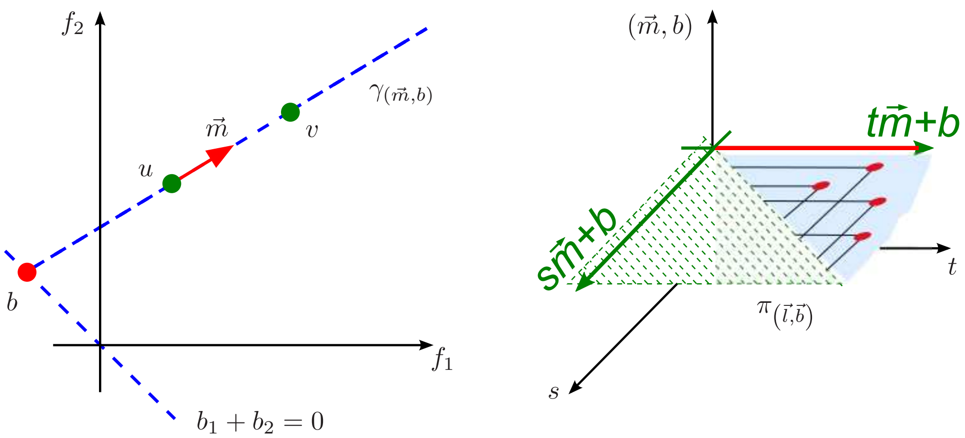

Fig. 2. Dimensional reduction for the PBNs function of $f : X \to \mathbb{R}^2$. Left: the line $\gamma_{(\vec{m},b)}$ through u and v. A unit vector $\vec{m}$ and a point b are used to parameterize this line as $\gamma_{(\vec{m},b)}(\tau) = \tau\vec{m} + b$. Right: the 1-dimensional PBNs function associated with $\gamma_{(\vec{m},b)}$ can be found on the leaf $\pi_{(\vec{m},b)}$ of the foliation.

The Reduction Theorem 1.3 implies that each k-dimensional PBNs function can be represented as a parameterized family of persistence diagrams, following the description introduced in Subsection 1.1 for the case $k = 1$. Indeed, each admissible pair $(\vec{m}, b)$ can be associated with a persistence diagram $\mathrm{Dgm}(f_{(\vec{m},b)})$ describing the PBNs of $f_{(\vec{m},b)}$. Therefore, on each $\pi_{(\vec{m},b)}$, the Matching Stability Theorem 1.2 can be applied. Moreover, the family $\left\{\mathrm{Dgm}(f_{(\vec{m},b)}) : (\vec{m}, b) \in Adm_k\right\}$ turns out to be a complete descriptor for β_f, because two k-dimensional PBNs functions coincide if and only if the corresponding parameterized families of persistence diagrams coincide.

The next result proves the stability of d_{match} with respect to the choice of the half-planes of the foliation. Indeed, small enough changes in $(\vec{m}, b)$ with respect to the max-norm induce only small changes in $\beta_{f_{(\vec{m},b)}}$ with respect to the matching distance [13, Thm. 4.5].

Proposition 1.3. *If $(\vec{m}, b) \in Adm_k$ and ε is a real number with $0 < \varepsilon < \min_{i=1,\ldots,k} m_i$, then for every admissible pair $(\vec{m}', b')$ with $\|(\vec{m}, b) - (\vec{m}', b')\|_\infty \leq \varepsilon$,* *it holds that*

$$d_{match}(\beta_{f_{(\vec{m},b)}}, \beta_{f_{(\vec{m}',b')}}) \le \varepsilon \cdot \frac{\max_{x\in X} \|f(x)\|_\infty + \|\vec{m}\|_\infty + \|b\|_\infty}{\min_{i=1,\dots,k}\{m_i(m_i-\varepsilon)\}}.$$

Analogously, it is possible to prove [13, Thm. 4.4] that d_{match} is stable with respect to the chosen filtering function, i.e. small enough changes of $f : X \to \mathbb{R}^k$ with respect to the max-norm induce small changes of $\beta_{f_{(\vec{m},b)}}$ with respect to the matching distance. Together with Proposition 1.3, this guarantees the stability of the whole approach.

2. Main Results

In this section we present new results about the discontinuities of the PBNs of a function $f : X \to \mathbb{R}^k$. We will start with some preliminary results for the case $k = 1$, which will be preparatory to prove our main theorems for the case $k > 1$.

2.1. *Case $k = 1$: Cornerpoints and discontinuity points*

A consequence of the Representation Theorem 1.1 is the following corollary. Its proof mimics the one of [31, Cor. 6], which is confined to the 0th homology situation.

Corollary 2.1. *Each discontinuity point (u, v) for β_f is such that either u is a discontinuity point for $\beta_f(\cdot, v)$, or v is a discontinuity point for $\beta_f(u, \cdot)$, or both these conditions hold.*

If we assume that $f : X \to \mathbb{R}$ is C^1, then the (finite) coordinates of a cornerpoint for β_f are critical values of f. Even if this result can be easily deduced from the related literature, to the best of our knowledge it has never been explicitly proved until now. Therefore, for the sake of completeness we formalize here this statement, which will be useful later.

Theorem 2.1. *Let X be a closed, C^1 Riemannian manifold, and let $f \in C^1(X, \mathbb{R})$. Then if $(\bar{u}, \bar{v})$ is a proper cornerpoint for β_f, it follows that both $\bar{u}$ and $\bar{v}$ are critical values of f. If $(\bar{u}, \infty)$ is a cornerpoint at infinity for β_f, it follows that $\bar{u}$ is a critical value of f.*

Proof. We confine ourselves to prove the former statement, since the proof of the latter is analogous. First of all, note that there exists a closed C^∞ Riemannian manifold $\widetilde{X}$ that is C^1-diffeomorphic to X through a C^1-diffeomorphism $h : \widetilde{X} \to X$ [36, Thm. 2.9]. Set $\tilde{f} = f \circ h$. Obviously, the PBNs functions associated with $\tilde{f} : \widetilde{X} \to \mathbb{R}$ and $f : X \to \mathbb{R}$ coincide. Therefore, $(\bar{u}, \bar{v})$ is also a cornerpoint for $\beta_{\tilde{f}}$.

We observe that the claim of our theorem holds for a closed C^∞ Riemannian manifold endowed with a Morse filtering function. This is a consequence of [25, Prop. 29] and the correspondence between critical values and homological critical values for a Morse function [2].

