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DISCRETE COMPUTATION OF SIZE FUNCTIONS

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A discrete approximation theory for computing size functions of submanifolds of Euclidean spaces is developed, both for D^k-homotopy and V-homotopy. Some concrete exampl,es of computation are provided.

Introduction.

Mathematicians are becoming more and more involved in the task of giving a well-structured mathematical answer to the problems of Pattern Recognition (as stressed, e.g., in [J] and [YWB]). The introduction of metric homotopies [Fr1] and of a distance for similarity classes of submanifolds of Euclidean spaces [Fr2] were aimed to provide theoretical techniques in that direction. The present paper takes the first idea to the computational side, by developing a "parallel" theory for discrete spaces and using it in an approximation process.

Given a submanifold $\mathcal{M}$ of a Euclidean space, the "size" function to be computed is an integral function of two real variables $\ell^{D^k}(x,y)$. This roughly describes the number of ways in which a (k+1)−fingered hand can grasp $\mathcal{M}$ with fingers not farther apart than x, where two grasps are considered equivalent if they can be deformed into each other without ever pulling the fingers apart beyond a diameter y. The computation is carried out by computing a discrete analogous $\ell^{D^k}_{\mathcal{P}}$ on a finite

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set $\mathcal{P}$ of points approximating $\mathcal{M}$. A second function $\ell^V(x,y)$ is computed, which is defined analogously, with volume replacing diameter.

This approximation process is far from being new, as it is the core of the "Regge calculus" [MTW, Ch. 42] of Gaussian curvature and, in general, of the geometry of spacetime. It was later used in [CMS] for generalizations, and in [BK] the other way around, i.e. for approximation of polyhedra by means of smooth manifolds.

Section 1 gives the basic definitions. Section 2 studies the relations between the size functions in the smooth case and their discrete counterparts, with diameter measuring the size. Section 3 does the same for volume instead of diameter. Section 4 presents some examples of explicit computation.

1. The basic definitions.

In this Section we shall give some preliminar definitions: our purpose is that of constructing some tools suitable to describe in one sense the "shape" of a submanifold of a Euclidean space. We begin by a concise description of $\mathbf{D}^k$−homotopic theory. In this paper $\mathcal{M}$ will be always a piecewise C^∞ and closed n−submanifold (n>0) of $\mathbf{E}^m$ and for every $k \in \mathbf{N}$ we shall denote by $\mathcal{M}^{k+1}$ the set of all ordered (k+1)−tuples of points in $\mathcal{M}$. In a natural way $\mathcal{M}^{k+1}$ is a metric space with respect to the distance d_k defined by setting $d_k\Big((Q_0,...,Q_k),(R_0,...,R_k)\Big)=\max_{0\le i\le k}\|Q_i-R_i\|$ for every pair $\Big((Q_0,...,Q_k),(R_0,...,R_k)\Big)\in\mathcal{M}^{k+1}\times\mathcal{M}^{k+1}$.

Definition (1.1). For every real number y we can define an equivalence relation on $\mathcal{M}^{k+1}$ and all its subsets: for every pair $\left((Q_0,\dots,Q_k),(R_0,\dots,R_k)\right)$ in $\mathcal{M}^{k+1}\times\mathcal{M}^{k+1}$ we shall say that $(Q_0,\dots,Q_k)$ and $(R_0,\dots,R_k)$ are $y-D^k$-*homotopic* if either they are the same ordered $(k+1)$-tuple or a function H exists such that:

i) $H\in C^0\left([0,1],\mathcal{M}^{k+1}\right)$

ii) $H(0)=(Q_0,\dots,Q_k)$ and $H(1)=(R_0,\dots,R_k)$

iii) for every $\tau\in[0,1]$ $\Delta\left(H(\tau)\right)\le y$, Δ being the function that takes every ordered $(k+1)$-tuple of points of E^m to the diameter of the convex hull of the $(k+1)$-tuple. If $(Q_0,\dots,Q_k)$ and $(R_0,\dots,R_k)$ are $y-D^k$-homotopic we shall write $(Q_0,\dots,Q_k)\overset{D^k}{\underset{y}{=}}(R_0,\dots,R_k)$ and, in case it exists, H will be said to be a $y-D^k$-*homotopy* from $(Q_0,\dots,Q_k)$ to $(R_0,\dots,R_k)$.

