Some research notes on G-invariant Persistent Homology

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Dagstuhl Seminar
"Applications of Combinatorial Topology to Computer Science"
LZI Schloss Dagstuhl, Germany
18 - 23 March 2012

Outline

1. **A Metric Approach to Shape Comparison**

2. ***G*-invariant Persistent Homology**

1. **A Metric Approach to Shape Comparison**

2. *G*-invariant Persistent Homology

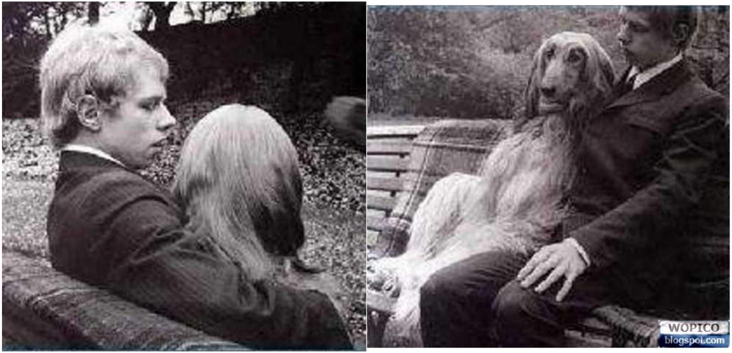
Informal position of the problem

Every comparison of properties involves the presence of

- an observer perceiving the properties
- a methodology to compare the properties

Informal position of the problem

The perception properties depend on the subjective interpretation of an observer:



Informal position of the problem

The perception properties depend on the subjective interpretation of an observer:

Impossible Ring and Pillars

by Guido Moretti

Our formal setting

Our formal setting:

- Each perception is formalized by a pair $(X, \vec{\varphi})$, where X is a topological space and $\vec{\varphi} : X \to \mathbb{R}^k$ is a continuous function.
- X represents the set of observations made by the observer, while $\vec{\varphi}$ describes how each observation is interpreted by the observer.

Our formal setting

Example a Let us consider Computerized Axial Tomography, where for each unit vector v in the real plane a real number is obtained, representing the total amount of mass $\varphi(v)$ encountered by an X-ray beam directed like v. In this case the topological space X equals the set of all unit vectors in $\mathbb{R}^2$, i.e. S^1. The filtering function is $\varphi : S^1 \to \mathbb{R}$.

Example b Let us consider a rectangle R containing an image, represented by a function $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3) : R \to \mathbb{R}^3$ that describes the RGB components of the colour for each point in the image. The filtering function is $\vec{\varphi} : R \to \mathbb{R}^3$.

Our formal setting

Assume that two "perceptions" $(X, \vec{\varphi})$, $(X, \vec{\psi})$ are given. We can define the following pseudo-metric:

$$d_G\left(\vec{\varphi}, \vec{\psi}\right) = \inf_{g \in G} \max_i \max_{x \in X} |\varphi_i(x) - \psi_i \circ g(x)|$$

where G is a fixed subgroup of the group $Hom(X)$ of all homeomorphisms from X onto X.

We shall call $d_G\left(\vec{\varphi}, \vec{\psi}\right)$ the natural pseudo-distance between $\vec{\varphi}$ and $\vec{\psi}$, associated with the group G.

The functional $\Theta(g) = \max_i \max_{x \in X} |\varphi_i(x) - \psi_i \circ g(x)|$ represents the "cost" of the matching between observations induced by g. The lower this cost, the better the matching between the two observations.

Our formal setting

- The natural pseudo-distance d_G measures the dissimilarity between the perceptions expressed by the pairs $(X, \vec{\varphi})$, $(X, \vec{\psi})$.
- The value d_G is small if and only if we can find a homeomorphism from X onto X in G that induces a small change of the filtering function (i.e., of the shape property we are interested to study).

- For more information:
- P. Donatini, P. Frosini, *Natural pseudodistances between closed manifolds*, Forum Mathematicum, 16 (2004), n. 5, 695-715.
- P. Donatini, P. Frosini, *Natural pseudodistances between closed surfaces*, Journal of the European Mathematical Society, 9 (2007), 331-353.