Now, for every real value $\varepsilon > 0$ we can find a Morse function $f_\varepsilon : \widetilde{X} \to \mathbb{R}$ such that $\max_{y\in\widetilde{X}} |\tilde{f}(y) - f_\varepsilon(y)| \le \varepsilon$ and $\max_{y\in\widetilde{X}} \left\|\nabla\tilde{f}(y) - \nabla f_\varepsilon(y)\right\| \le \varepsilon$: We can obtain f_ε by considering first the smooth function given by the convolution of $\tilde{f}$ and an opportune "regularizing" function, and then a Morse function f_ε approximating in $C^1(\widetilde{X}, \mathbb{R})$ the previous function [38, Cor. 6.8]. Therefore, from the Matching Stability Theorem 1.2 it follows that for every $\varepsilon > 0$ we can find a cornerpoint $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ for the PBNs function β_{f_ε} with $\|(\bar{u}, \bar{v}) - (\bar{u}_\varepsilon, \bar{v}_\varepsilon)\|_\infty \le \varepsilon$ and $\bar{u}_\varepsilon, \bar{v}_\varepsilon$ as critical values for f_ε. Passing to the limit for $\varepsilon \to 0$ we obtain that both $\bar{u}$ and $\bar{v}$ are critical values for $\tilde{f}$. The claim follows by observing that, since $\tilde{f}$ and f have the same critical values, both $\bar{u}$ and $\bar{v}$ are also critical values for f. □

From the Representation Theorem 1.1 and Theorem 2.1 we obtain the following corollary, refining Corollary 2.1 in the C^1 case (we skip the easy proof):

Corollary 2.2. *Let X be a closed, C^1 Riemannian manifold, and let $f \in C^1(X, \mathbb{R})$. Let also $(\bar{u}, \bar{v})$ be a discontinuity point for β_f. Then at least one of the following statements holds:*

(i) *$\bar{u}$ is both a discontinuity point for $\beta_f(\cdot, \bar{v})$ and a critical value for f;*
(ii) *$\bar{v}$ is both a discontinuity point for $\beta_f(\bar{u}, \cdot)$ and a critical value for f.*

2.2. *Case $k > 1$: necessary conditions for discontinuities of PBNs*

The generalization of Corollary 2.2 to the case $k > 1$ is not straightforward, and requires new ideas which will be given in this section, together with our main results. In order to do that, we will confine ourselves to the case when X is a closed (i.e. compact and without boundary) C^1 Riemannian n-manifold.

From now to Theorem 2.5, we shall assume that an admissible pair $(\vec{m}, b) \in Adm_k$ is fixed, and consider the PBNs function β_F with $F : X \to \mathbb{R}$ defined by setting $F(x) = \max_{i=1,\dots,k} \frac{f_i(x)-b_i}{m_i}$. We shall say that F and β_F are the filtering function and the 1-dimensional PBNs function corresponding to the half-plane $\pi_{(\vec{m},b)}$, respectively.

The main results of this section are stated in Theorem 2.4 and Theorem 2.5, showing a necessary condition for a point $(u, v) \in \Delta^+$ to be a discontinuity point for the PBNs function β_f, under the assumption that f is C^1 and C^0, respectively. For the sake of clarity, we will provide now a sketch of the arguments that will lead us to the proof of our main results.

Theorem 2.4 is a generalization for $k > 1$ of Corollary 2.2. In order to prove it, the first step is to adapt Theorem 2.1 to the 1-dimensional PBNs function β_F corresponding to the half-plane $\pi_{(\vec{m},b)}$. We recall that, according to Theorem 2.1, each finite coordinate of a cornerpoint has to be a critical value for the considered C^1 filtering function. However, in our case the scalar function F is not C^1 in general (even if f is C^1), and therefore we need to prove a modified version of Theorem 2.1. To this end, we generalize the concepts of critical point and critical value by introducing the definitions of $(\vec{m}, b)$-*pseudocritical point* and $(\vec{m}, b)$-*pseudocritical value* for a C^1 function (Definition 2.1). These notions, together with an approximation in $C^0(X, \mathbb{R})$ of the function F by C^1 functions, are used to prove that, if $f \in C^1(X, \mathbb{R}^k)$, each finite coordinate of a cornerpoint for β_F has to be an $(\vec{m}, b)$-pseudocritical value for f (Theorem 2.2).

Next, we show (Proposition 2.1) that a correspondence exists between the discontinuity points of β_F and the ones of β_f. Theorem 2.2 and Proposition 2.1 lead us to the relation (Theorem 2.3) between those discontinuity points for β_f lying on the half-plane $\pi_{(\vec{m},b)}$, and the $(\vec{m}, b)$-pseudocritical values for f. This last result is refined in Theorem 2.4 under the assumption that f is C^1, providing a necessary condition for discontinuities of β_f that does not depend on the half-planes of the foliation. This can be done by introducing the concepts of *pseudocritical point* and *pseudocritical value* (a.k.a. *Pareto critical point* and *Pareto critical value*) for an $\mathbb{R}^k$-valued C^1 function (Definition 2.2), and considering a suitable projection $\rho : \mathbb{R}^k \to \mathbb{R}^h$. The necessary condition given in Theorem 2.4 is finally extended to the case of continuous filtering functions (Theorem 2.5), once more by means of an approximation technique, and the notions of *special point* and *special value*.

Before going on, we need the following definition:

Definition 2.1. Assume that $f \in C^1(X, \mathbb{R}^k)$. For every $x \in X$, set $I_x = \left\{i \in \{1, \ldots, k\} : \frac{f_i(x)-b_i}{m_i} = F(x)\right\}$. We shall say that x is an $(\vec{m}, b)$-*pseudocritical point for* f if the convex hull of the gradients $\nabla f_i(x)$, $i \in I_x$, contains the null vector, i.e. for every $i \in I_x$ there exists a real value λ_i such that $\sum_{i \in I_x} \lambda_i \nabla f_i(x) = \mathbf{0}$, with $0 \le \lambda_i \le 1$ for $i \in I_x$ and $\sum_{i \in I_x} \lambda_i = 1$. If x is an $(\vec{m}, b)$-pseudocritical point for f, the value $F(x)$ will be called an $(\vec{m}, b)$-*pseudocritical value for* f.