Obviously the diameter we spoke about in the previous definition is the one inherited from the embedding of $\mathcal{M}$ in E^m. In the case $y\ge\Delta\left((Q_0,\dots,Q_k)\right)$ the symbol $(Q_0,\dots,Q_k)\overset{D^k}{\underset{y}{=}}(R_0,\dots,R_k)$ means, in plain words, that we can "transform $(Q_0,\dots,Q_k)$ in $(R_0,\dots,R_k)$ without exceeding the diameter y". Diameter can be thought of as a real function defined on E^{k+1}. A $y-D^k$-homotopy is actually just a path in $\mathcal{M}^{k+1}$, where no point of the path has a value greater than y.

Definition (1.2). Let x and y be real numbers. We shall denote by $\mathcal{M}_\Delta^{k+1}\langle x\rangle$ the subset of $\mathcal{M}^{k+1}$ containing the $(k+1)$-tuples on which the function Δ takes a value less than or equal to x. Moreover, we shall denote by $\ell^{D^k}(x,y)$ the number of equivalence classes into which $\mathcal{M}_\Delta^{k+1}\langle x\rangle$ is divided by the relation of $y-D^k$-homotopy if such a number is finite, $+\infty$ otherwise.

Another formalization of the idea described at the beginning of this section can be the one that we obtain by substituting in the above-mentioned definitions the concept of $(k+1)$-tuple by the one of

$(m+1)$-tuple (with m determined by the dimension of E^m) and the function Δ by the function *vol* which takes every ordered $(m+1)$-tuple to the oriented volume of its convex hull. We shall call V-homotopy the corresponding theory. This procedure leads in a natural way to the function $\ell^V(x,y)$ (which corresponds naturally to the function $\ell^{D^k}(x,y)$). We give in the following the exact formalization of the above-mentioned concepts. On $\mathcal{M}^{m+1}$ we shall consider the distance d_m.

Definition (1.3). For every real number y we can define an equivalence relation on $\mathcal{M}^{m+1}$ and all its subsets: for every pair $\left((Q_0,\dots,Q_m),(R_0,\dots,R_m)\right)$ in $\mathcal{M}^{m+1}\times\mathcal{M}^{m+1}$ we shall say that $(Q_0,\dots,Q_m)$ and $(R_0,\dots,R_m)$ are $y-V$-*homotopic* if either they are the same ordered $(m+1)$-tuple or a function H exists such that:

i) $H\in C^0\left([0,1],\mathcal{M}^{m+1}\right)$

ii) $H(0)=(Q_0,\dots,Q_m)$ and $H(1)=(R_0,\dots,R_m)$

iii) for every $\tau\in[0,1]$ it results $vol\left(H(\tau)\right)\le y$, *vol* being the function that takes every ordered $(m+1)$-tuple $A=\left(A_0,\ A_1,\ A_2,\dots,\ A_m\right)$ of points of E^m into the value that the form $\frac{1}{m!}dx_1\wedge dx_2\wedge\dots\wedge dx_m$ takes on the ordered m-tuple of vectors of R^m $\left(A_1-A_0,\ A_2-A_0,\dots,\ A_m-A_0\right)$. If $(Q_0,\dots,Q_m)$ and $(R_0,\dots,R_m)$ are $y-V$-homotopic we shall write $(Q_0,\dots,Q_m)\overset{V}{\underset{y}{=}}(R_0,\dots,R_m)$ and, in case it exists, H will be said to be a $y-V$-*homotopy* from $(Q_0,\dots,Q_m)$ to $(R_0,\dots,R_m)$.