1. A Metric Approach to Shape Comparison

2. **G-invariant Persistent Homology**

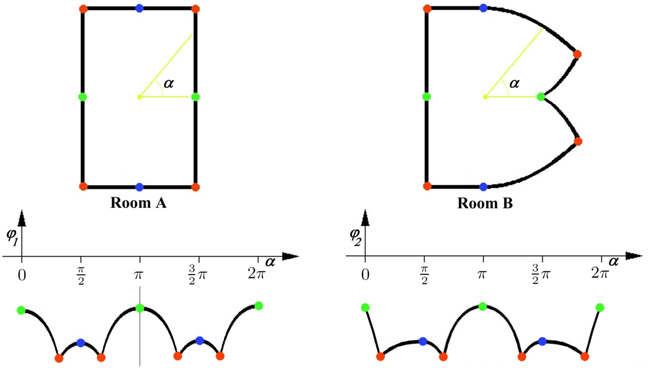
Natural pseudo-distance and Persistent Homology

- The natural pseudo-distance is usually difficult to compute.
- Lower bounds for the natural pseudo-distance $d_{Hom(X)}$ can be obtained by computing Persistent Homology.

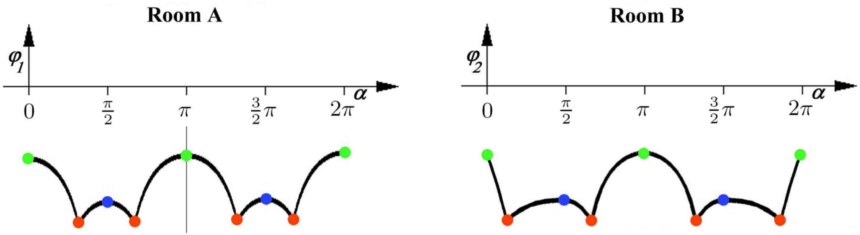
- The same does not hold for the natural pseudo-distance d_G.

Classical Persistent Homology is not tailored for invariance with respect to generic groups of transformations

Example. A sensor is in the middle of a room, measuring its distance from the surrounding walls, for each direction and verse. We consider the function $\varphi : S^1 \to \mathbb{R}$, where $\varphi(v)$ equals minus the distance from the wall in the direction of the unit vector v.



Classical Persistent Homology is not tailored for invariance with respect to generic groups of transformations



Let R denote the group of rigid motions of S^1. We observe that $d_{Hom(S^1)}(\varphi_1, \varphi_2) = 0$, while $d_R(\varphi_1, \varphi_2) > 0$, so that classical Persistent Homology cannot distinguish φ_1 from φ_2. While there exists a homeomorphism taking the observations of room A into the observations of room B (and hence φ_1 and φ_2 produce the same persistent homology), there is no rotation doing the same.

The idea of *G*-invariant Persistent Homology

Definition

Let us consider a topological space X and fix a subgroup G of the group $Hom(X)$ of all homeomorphisms from X onto X. Let $(\hat{C}, \partial)$ be a subcomplex of the singular chain complex $(C(X), \partial)$ of X, verifying the following property in any degree n:

$(*)$ If $c = \sum_i a_i \sigma_i \in \hat{C}_n$ then $g(c) = \sum_i a_i (g \circ \sigma_i) \in \hat{C}_n$ for every $g \in G$ (i.e. $\hat{C}$ is invariant under the action of the group G).

The chain complex $\hat{C}$ will be said a *G-invariant chain subcomplex of* $C(X)$. We shall call the group $H_n(\hat{C}) = \ker \partial_n / \mathrm{im}\, \partial_{n+1}$ the *n-th homology group associated with the G-invariant chain complex* $(\hat{C}, \partial)$.

The idea of G-invariant Persistent Homology

Let $\hat{C}$ be a G-invariant chain subcomplex of $C(X)$. For every topological subspace X' of X, we can consider the new chain complex $\cdots \stackrel{\partial_{n+1}}{\rightarrow} \hat{C}_n \bigcap C_n(X') \stackrel{\partial_n}{\rightarrow} \hat{C}_{n-1} \bigcap C_{n-1}(X') \stackrel{\partial_{n-1}}{\rightarrow} \cdots$. (This new chain complex is not requested to be G-invariant).

Definition

This chain complex will be called *chain subcomplex of $\hat{C}$ induced by restriction to the topological subspace X'*.

Now we can apply this idea to Persistent Homology.