The concept of $(\vec{m}, b)$-pseudocritical point is strongly connected, via the function F introduced in Definition 2.1, with the notion of generalized gradient by F. H. Clarke [21]. For a point $x \in X$, the condition of being $(\vec{m}, b)$-pseudocritical for f corresponds to the one of being "critical" for the generalized gradient of F [21, Prop. 2.3.12]. However, in this context we prefer to adopt a terminology highlighting the dependence on the considered half-plane.

Theorem 2.2. *Assume that* $f \in C^1(X, \mathbb{R}^k)$. *If* (σ, τ) *is a proper cornerpoint of* β_F, *then both* σ *and* τ *are* $(\vec{m}, b)$-*pseudocritical values for* f. *If* (σ, ∞) *is a cornerpoint at infinity of* β_F, *then* σ *is an* $(\vec{m}, b)$-*pseudocritical value for* f.

Proof. We confine ourselves to proving the former statement. Indeed, the proof of the latter is analogous. The idea is to show that our statement holds for a C^1 function approximating the filtering function $F : X \to \mathbb{R}$ in $C^0(X, \mathbb{R})$, and verify that this property passes to the limit. Let us now set $g_i(x) = \frac{f_i(x)-b_i}{m_i}$ and choose $c \in \mathbb{R}$ such that $\min_{x \in X} g_i(x) > -c$, for every $i = 1, \ldots, k$. Consider the function sequence (F_p), $p \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$, whith $F_p : X \to \mathbb{R}$ and $F_p(x) = \left(\sum_{i=1}^k (g_i(x) + c)^p\right)^{\frac{1}{p}} - c$: Such a sequence converges uniformly to the function F.

Indeed, for every $x \in X$ and for every index p we have that

$$\begin{aligned}|F(x)-F_p(x)| &= \left|\max_i g_i(x) - \left(\left(\sum_{i=1}^k (g_i(x)+c)^p\right)^{\frac{1}{p}} - c\right)\right| = \\ &= \left|\max_i\{g_i(x)+c\} - \left(\sum_{i=1}^k (g_i(x)+c)^p\right)^{\frac{1}{p}}\right| = \\ &= \left(\sum_{i=1}^k (g_i(x)+c)^p\right)^{\frac{1}{p}} - \max_i\{g_i(x)+c\} \le \\ &\le \max_i\{g_i(x)+c\}\cdot(k^{\frac{1}{p}}-1).\end{aligned}$$

Let us now consider a proper cornerpoint $\bar{C} = (\bar{u}, \bar{v})$ of β_F. By the Matching Stability Theorem 1.2 it follows that it is possible to find a large enough p and a proper cornerpoint $C_p = (u_p, v_p)$ of the 1-dimensional PBNs function β_{F_p} such that C_p is arbitrarily close to $\bar{C}$. Being C_p a proper cornerpoint of β_{F_p}, Theorem 2.1 implies that its coordinates are critical values of the C^1 function F_p. By focusing on the abscissa of C_p (similar arguments hold for the ordinate of C_p), it follows that there exists $x^p \in X$ with $u_p = F_p(x^p)$ and (in local coordinates $x_1, \ldots, x_n$ of the n-manifold X)

$$\begin{aligned}0 &= \frac{\partial F_p}{\partial x_1}(x^p) = \left(\sum_{i=1}^k (g_i(x^p)+c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{i=1}^k (g_i(x^p)+c)^{p-1}\cdot\frac{\partial g_i}{\partial x_1}(x^p)\right)\\ &\vdots\\ 0 &= \frac{\partial F_p}{\partial x_n}(x^p) = \left(\sum_{i=1}^k (g_i(x^p)+c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{i=1}^k (g_i(x^p)+c)^{p-1}\cdot\frac{\partial g_i}{\partial x_n}(x^p)\right) .\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{i=1}^k (g_i(x^p)+c)^{p-1}\cdot\frac{\partial g_i}{\partial x_1}(x^p) = 0\\ &\vdots\\ &\sum_{i=1}^k (g_i(x^p)+c)^{p-1}\cdot\frac{\partial g_i}{\partial x_n}(x^p) = 0 .\end{aligned}$$

Therefore, by setting

$$w^p = (w_1^p, \ldots, w_k^p) = \left((g_1(x^p)+c)^{p-1}, \ldots, (g_k(x^p)+c)^{p-1}\right),$$

we can write ${}^tJ(x^p) \cdot {}^t w^p = \mathbf{0}$, where $J(x^p)$ is the Jacobian matrix of $g = (g_1, \ldots, g_k)$ computed at the point x^p. By the compactness of X, we can assume (possibly by extracting a subsequence) that (x^p) converges to a point $\bar{x}$. Let us define $z^p = \frac{w^p}{\|w^p\|_\infty}$. Again by compactness (indeed, $\|z^p\|_\infty = 1$) we can also assume, possibly by considering a subsequence, that the sequence (z^p) converges to a vector $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_k)$, where $\bar{z}_i = \lim_{p\to\infty} \frac{w_i^p}{\|w^p\|_\infty}$ and $\|\bar{z}\|_\infty = 1$. Obviously ${}^t\mathcal{J}(x^p) \cdot {}^t z^p = \mathbf{0}$ and hence we have