In the case $y\ge vol\left((Q_0,\dots,Q_m)\right)$ the symbol $(Q_0,\dots,Q_m)\overset{V}{\underset{y}{=}}(R_0,\dots,R_m)$ means, in plain words, that we can "transform $(Q_0,\dots,Q_m)$ in $(R_0,\dots,R_m)$ without exceeding the volume y". Volume can be thought of as a real function defined on E^{m+1}. A $y-V$-homotopy is actually just a path in $\mathcal{M}^{m+1}$, where no point of the path has a value greater than y.

Definition (1.4). Let x and y be real numbers. We shall denote by $\mathcal{M}_{vol}^{m+1}\langle x\rangle$ the subset of $\mathcal{M}^{m+1}$ containing the (m+1)−tuples on which the function *vol* takes a value less than or equal to x. Moreover, we shall denote by $\ell^{V}(x,y)$ the number of equivalence classes into which $\mathcal{M}_{vol}^{m+1}\langle x\rangle$ is divided by the relation of y −V−homotopy if such a number is finite, $+\infty$ otherwise.

Remark (1.1). In the following the functions ℓ^{D^k} and ℓ^{V} will be called also size functions. For the main results concerning D^k− and V−homotopic theory (whose core is the study of the corresponding size functions) and a more general view of this subject we refer to [Fr2] and [Fr3]. We confine the discussion to pointing out that if $\mathcal{M}$ is a closed, connected and piecewise C^∞ n−submanifold of $\mathbf{E}^m$ and μ is the minimum of the function *vol* on $\mathcal{M}^{m+1}$ then the values of ℓ^{D^k} and ℓ^{V} are interesting (that is not trivial) respectively for $0<x\leq y$ and for either $\mu<x\leq y$ or $\mu=x$.

2. Size functions in $\mathbf{D}^k$−homotopic theory.

In this Section we shall give a technique to compute the function $\ell^{D^k}(x,y)$. In order to do it we need some preliminary definitions. Our purpose is that of "approximating" the considered manifold $\mathcal{M}$ and the corresponding function ℓ^{D^k} respectively with a finite set $\mathcal{P}$ and a function $\ell_{\mathcal{P}}^{D^k}$. The function $\ell_{\mathcal{P}}^{D^k}$ will be related to the function ℓ^{D^k} but much simpler to be computed.

From now on ϵ will be a positive real number.

Definition (2.1). Let $\mathcal{P}=\{P_0,P_1,\dots,P_h\}$ be a finite set of points of $\mathbf{E}^m$ and let us denote by $\mathfrak{B}_\epsilon$ the set of the h+1 open balls $B^\circ(P_i,\epsilon)$ of radius ϵ centered in the points of $\mathcal{P}$. Suppose that $\mathfrak{B}_\epsilon$ verifies the following properties:

i) $\mathcal{M}$ is contained in $\bigcup_{i=0}^{h} B^\circ(P_i,\epsilon)$

ii) for every index i $(0\leq i\leq h)$ $B^\circ(P_i,\epsilon)\cap\mathcal{M}$ is a non−empty connected set.

We shall call $\mathfrak{B}_\epsilon$ an *ϵ−covering* of $\mathcal{M}$. The set $\mathcal{P}$ will be called ***the set of the centers of*** $\mathfrak{B}_\epsilon$. We can define the following relation $\sim$ on the set $\mathcal{P}$: $P_i\sim P_j$ if $\left(B^\circ(P_i,\epsilon)\cup B^\circ(P_j,\epsilon)\right)\cap\mathcal{M}$ is a connected set. For every positive integer k we define on the set $\mathcal{P}^{k+1}$ of ordered (k+1)−tuples of points of $\mathcal{P}$ a relation. Let $\alpha=(Q_0,Q_1,\dots,Q_k)$, $\beta=(R_0,R_1,\dots,R_k)\in\mathcal{P}^{k+1}$: if for every index i it results that $Q_i\sim R_i$ then we shall say that α and β are ***adjacent*** (in symbols $\alpha\simeq\beta$).