The idea of *G*-invariant Persistent Homology

Definition

Let $\hat{C}$ be a G-invariant chain subcomplex of $C(X)$. Assume that a continuous function $\vec{\varphi} : X \to \mathbb{R}^k$ is given and consider the chain subcomplex $\hat{C}^{\vec{\varphi} \prec u}$ of $\hat{C}$ induced by restriction to the topological subspace given by the sublevel set $X^{\vec{\varphi} \prec u} = \{x \in X : \vec{\varphi}(x) \prec u\}$.
If $u, v \in \mathbb{R}^k$ and $u \prec v$ (i.e., $u_j < v_j$ for every index j), we can consider the inclusion $i_{(u,v)}$ of the chain complex $C^{\vec{\varphi} \prec u}$ into the chain complex $C^{\vec{\varphi} \prec v}$. Such an inclusion induces a homomorphism $i^*_{(u,v)} : H_n\left(\hat{C}^{\vec{\varphi} \prec u}\right) \to H_n\left(\hat{C}^{\vec{\varphi} \prec v}\right)$. We shall call the group $H_n(u, v) = i^*_{(u,v)}\left(H_n\left(\hat{C}^{\vec{\varphi} \prec u}\right)\right)$ the *n*-th persistent homology group associated with the G-invariant chain complex $\hat{C}$, computed at point (u, v). The rank $\rho_{\vec{\varphi}}(u, v)$ of this group will be said *n-th persistent Betti number associated with the G-invariant chain complex $\hat{C}$, computed at point (u, v)*.

The idea of *G*-invariant Persistent Homology

The following result holds:

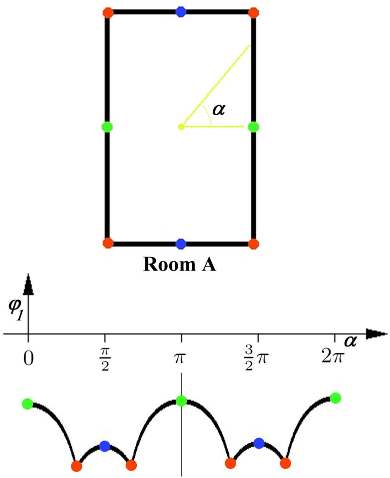
Theorem

Let us consider the persistent Betti number functions $\rho_{\vec{\varphi}}$, $\rho_{\vec{\psi}}$ associated with a G-invariant chain complex $\hat{C}$, with respect to two continuous functions $\vec{\varphi}: X \to \mathbb{R}^k$, $\vec{\psi}: X \to \mathbb{R}^k$. Let us assume that the groups $H_n(\hat{C})$ are finitely generated. Then

$$d_G(\vec{\varphi}, \vec{\psi}) \geq d_{match}(\rho_{\vec{\varphi}}, \rho_{\vec{\psi}}).$$

The idea of G-invariant Persistent Homology

Let us go back to our example to make this idea clear.



The idea of *G*-invariant Persistent Homology

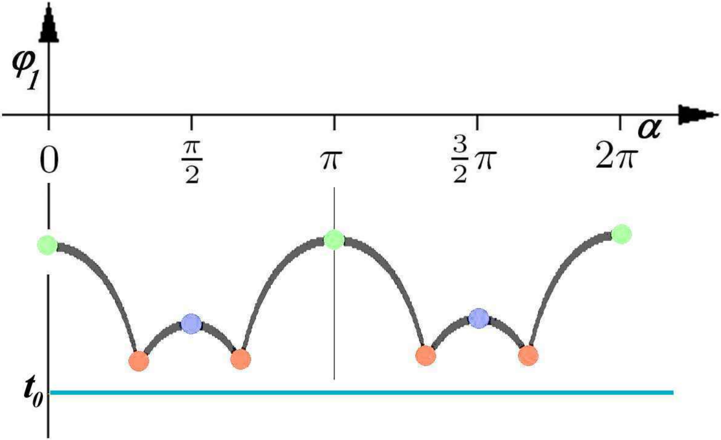
Now, let us consider a 0-th R-invariant homology of the sub-level sets $\{v \in S^1 : \varphi_i(v) \leq t\}$, varying t. R is the group of all rotations of S^1. We consider the set of 0-chains where a 0-simplex appears if and only if its antipodal 0-simplex appears too, with the same multiplicity. In plain words, we just take the linear combinations of points of S^1 of the kind $\sum_i a_i(P_i + P_i')$, where P_i and P_i' are antipodal. (Remember that $P_i + P_i'$ is a formal sum, so it doesn't vanish.) The other linear combinations of points are not considered legitimate 0-chains. We call $\hat{C}_0$ the set of the 0-chains we have chosen.

Analogously, we choose to consider the set of 1-chains where a 1-simplex appears if and only if its antipodal 1-simplex appears too, with the same multiplicity. We call $\hat{C}_1$ the set of the 1-chains we have chosen.

For any level t we can consider the 0-th homology of the sub-level set under t.