$$ {}^t\mathcal{J}(\bar{x}) \cdot {}^t\bar{z} = \mathbf{0}. \tag{2.1} $$

Now, for every index p and for every $i = 1, \ldots, k$ the relation $0 < z_i^p \leq 1$ holds, thus the condition $0 \leq \bar{z}_i = \lim_{p\to\infty} z_i^p \leq 1$ is satisfied for each $i = 1, \ldots, k$. Let us now recall that $F(\bar{x}) = \max_i g_i(\bar{x})$ by definition, and consider the set $I_{\bar{x}} = \{i \in \{1, \ldots, k\} : g_i(\bar{x}) = F(\bar{x})\} = \{i_1, \ldots, i_h\}$. For every $r \notin I_{\bar{x}}$ the component $\bar{z}_r$ is equal to 0, because $0 \leq z_r^p = \left(\frac{g_r(x^p)+c}{\max_i\{g_i(x^p)+c\}}\right)^{p-1}$ and $\lim_{p\to\infty} \frac{g_r(x^p)+c}{\max_i\{g_i(x^p)+c\}} = \frac{g_r(\bar{x})+c}{F(\bar{x})+c}$, which is strictly less than 1 for $g_r(\bar{x}) < F(\bar{x})$. Hence we have $\bar{z} = \bar{z}_{i_1} \cdot e_{i_1} + \cdots + \bar{z}_{i_h} \cdot e_{i_h}$, where e_i is the i^{th} vector of the standard basis of $\mathbb{R}^k$. Thus, from equality (2.1) we have $\sum_{j=1}^h \bar{z}_{i_j} \cdot \frac{\partial g_{i_j}}{\partial x_1}(\bar{x}) = 0, \ldots, \sum_{j=1}^h \bar{z}_{i_j} \cdot \frac{\partial g_{i_j}}{\partial x_n}(\bar{x}) = 0$, that is, $\sum_{j=1}^h \frac{\bar{z}_{i_j}}{m_{i_j}} \cdot \frac{\partial f_{i_j}}{\partial x_1}(\bar{x}) = 0, \ldots, \sum_{j=1}^h \frac{\bar{z}_{i_j}}{m_{i_j}} \cdot \frac{\partial f_{i_j}}{\partial x_n}(\bar{x}) = 0$, since $g_i = \frac{f_i - b_i}{m_i}$. Hence, $\sum_{j=1}^h \frac{\bar{z}_{i_j}}{m_{i_j}} \nabla f_{i_j}(\bar{x}) = \mathbf{0}$. By recalling that $\bar{z}_{i_j} \geq 0$, $m_{i_j} > 0$ and $\bar{z}$ is a non–vanishing vector, it follows immediately that $\sum_{j=1}^h \frac{\bar{z}_{i_j}}{m_{i_j}} > 0$ and therefore the convex hull of the gradients $\nabla f_{i_1}(\bar{x}), \ldots, \nabla f_{i_h}(\bar{x})$ contains the null vector. Thus, $\bar{x}$ is an $(\vec{m}, b)$-pseudocritical point for f and hence $F(\bar{x})$ is an $(\vec{m}, b)$-pseudocritical value for f. Moreover, from the uniform convergence of the sequence (F_p) to F and from the continuity of the function F, we have (recall that $\bar{C} = \lim_{p\to\infty} C_p$)

$$ \bar{u} = \lim_{p\to\infty} u_p = \lim_{p\to\infty} F_p(x_p) = F(\bar{x}). $$

In other words, the abscissa $\bar{u}$ of a proper cornerpoint of β_F is the image of an $(\vec{m}, b)$-pseudocritical point $\bar{x}$ through F, i.e. an $(\vec{m}, b)$-pseudocritical value for f. An analogous reasoning holds for the ordinate $\bar{v}$ of a proper cornerpoint. □

Our next result shows that each discontinuity of β_f corresponds to a discontinuity of the 1-dimensional PBNs function associated with a suitable half-plane of the foliation.

Proposition 2.1. *A point $(u, v) = (s \cdot \vec{m} + b, t \cdot \vec{m} + b) \in \pi_{(\vec{m},b)}$ is a discontinuity point for β_f if and only if (s, t) is a discontinuity point for β_F.*

Proof. Obviously, if (s, t) is a discontinuity point for β_F, then $(u, v) = (s \cdot \vec{m} + b, t \cdot \vec{m} + b) \in \pi_{(\vec{m},b)}$ is a discontinuity point for β_f, because of the Reduction Theorem 1.3. In order to prove the inverse implication, we shall verify the contrapositive statement, i.e. if (s, t) is not a discontinuity point for β_F, then $(s\cdot\vec{m}+b, t\cdot\vec{m}+b)$ is not a discontinuity point for β_f. Indeed, if (s, t) is not a discontinuity point for β_F, then β_F is locally constant at (s, t) (recall that each PBNs function is natural-valued).

Therefore it will be possible to choose a real number $\eta > 0$ such that

$$\beta_F(s-\eta, t+\eta) = \beta_F(s+\eta, t-\eta). \tag{2.2}$$

Before proceeding in our proof, we need the following result:

Lemma 2.1. *Let $g, g' : X \to \mathbb{R}$. If $d_{match}(\beta_g, \beta_{g'}) \leq 2\varepsilon$, then it holds that*

$$\beta_g(s-\varepsilon, t+\varepsilon) \leq \beta_{g'}(s+\varepsilon, t-\varepsilon),$$

for every (s,t) with $s+\varepsilon < t-\varepsilon$.

Proof of Lemma 2.1. Recall that Δ^* is the set $\Delta^+ \cup \{(a, \infty) : a \in \mathbb{R}\}$. For every (s,t) with $s < t$, let us define the set $L_{(s,t)} = \{(\sigma, \tau) \in \Delta^* : \sigma \leq s, \tau > t\}$. By the Representation Theorem 1.1 we have that $\beta_g(s-\varepsilon, t+\varepsilon)$ equals the number of cornerpoints (both proper and at infinity) for β_g belonging to the set $L_{(s-\varepsilon,t+\varepsilon)}$. Being $d_{match}(\beta_g, \beta_{g'}) \leq 2\varepsilon$, the number of proper cornerpoints and cornerpoints at infinity for $\beta_{g'}$ in the set $L_{(s+\varepsilon,t-\varepsilon)}$ is not less than $\beta_g(s-\varepsilon, t+\varepsilon)$. The reason is that the change from g to g' does not move the cornerpoints more than 2ε, with respect to the max-norm, because of the Matching Stability Theorem 1.2. By applying the Representation Theorem 1.1 once again to $\beta_{g'}$, we get our claim. □

Let us go back to the proof of Proposition 2.1. By Proposition 1.3, we can then consider a real value $\varepsilon = \varepsilon(\eta)$ with $0 < \varepsilon < \min_{i=1,\dots,k} m_i$ such that for every admissible pair $(\vec{m}', b')$ with $\|(\vec{m}, b) - (\vec{m}', b')\|_\infty \leq \varepsilon$, the relation $d_{match}(\beta_F, \beta_{F'}) \leq \frac{\eta}{2}$ holds, where $\beta_{F'}$ is the 1-dimensional PBNs function corresponding to the half-plane $\pi_{(\vec{m}',b')}$. By applying Lemma 2.1 twice and the monotonicity of $\beta_{F'}$ in each variable (cf. Proposition 1.1), we get the inequalities

$$\begin{aligned}\beta_F(s-\eta, t+\eta) &\leq \beta_{F'}(s-\frac{\eta}{2}, t+\frac{\eta}{2}) \\ &\leq \beta_{F'}(s+\frac{\eta}{2}, t-\frac{\eta}{2}) \leq \beta_F(s+\eta, t-\eta). \end{aligned} \tag{2.3}$$