Remark (2.1). Obviously $\simeq$ is a reflexive and symmetric relation.

In the following of this paper we shall suppose that an ϵ−covering $\mathfrak{B}_\epsilon$ of $\mathcal{M}$ is given and denote by $\mathcal{P}$ the set $\{P_0,P_1,\dots,P_h\}$ of the centers of $\mathfrak{B}_\epsilon$.

Definition (2.2). For every $x,y\in\mathbf{R}$ let us denote by $\mathcal{P}_\Delta^{k+1}\langle x\rangle$ the set of the elements of $\mathcal{P}^{k+1}$ on which the function diameter Δ takes a value not greater than x and by $\overset{D^k}{\underset{y}{\approx}}$ the equivalence relation on $\mathcal{P}_\Delta^{k+1}\langle x\rangle$ defined this way: if $\alpha,\beta\in\mathcal{P}_\Delta^{k+1}\langle x\rangle$ we write $\alpha\overset{D^k}{\underset{y}{\approx}}\beta$ if either $\alpha=\beta$ or there exists a finite sequence $(\gamma(i))_{i=0,\dots,r}$ of (k+1)−tuples in $\mathcal{P}_\Delta^{k+1}\langle y\rangle$ such that $\gamma(0)=\alpha$, $\gamma(r)=\beta$ and for every index i with $0\leq i\leq r-1$ $\gamma(i)$ and $\gamma(i+1)$ are adjacent (k+1)−tuples. This equivalence relation will be called $y-D^k$*−equivalence* and the sequence $(\gamma(i))_{i=0,\dots,r}$ will be said to be a $y-D^k$*−equivalence sequence from α to β of length r with respect to* $\mathfrak{B}_\epsilon$. We shall denote by $\ell_{\mathcal{P}}^{D^k}(x,y)$ the number of equivalence classes in which $\mathcal{P}_\Delta^{k+1}\langle x\rangle$ is divided by $y-D^k$−equivalence.

The following two definitions will allow us to substitute every Y-D^k-homotopy H between two (k + 1)-tuples of points with a (y + 2ε)-D^k-equivalence sequence which "approximates" H in some sense. For sake of clearness, before defining precisely the concepts we need, we explain in plain words (but with some imprecision) the idea underlying the next definitions. In order to construct a $(y+2\varepsilon)$-D^k-equivalence sequence $S(H)=\left(\gamma(i)\right)_{i=0,\dots,r}$ naturally corresponding from the discrete point of view to the homotopy H we proceed the following way. First of all we define $\gamma(0)$ as the ordered $(k+1)$-tuple of centers of $\mathcal{B}_\epsilon$ nearest to the $(k+1)$-tuple $H(0)$. Then let us suppose that the $(k+1)$-tuple $\gamma(i)=\left(C_0(i),C_1(i),\dots,C_{k-1}(i),C_k(i)\right)$ and the "time" $\tau(i)$ have been defined and that for every j the $(j+1)$-th component $H_j\left(\tau(i)\right)$ of $H\left(\tau(i)\right)$ is contained in the open ball $B_j(i)$ of center $C_j(i)$ and radius ϵ. While the "time" parameter τ is varying from $\tau(i)$ it may happen that one (or more) of the points $H_j(\tau)$ reaches the boundary of $B_j(i)$. When it happens we call $\tau(i+1)$ the corresponding "time" and construct the following term $\gamma(i+1)$ of $S(H)$ by defining it as the ordered $(k+1)$-tuple of centers of $\mathcal{B}_\epsilon$ nearest to $H\left(\tau(i+1)\right)$. Now let us give the precise definitions.