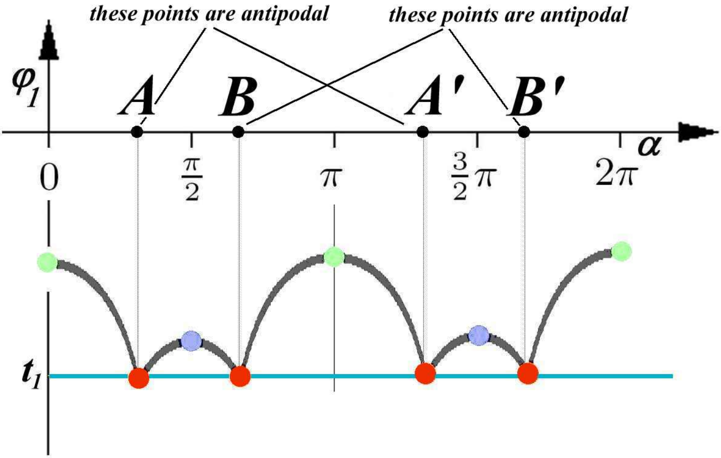
The idea of *G*-invariant Persistent Homology

First of all, let us consider the filtering function φ_1 on S^1.
When $t = t_0$ the set $\hat{C}_0$ is empty and hence the group $H_0(\hat{C}^{\vec{\varphi} \prec t_0})$ is trivial.



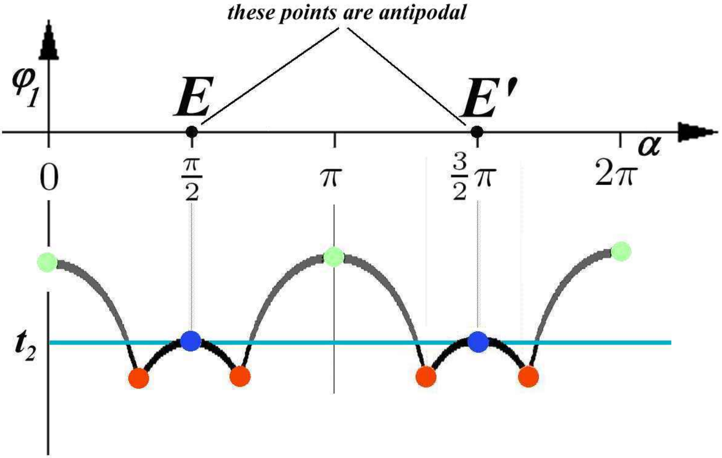
The idea of G-invariant Persistent Homology

When $t = t_1$ the 0-chains $A + A'$ and $B + B'$ are born, so that $H_0(\hat{C}^{\vec{\varphi} \prec t_1}) = \mathbb{Z} \oplus \mathbb{Z}$.



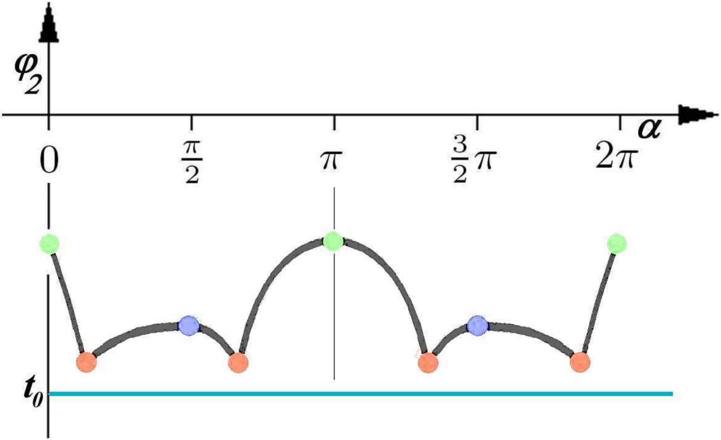
The idea of G-invariant Persistent Homology

When $t = t_2$ the 0-chains $A + A'$ and $B + B'$ become homologous to the 0-chain $E + E'$, so that $H_0(\hat{C}^{\vec{\varphi} \prec t_2}) = \mathbb{Z}$.



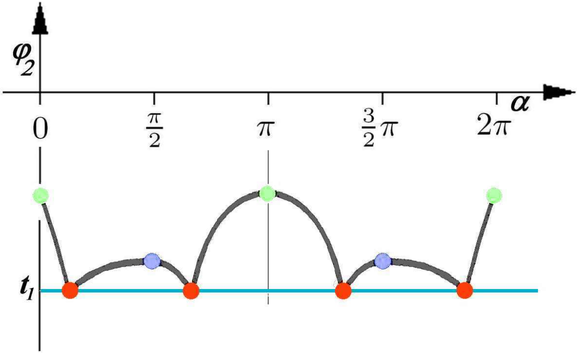
The idea of *G*-invariant Persistent Homology

Now, let us consider the filtering function φ_2 on S^1.
When $t = t_0$ the set $\hat{C}_0$ is empty and hence the group $H_0(\hat{C}^{\vec{\varphi} \prec t_0})$ is trivial.