By equality (2.2) we have that the inequalities (2.3) imply

$$\begin{aligned}\beta_F(s-\eta, t+\eta) &= \beta_{F'}(s-\frac{\eta}{2}, t+\frac{\eta}{2}) \\ &= \beta_{F'}(s+\frac{\eta}{2}, t-\frac{\eta}{2}) = \beta_F(s+\eta, t-\eta). \end{aligned} \tag{2.4}$$

Therefore, once again because of the monotonicity of $\beta_{F'}$ in each variable, for every (s', t') with $\|(s,t) - (s',t')\|_\infty \leq \frac{\eta}{2}$ and for every $(\vec{m}', b')$ with $\|(\vec{m}, b) - (\vec{m}', b')\|_\infty \leq \varepsilon$ the equality $\beta_{F'}(s', t') = \beta_F(s,t)$ holds. By applying the Reduction Theorem 1.3 we get $\beta_f(s' \cdot \vec{m}' + b', t' \cdot \vec{m}' + b') = \beta_f(s \cdot \vec{m} + b, t \cdot \vec{m} + b)$. In other words, β_f is locally constant at the point (u, v), and hence (u, v) is not a discontinuity point for β_f. □

Let us observe that Proposition 2.1 holds under weaker hypotheses, i.e. in the case that X is a non-empty, compact and locally connected Hausdorff space. However, for the sake of simplicity, we prefer here to confine ourselves to the setting assumed at the beginning of the present section.

The following theorem associates the discontinuities of β_f to the $(\vec{m}, b)$-pseudocritical values of f.

Theorem 2.3. *Assume that* $f \in C^0(X, \mathbb{R}^k)$. *Let* $(u, v) \in \Delta^+$ *with* $(u, v) = (s \cdot \vec{m} + b, t \cdot \vec{m} + b) \in \pi_{(\vec{m},b)}$. *If* (u, v) *is a discontinuity point for* β_f *then at least one of the following statements holds:*

(i) *s is a discontinuity point for* $\beta_F(\cdot, t)$*;*
(ii) *t is a discontinuity point for* $\beta_F(s, \cdot)$.

Moreover, (i) *and* (ii) *are equivalent to*

(i') *u is a discontinuity point for* $\beta_f(\cdot, v)$*;*
(ii') *v is a discontinuity point for* $\beta_f(u, \cdot)$*,*

respectively. If $f \in C^1(X, \mathbb{R}^k)$*, statement* (i) *implies that s is an* $(\vec{m}, b)$*-pseudocritical value for f, and statement* (ii) *implies that t is an* $(\vec{m}, b)$*-pseudocritical value for f.*

Proof. By Proposition 2.1 we have that (s, t) is a discontinuity point for β_F, and from Corollary 2.1 it follows that either s is a discontinuity point for $\beta_F(\cdot, t)$ or t is a discontinuity point for $\beta_F(s, \cdot)$, or both these conditions hold, thus proving the first part of the theorem.

Let us now suppose that s is a discontinuity point for $\beta_F(\cdot, t)$. Being the function $\beta_F(\cdot, t)$ monotonic, for every real value $\varepsilon > 0$ we have that $\beta_F(s-\varepsilon, t) \neq \beta_F(s+\varepsilon, t)$. Moreover, the following equalities hold from the Reduction Theorem 1.3:

$$\begin{aligned} \beta_F(s-\varepsilon, t) &= \beta_f((s-\varepsilon) \cdot \vec{m} + b, t \cdot \vec{m} + b) = \beta_f(u - \varepsilon \cdot \vec{m}, v) \\ \beta_F(s+\varepsilon, t) &= \beta_f((s+\varepsilon) \cdot \vec{m} + b, t \cdot \vec{m} + b) = \beta_f(u + \varepsilon \cdot \vec{m}, v). \end{aligned} \tag{2.5}$$

By setting $\vec{\varepsilon} = \varepsilon \cdot \vec{m}$, we get $\beta_f(u - \vec{\varepsilon}, v) \neq \beta_f(u + \vec{\varepsilon}, v)$. Therefore u is a discontinuity point for $\beta_f(\cdot, v)$, thus proving that $(i) \Rightarrow (i')$.

Let us now prove that $(i') \Rightarrow (i)$. If u is a discontinuity point for $\beta_f(\cdot, v)$, from the monotonicity in the variable u (cf. Proposition 1.1) it follows that $\beta_f(u - \varepsilon \cdot \vec{m}, v) \neq \beta_f(u + \varepsilon \cdot \vec{m}, v)$ for every $\varepsilon > 0$. Therefore, from the equalities (2.5) we get $\beta_F(s-\varepsilon, t) \neq \beta_F(s+\varepsilon, t)$, proving that $(i') \Rightarrow (i)$. Analogously, we can show that $(ii) \Leftrightarrow (ii')$.

Furthermore, if s is a discontinuity point for $\beta_F(\cdot, t)$, from the Representation Theorem 1.1 it follows that s is the abscissa of a cornerpoint (possibly at infinity). Hence, if $f \in C^1(X, \mathbb{R}^k)$ then by Theorem 2.2 we have that s is an $(\vec{m}, b)$-pseudocritical value for f.

In a similar way, we can examine the case that t is a discontinuity point for $\beta_F(s, \cdot)$, and get the final statement. □

Before giving our first main result, we need the following definition.

Definition 2.2. Let $\mathcal{L} : X \to \mathbb{R}^h$, and suppose that $\mathcal{L}$ is C^1 at a point $x \in X$. The point x is said to be a *pseudocritical point for* $\mathcal{L}$ if the convex hull of the gradients $\nabla\mathcal{L}_i(x)$, $i = 1, \dots, h$, contains the null vector, i.e. there exist $\lambda_1, \dots, \lambda_h \in \mathbb{R}$ such that $\sum_{i=i}^h \lambda_i \cdot \nabla\mathcal{L}_i(x) = \mathbf{0}$, with $0 \le \lambda_i \le 1$ for $1 \le i \le h$, and $\sum_{i=1}^h \lambda_i = 1$. If x is a pseudocritical point of $\mathcal{L}$, then $\mathcal{L}(x)$ will be called a *pseudocritical value for* $\mathcal{L}$.