Definition (2.3). Let an ordered $(k+1)$-tuple $\alpha=\left(Q_0,Q_1,\dots,Q_k\right)\in\mathcal{M}^{k+1}$ be given. For every j $(0\le j\le k)$ let us define $\overline{Q}_j$ as the point $P_{\bar{s}}\in\mathcal{P}$ which has minimum index $\bar{s}$ among those points $P_s\in\mathcal{P}$ that minimize the distance from Q_j $\left(\text{obviously } \overline{Q}_j\in B^\circ\left(P_{\bar{s}},\epsilon\right)\right)$. We shall set $\nu(\alpha)=\left(\overline{Q}_0,\overline{Q}_1,\dots,\overline{Q}_k\right)$.

Definition (2.4). Let $H:[0,1]\to\mathcal{M}$ be a homotopy between two ordered $(k+1)$-tuples of points of $\mathcal{M}$ α and β $\left(\text{i.e. a continuous function } H(\tau)=\left(H_0(\tau),H_1(\tau),\dots,H_{k-1}(\tau),H_k(\tau)\right):[0,1]\to\mathcal{M}^{k+1} \text{ with } H(0)=\alpha,\ H(1)=\beta\right)$. Now we can inductively define a finite sequence $S(H)=\left(\gamma(i)\right)_{i=0,\dots,r}$ of $(k+1)$-tuples of $\mathcal{P}^{k+1}$. We set $\tau(0)=0$ and define $\left(C_0(0),C_1(0),\dots,C_{k-1}(0),C_k(0)\right)=\nu\left(H\left(\tau(0)\right)\right)$ $\left(\text{obviously for every j } (0\le j\le k)\ H_j\left(\tau(0)\right)\in B^\circ\left(C_j(0),\epsilon\right)\right)$. Furthermore we set $\mathbf{B}(0)=\left(B_0(0),B_1(0),\dots,B_{k-1}(0),B_k(0)\right)$ where for every j $(0\le j\le k)$ $B_j(0)$ is the open ball $B^\circ\left(C_j(0),\epsilon\right)$. Moreover let us set $\gamma(0)=\left(C_0(0),C_1(0),\dots,C_{k-1}(0),C_k(0)\right)$. Now suppose $\tau(i)$, $\gamma(i)$ and $\mathbf{B}(i)$ defined for the natural index i. If $\tau(i)=1$ we stop the procedure. Otherwise, we choose in $[0,1]$ a point $\tau(i+1)>\tau(i)$ such that the following two conditions hold:

i) $H_j\left([\tau(i),\tau(i+1))\right)\subset B_j(i)$ for $0\le j\le k$

ii) either $\tau(i+1)=1$ or there exists a non-empty set J_i of indexes such that $j\in J_i$ implies $H_j\left(\tau(i+1)\right)\in\partial B_j(i)$.

Such a value $\tau(i+1)$ exists because of the continuity of H. We define $\left(C_0(i+1),C_1(i+1),\dots,C_{k-1}(i+1),C_k(i+1)\right)=\nu\left(H\left(\tau(i+1)\right)\right)$ and for every index j we set $B_j(i+1)=B^\circ\left(C_j(i+1),\epsilon\right)$. Moreover let us set $\gamma(i+1)=\left(C_0(i+1),C_1(i+1),\dots,C_{k-1}(i+1),C_k(i+1)\right)$. There will exist an index r such that $\tau(r)=1$: let us define $S(H)=\left(\gamma(i)\right)_{i=0,\dots,r}$.

Remark (2.2). The finiteness of the sequence $\left(\gamma(i)\right)$ defined in Definition (2.4) $\left(\text{that is the existence of an index r such that } \tau(r)=1\right)$ can be deduced from the compactness of $\mathcal{M}$. In fact this condition implies the existence of a positive η such that the set $\mathcal{B}_{\epsilon-\eta}$ of the open balls with center belonging to $\mathcal{P}$ and radius $\epsilon-\eta$ is again a covering $\left(\text{not necessarily an } (\epsilon-\eta)\text{-covering}\right)$ of $\mathcal{M}$. So if $Q\in\mathcal{M}$ and P_r is one of the centers of $\mathcal{B}_\epsilon$ that has minimum distance from Q then we have $d\left(Q,\partial B^\circ(P_r,\epsilon)\right)>\eta$. Therefore for every index i such that $\tau(i+1)$ is defined it results $d_k\left(H\left(\tau(i+1)\right),\ H\left(\tau(i)\right)\right)>\eta$. Hence, if the sequence $\left(\tau(i)\right)$ were infinite then $\left(H\left(\tau(i)\right)\right)$ would be a non-convergent infinite sequence, against the hypothesis that H is continuous in the point $\bar{\tau}=\lim_{i\to\infty}\tau(i)$.