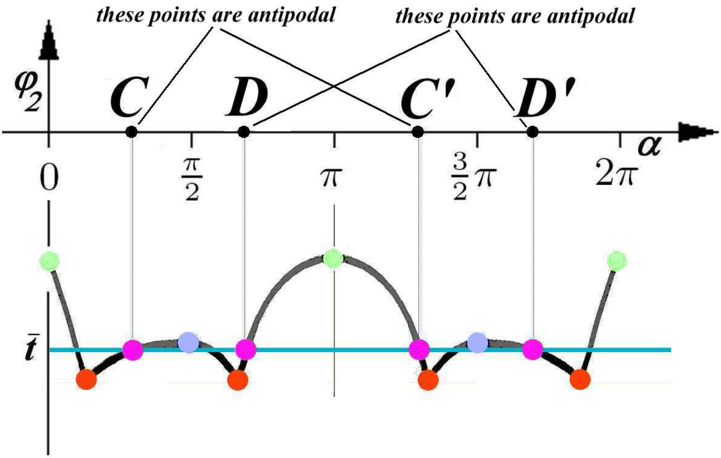
The idea of G-invariant Persistent Homology

When $t = t_1$ the set $\hat{C}_0$ is still empty, so that the group $H_0(\hat{C}^{\vec{\varphi} \prec t_1})$ is still trivial. We observe that there does not exist any legitimate 0-chain under t_1 because every pair of red points have a distance that is different from π (i.e. no pair of red points corresponds to a pair of antipodal points in S^1).



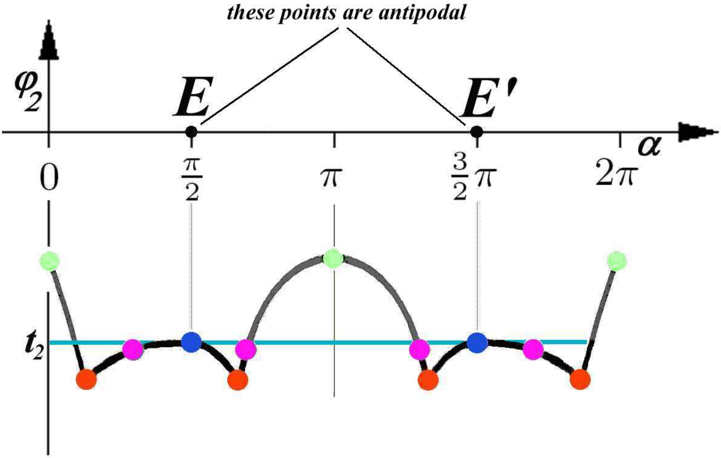
The idea of G-invariant Persistent Homology

When $t = \bar{t}$ the 0-chains $C + C'$ and $D + D'$ are born, so that $H_0(\hat{C}^{\vec{\varphi} \prec \bar{t}}) = \mathbb{Z} \oplus \mathbb{Z}$.



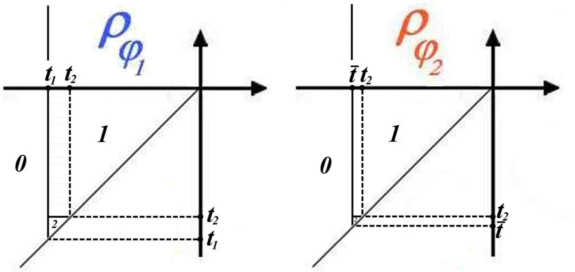
The idea of G-invariant Persistent Homology

When $t = t_2$ the 0-chains $C + C'$ and $D + D'$ become homologous to the 0-chain $E + E'$, so that $H_0(\hat{C}^{\vec{\varphi} \prec t_2}) = \mathbb{Z}$.



The idea of *G*-invariant Persistent Homology

Here are the *R*-invariant persistent Betti number functions ρ_{φ_1} and ρ_{φ_2} of φ_1 and φ_2.



We see that, while Persistent Homology cannot distinguish rooms *A* and *B*, *G*-invariant persistent homology can do it. In other words, *G*-invariant persistent homology (varying *G*) is strictly more discriminative than classical persistent homology.

Conclusions

We have illustrated the definition of G-invariant Persistent Homology, showing that it allows to obtain lower bounds for the natural pseudo-distance d_G.

We have also shown an example, suggesting that G-invariant Persistent Homology could be suitable for applications where an invariance group G is involved, different from the group of all homeomorphisms.