Definition 2.2 corresponds to the Fritz John necessary condition for optimality in Nonlinear Programming [1]. We shall use the term "pseudocritical" just for the sake of conciseness. For further references see [40]. The concept of pseudocritical point is strongly related also to the one of Jacobi Set [27]. In literature, pseudocritical points are also called Pareto critical points.

The next example makes Definition 2.2 clearer.

Example 2.1. Let us compute the pseudocritical points and values for the function $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : X \to \mathbb{R}^2$, where X is the surface coinciding with the unit sphere $S^2 \subset \mathbb{R}^3$, and $\mathcal{L}$ is obtained as the restriction to X of the function $L = (L_1, L_2) : \mathbb{R}^3 \to \mathbb{R}^2$, with $L(x_1, x_2, x_3) = (x_1, x_3)$ (see Figure 3). According to Definition 2.2, it follows that a point $x \in X$ is pseudocritical for $\mathcal{L}$ if and only if either $\nabla\mathcal{L}_1(x) = \mathbf{0}$, or $\nabla\mathcal{L}_2(x) = \mathbf{0}$, or these two gradient vectors are parallel with opposite verse. Referring to our example, $\nabla\mathcal{L}_1(x)$ and $\nabla\mathcal{L}_2(x)$ are the orthogonal projections of $\nabla L_1(x) = (1, 0, 0)$ and $\nabla L_2(x) = (0, 0, 1)$ onto the tangent space of X at x, respectively. Therefore, it can be easily verified that the pseudocritical points of X for the function $\mathcal{L}$ are given by the set $\{(\cos\alpha, 0, \sin\alpha) : 0 \le \alpha \le \frac{\pi}{2} \ \vee \ \pi \le \alpha \le \frac{3}{2}\pi\}$. Hence, the corresponding pseudocritical values are the elements of the set $\{(\cos\alpha, \sin\alpha) : 0 \le \alpha \le \frac{\pi}{2} \ \vee \ \pi \le \alpha \le \frac{3}{2}\pi\}$.

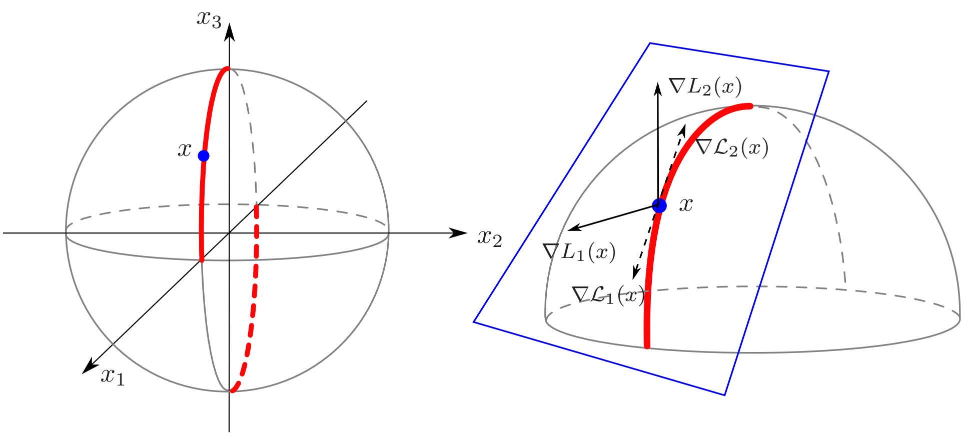

Fig. 3. (*a*) The sphere $S^2 \subseteq \mathbb{R}^3$ endowed with the filtering function $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : S^2 \to \mathbb{R}^2$, defined as $\mathcal{L}(x_1, x_2, x_3) = (x_1, x_3)$ for each $(x_1, x_2, x_3) \in S^2$. The pseudocritical points of $\mathcal{L}$ are depicted in bold red. (*b*) The point x is a pseudocritical point for $\mathcal{L}$, because the vectors $\nabla\mathcal{L}_1(x)$ and $\nabla\mathcal{L}_2(x)$ are parallel with opposite verse.

Let $(i_1, ..., i_h) \in \{1, ..., k\}^h$ with $h \leq k$ and $i_1 < ... < i_h$. In the following, we shall say that $\rho : \mathbb{R}^k \to \mathbb{R}^h$ is a *projection associated with the h-tuple* $(i_1, \ldots, i_h)$ if $\rho((u_1, \ldots, u_k)) = (u_{i_1}, \ldots, u_{i_h})$, for every $u = (u_1, \ldots, u_k) \in \mathbb{R}^k$. In other words, such a function ρ is used to delete some components of $u \in \mathbb{R}^k$.

We are now ready to give the first main result of this paper.

Theorem 2.4. *Assume that $f \in C^1(X, \mathbb{R}^k)$. Let $(u, v) \in \Delta^+$ be a discontinuity point for β_f. Then at least one of the following statements holds:*

(i) u is a discontinuity point for $\beta_f(\cdot, v)$;
(ii) v is a discontinuity point for $\beta_f(u, \cdot)$.

Moreover, if (i) holds, then a projection ρ exists such that $\rho(u)$ is a pseudocritical value for $\rho \circ f$. If (ii) holds, then a projection ρ exists such that $\rho(v)$ is a pseudo-critical value for $\rho \circ f$.

Proof. By Proposition 1.2, an admissible pair $(\vec{m}, b)$ exists, such that $(u, v) = (s \cdot \vec{m} + b, t \cdot \vec{m} + b)$ for a suitable pair (s, t). Statements (i) and (ii) are guaranteed by Theorem 2.3, assuring that either u is a discontinuity point for $\beta_f(\cdot, v)$ and s is an $(\vec{m}, b)$- pseudocritical value for f, or v is a discontinuity point for $\beta_f(u, \cdot)$ and t is an $(\vec{m}, b)$-pseudocritical value for f, or both these conditions hold.