Remark (2.3). Referring to Definition (2.4) it is easy to prove that for every index i with $0\le i\le r-1$ it results $\gamma(i)\simeq\gamma(i+1)$.

Lemma (2.1). *Let* $\alpha=(Q_0,Q_1,\ldots,Q_k)$, $\beta=(R_0,R_1,\ldots,R_k)\varepsilon E^{m(k+1)}$ *with* $d_k(\alpha,\beta)<\epsilon$. *Then* $|\Delta(\alpha)-\Delta(\beta)|<2\epsilon$.

Proof. Trivial. □

Lemma (2.2). *Let* $y\varepsilon\mathbf{R}$, $\alpha_1,\alpha_2\varepsilon\mathcal{M}^{k+1}$, $\beta_1,\beta_2\varepsilon\mathcal{P}^{k+1}$, $d_k(\alpha_1,\beta_1)<\epsilon$ *and* $d_k(\alpha_2,\beta_2)<\epsilon$. *The following statements hold:*

i) If $\alpha_1,\alpha_2\varepsilon\mathcal{M}^{k+1}\langle y\rangle$ *and* $\alpha_1\overset{D^k}{\underset{y}{=}}\alpha_2$ *then* $\beta_1,\beta_2\varepsilon\mathcal{P}_\Delta^{k+1}\langle y+2\epsilon\rangle$ *and* $\beta_1\overset{D^k}{\underset{y+2\epsilon}{\approx}}\beta_2$.

ii) If $\beta_1,\beta_2\varepsilon\mathcal{P}_\Delta^{k+1}\langle y\rangle$ *and* $\beta_1\overset{D^k}{\underset{y}{\approx}}\beta_2$ *then* $\alpha_1,\alpha_2\varepsilon\mathcal{M}^{k+1}\langle y+2\epsilon\rangle$ *and* $\alpha_1\overset{D^k}{\underset{y+2\epsilon}{=}}\alpha_2$.

Proof. i) On the one hand consider a $y-D^k$ −homotopy H from α_1 to α_2: our definitions, Lemma (2.1) and Remark (2.3) imply that $S(H)$ is a $(y+2\varepsilon)-D^k$ −equivalence sequence from $\nu(\alpha_1)$ to $\nu(\alpha_2)$ with respect to the ε −covering $\mathcal{B}_\epsilon$. On the other hand, since for $i=1,2$ the hypothesis $d_k(\alpha_i,\beta_i)<\varepsilon$ and the condition $d_k(\alpha_i,\nu(\alpha_i))<\varepsilon$ hold it results $\nu(\alpha_i)=\beta_i$ because of the definition of $\simeq$ and $\Delta(\beta_i)\le y+2\varepsilon$ because of Lemma (2.1). So $\beta_1,\beta_2\in\mathcal{P}_\Delta^{k+1}\langle y+2\varepsilon\rangle$ and $\beta_1\overset{D^k}{\underset{y+2\varepsilon}{\approx}}\nu(\alpha_1)\overset{D^k}{\underset{y+2\varepsilon}{\approx}}\nu(\alpha_2)\overset{D^k}{\underset{y+2\varepsilon}{\approx}}\beta_2$: the statement is proved.