Let us now confine ourselves to assume that u is a discontinuity point for $\beta_f(\cdot, v)$ and s is an $(\vec{m}, b)$-pseudocritical value for f. We shall prove that a projection ρ exists such that $\rho(u)$ is a pseudocritical value for $\rho \circ f$. The proof in the case that v is a discontinuity point for $\beta_f(u, \cdot)$ and t is an $(\vec{m}, b)$-pseudocritical value for f works in quite a similar way. Being s an $(\vec{m}, b)$-pseudocritical value for f, by Definition 2.1 there exist a point $x \in X$, some indices $i_1, \ldots, i_h \in \{1, \ldots, k\}$ with $h \leq k$ and $i_1 < \cdots < i_h$, and an h-tuple $(\lambda_1, \ldots, \lambda_h)$ such that $s = F(x) = \frac{f_{i_1}(x) - b_{i_1}}{m_{i_1}} = \cdots = \frac{f_{i_h}(x) - b_{i_h}}{m_{i_h}}$ and $\sum_{j=1}^h \lambda_j \cdot \nabla f_{i_j}(x) = \mathbf{0}$, with $0 \leq \lambda_j \leq 1$ for $j = 1, \ldots, h$, and $\sum_{j=1}^h \lambda_j = 1$. Let us now consider the projection $\rho : \mathbb{R}^k \to \mathbb{R}^h$ defined by setting $\rho(u) = (u_{i_1}, \ldots, u_{i_h})$. Being $(u, v) = ((u_1, \ldots, u_k), (v_1, \ldots, v_k)) = ((s \cdot m_1 + b_1, \ldots, s \cdot m_k + b_k), (t \cdot m_1 + b_1, \ldots, t \cdot m_k + b_k))$, we observe that $u_{i_j} = \left(\frac{f_{i_j}(x) - b_{i_j}}{m_{i_j}}\right) \cdot m_{i_j} + b_{i_j} = f_{i_j}(x)$, for every $j = 1, \ldots, h$. Therefore, $\rho(u) = \rho \circ f(x)$. Being x a pseudo-critical point for $\rho \circ f$, it follows that $\rho(u)$ is a pseudocritical value for $\rho \circ f$. □

Note that Theorem 2.4 refines the result obtained in Theorem 2.3, providing a relation between discontinuities of PBNs functions and pseudo-criticality without any reference to the foliation of Δ^+.

2.3. *Refining Theorem 2.4 to less regular filtering functions*

In this section we generalize Theorem 2.4 to the case of continuous filtering functions. In what follows, we shall call a *special point for a continuous function*

$\mathcal{L} : X \to \mathbb{R}^h$ any point in the closure of the set where $\mathcal{L}$ is not C^1. If x is a special point for $\mathcal{L}$, the value $\mathcal{L}(x)$ will be called a *special value for* $\mathcal{L}$.

Theorem 2.5. *Assume that* $f \in C^0(X, \mathbb{R}^k)$*. Let* $(u, v) \in \Delta^+$ *be a discontinuity point for* β_f*. Then at least one of the following statements holds:*

(i) *u is a discontinuity point for* $\beta_f(\cdot, v)$*;*
(ii) *v is a discontinuity point for* $\beta_f(u, \cdot)$*.*

Moreover, if (i) *holds, then a projection* ρ *exists such that* $\rho(u)$ *is either a special value or a pseudocritical value for* $\rho \circ f$*. If* (ii) *holds, then a projection* ρ *exists such that* $\rho(v)$ *is either a special value or a pseudocritical value for* $\rho \circ f$*.*

Proof. By Proposition 1.2, an admissible pair $(\vec{m}, b)$ exists, such that $(u, v) = (s \cdot \vec{m} + b, t \cdot \vec{m} + b)$ for a suitable pair (s, t). Statements (i) and (ii) are guaranteed by Theorem 2.3, assuring that either u is a discontinuity point for $\beta_f(\cdot, v)$ and s is a discontinuity point for $\beta_F(\cdot, t)$, or v is a discontinuity point for $\beta_f(u, \cdot)$ and t is a discontinuity point for $\beta_F(s, \cdot)$, or both these conditions hold. (We recall that F is given by $\max_{j=1,\ldots,k} \left\{ \frac{f_j - b_j}{m_j} \right\}$).

Let us now assume that u is a discontinuity point for $\beta_f(\cdot, v)$ and s is a discontinuity point for $\beta_F(\cdot, t)$. We shall prove that a projection ρ exists such that $\rho(u)$ is either a special value or a pseudocritical value for $\rho \circ f$.

Call $\mathbb{S}_j$ the set of special points of $f_j : X \to \mathbb{R}$, for $j = 1, \ldots, k$. By definition, the set $\mathbb{S}_j$ is closed. For every $i \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ and $j = 1, \ldots, k$, consider the compact set K_j^i defined as $\{x \in X : d(x, \mathbb{S}_j) \geq \frac{1}{i}\}$ if $\mathbb{S}_j$ is non-empty, and as X otherwise. Furthermore, take a C^1 function $f_j^i : X \to \mathbb{R}$ such that

(1) $\max_{x \in X} |f_j(x) - f_j^i(x)| \leq \frac{1}{i}$;
(2) if $K_j^i \neq \emptyset$, $\max_{x \in K_j^i} \|\nabla f_j(x) - \nabla f_j^i(x)\| \leq \frac{1}{i}$.

This can be done by considering the convolution of each component f_j, $j = 1, \ldots, k$, with a suitable “regularizing” function.

We now set $F^i = \max_{j=1,\ldots,k} \left\{ \frac{f_j^i - b_j}{m_j} \right\}$, with $f^i = (f_1^i, \ldots, f_k^i)$ for every $i \in \mathbb{N}^+$.

Being s a discontinuity point for $\beta_F(\cdot, t)$, by the Representation Theorem 1.1 it follows that a cornerpoint of β_F (proper or at infinity) of coordinates $(s, \bar{t})$ exists, with $\bar{t} > t$. Moreover, by condition (1) we have that the sequence (F^i) uniformly converges to F. Therefore, the Matching Stability Theorem 1.2 implies that a sequence $((s^i, \bar{t}^i))$ exists, such that $(s^i, \bar{t}^i)$ is a cornerpoint for β_{F^i} and $((s^i, \bar{t}^i))$ converges to $(s, \bar{t})$. For every large enough index i, once more by the Representation Theorem 1.1, s^i is then a discontinuity point for $\beta_{F^i}(\cdot, t)$, and hence by Theorem 2.3 we have that $u^i = s^i \cdot \vec{m} + b$ is a discontinuity point for $\beta_{f^i}(\cdot, v)$. From Theorem 2.4 it follows that a projection ρ^i exists, such that $\rho^i(u^i)$ is a pseudocritical value for $\rho^i \circ f^i$. Possibly by considering a subsequence, we can suppose that all the ρ^i equal a projection ρ associated with the h-tuple $(j_1, \ldots, j_h)$. Moreover, we can consider a sequence (x^i) such that $x^i \in X$, $\rho \circ f^i(x^i) = \rho(u^i)$ and x^i is a pseudocritical point for $\rho \circ f^i$. Furthermore, by the compactness of X, possibly by extracting a subsequence we can assume (x^i) converging to a point $x \in X$. From the continuity of f and from the uniform convergence of (f^i) to f, we deduce

(3) $\rho \circ f(x) = \lim_{i\to\infty} \rho \circ f(x^i) = \lim_{i\to\infty} \rho \circ f^i(x^i) = \lim_{i\to\infty} \rho(u^i) = \rho(u)$.

If $\rho(u)$ is a special value for $\rho \circ f$ then our claim is proved.

If $\rho(u) = (u_{j_1}, \ldots, u_{j_h})$ is not a special value for $\rho \circ f$ then $x \notin \mathbb{S}_{j_1} \cup \ldots \cup \mathbb{S}_{j_h}$. Hence, for any large enough index i, it follows that $x, x^i \in K^i_{j_1} \cap \ldots \cap K^i_{j_h}$. By recalling that each point x^i is a pseudocritical point for $\rho \circ f^i$, and by observing that the property of being a pseudocritical point passes to the limit, we get that $\rho(u)$ is a pseudocritical value for $\rho \circ f$. In other words, we have just proved that if u is a discontinuity point for $\beta_f(\cdot, v)$, then a projection ρ exists such that $\rho(u)$ is either a special value or a pseudocritical value for $\rho \circ f$.

Analogously, it is possible to prove that if v is a discontinuity point for $\beta_f(u, \cdot)$, then a projection ρ exists such that $\rho(v)$ is either a special value or a pseudocritical value for $\rho \circ f$. □

2.4. *Toward applications*

The results proved in this paper imply several relevant consequences. First of all, they contribute to clarifying the structure of multidimensional PBNs. In order to explain this point let us consider the case of a compact smooth surface $\mathcal{S}$ endowed with a smooth function $f = (f_1, f_2) : \mathcal{S} \to \mathbb{R}^2$. It is immediate to verify that all pseudocritical points belong to the Jacobi set of f, that is the set where the gradients ∇f_1 and ∇f_2 are parallel. This implies [27] that in the generic case the pseudocritical points belong to a 1-submanifold $\mathcal{J}$ of $\mathcal{S}$ (in local coordinates such a manifold is determined by the vanishing of the Jacobian of f). For the computation of $\mathcal{J}$ and related algorithms we refer to [27]. Now, let $\mathcal{P}$ be the set of pseudocritical values for f, and let $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) be the set of critical values for f_1 (resp. f_2). Following these notations, if we assume that $\mathcal{A}_1 = \mathcal{C}_1 \times \mathbb{R}^3$, $\mathcal{A}_2 = \mathbb{R} \times \mathcal{C}_2 \times \mathbb{R}^2$, $\mathcal{B}_1 = \mathbb{R}^2 \times \mathcal{C}_1 \times \mathbb{R}$, $\mathcal{B}_2 = \mathbb{R}^3 \times \mathcal{C}_2$, $\mathcal{P}_1 = \mathcal{P} \times \mathbb{R}^2$ and $\mathcal{P}_2 = \mathbb{R}^2 \times \mathcal{P}$, then Theorem 2.5 allows us to claim that all discontinuity points (u_1, u_2, v_1, v_2) of the PBNs function β_f belong to the set $\mathcal{K} = \Delta^+ \cap (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{P}_1 \cup \mathcal{P}_2)$ (with a little abuse of notation, we identify our sets as subsets of $\mathbb{R}^4$).

In light of this, we can imagine the possibility of designing new algorithms to speed-up the computation of multidimensional PBNs [10]. Let us consider the connected components in which the domain of β_f is divided by the set $\mathcal{K}$. Being PBNs functions locally constant at each point of continuity (we recall that they are natural-valued), we immediately obtain that β_f is constant at each of those connected components. It follows that the computation of β_f just requires the computation of its value at only one point for each connected component. These arguments open the way to new and more efficient methods of computation for multidimensional PBNs.

Also, our results may be of help in case the computation is achieved through the use of the foliation method. Indeed, this alternative approach has revealed to be useful in the development of possible distances between multidimensional PBNs [4,14]. In this context, Example 2.1 suggests that, by virtue of Proposition 1.3 and our new Theorem 2.2, it could be possible to track *leaf by leaf* the movements of cornerpoints associated with the 1-dimensional restrictions of PBNs functions, thus avoiding to compute such restrictions from scratch every time a new leaf in the foliation is visited.

Last, our results also make new pseudodistances between PBNs functions computable in an easier way. Indeed, let us consider two functions $g : X \to \mathbb{R}^k$, $g' : Y \to \mathbb{R}^k$ and the value δ_H giving the Hausdorff distance between the sets where β_g and $\beta_{g'}$ are discontinuous. It is trivial to check that the function d_D defined by setting $d_D(\beta_g, \beta_{g'}) = \delta_H$ is a pseudodistance between multidimensional PBNs. Helping us to localize the discontinuities of PBNs functions, Theorem 2.5 makes the computation of d_D easier.

Conclusions and future work

In this paper we have investigated on the intrinsic structure of multidimensional persistent Betti numbers. In particular, we have proved that the discontinuity points of k-dimensional PBNs have at least one special or pseudocritical coordinate, under the hypothesis that the considered filtering function is (at least) continuous. We think that this work may contribute to fill the existing gap between the development of persistence theory for scalar- and vector-valued filtering functions, respectively, with potential implications also from the application viewpoint.

We conclude by observing that the results presented here are deeply connected with the fairly new concept of *persistence space* [16,17]. This is a generalization of a persistence diagram, providing a stable and complete representation of multidimensional PBNs. A persistence space is defined through the notion of *cornerpoint*, which are indeed introduced by extending Definitions 1.2 and 1.3 to the case of $\mathbb{R}^k$-valued filtering functions. In this context, our results imply that the cornerpoints of a persistence space have coordinates that are special or pseudocritical values of the associated filtering function.

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