ii) By definition $\beta_1\overset{D^k}{\underset{y}{\approx}}\beta_2$ implies that there exists a $y-D^k$ −equivalence sequence $(\beta^0=\beta_1,\beta^1,\ldots,\beta^r=\beta_2)$ with respect to $\mathcal{B}_\epsilon$. Let us denote by $(P_{i(0,h)},P_{i(1,h)},\ldots,P_{i(k,h)})$ the $(k+1)$ −tuple β^h $(0\le h\le r)$. For every t and h let us choose in $B^0(P_{i(t,h)},\varepsilon)\cap\mathcal{M}$ a point $Q_{(t,h)}$ and define $\alpha^h=(Q_{i(0,h)},Q_{i(1,h)},\ldots,Q_{i(k,h)})$. We can suppose $\alpha^0=\alpha_1$ and $\alpha^r=\alpha_2$. Because of the definition of $\overset{D^k}{\underset{y}{\approx}}$ we have that for $0\le t\le k$ and $0\le h\le r-1$ the set $(B^0(P_{i(t,h)},\varepsilon)\cup B^0(P_{i(t,h+1)},\varepsilon))\cap\mathcal{M}$ is non-empty and connected. So there exists a continuous path $\pi_{(t,h)}:[0,1]\to\mathcal{M}$ such that $\pi_{(t,h)}(0)=Q_{(t,h)}$, $\pi_{(t,h)}(1)=Q_{(t,h+1)}$, $\pi_{(t,h)}([0,\frac{1}{2}])\subset B^0(P_{i(t,h)},\varepsilon)$ and $\pi_{(t,h)}([\frac{1}{2},1])\subset B^0(P_{i(t,h+1)},\varepsilon)$. Define $H^h:[0,1]\to\mathcal{M}^{k+1}$ by setting $H^h(\tau)=(\pi_{i(0,h)}(\tau),\pi_{i(1,h)}(\tau),\ldots,\pi_{i(k,h)}(\tau))$: we have by construction that for every $\tau\in[0,1]$ and every index $h<r$ it results either $d_k(\beta^h,H^h(\tau))<\varepsilon$ or $d_k(\beta^{h+1},H^h(\tau))<\varepsilon$.

Therefore, because of Lemma (2.1) and the hypothesis $\Delta(\beta^h)\le y$ for every h, we have that H^h is a $(y+2\varepsilon)-D^k$ −homotopy from α^h to α^{h+1}. So the product of the homotopies H^h is a $(y+2\varepsilon)-D^k$ −homotopy from $\alpha^0=\alpha_1$ to $\alpha^r=\alpha_2$ and $\alpha_1,\alpha_2\in\mathcal{M}^{k+1}\langle y+2\varepsilon\rangle$: the statement is proved. □

Now we can prove the main results in this Section. They will allow us to actually compute the values of the function $\ell^{D^k}(x,y)$.

Theorem (2.1). *For every* $x,y\varepsilon\mathbf{R}$ *with* $x+2\epsilon\le y-2\epsilon$ *we have* $\ell_{\mathcal{P}}^{D^k}(x-2\epsilon,y+2\epsilon)\le\ell^{D^k}(x,y)\le\ell_{\mathcal{P}}^{D^k}(x+2\epsilon,y-2\epsilon)$.

Proof. Let us prove that $\ell_{\mathcal{P}}^{D^k}(x-2\epsilon,y+2\epsilon)\le\ell^{D^k}(x,y)$. In order to do it we shall construct an injective map F from $\mathcal{P}_\Delta^{k+1}\langle x-2\epsilon\rangle\Big/\overset{D^k}{\underset{y+2\epsilon}{\approx}}$ to $\mathcal{M}_\Delta^{k+1}\langle x\rangle\Big/\overset{D^k}{\underset{y}{=}}$ (we shall suppose that $\mathcal{P}_\Delta^{k+1}\langle x-2\epsilon\rangle\Big/\overset{D^k}{\underset{y+2\epsilon}{\approx}}$ is not empty because otherwise we would have $\ell_{\mathcal{P}}^{D^k}(x-2\epsilon,y+2\epsilon)=0$ and the thesis would be trivial). This fact will imply that the cardinality of $\mathcal{P}_\Delta^{k+1}\langle x-2\epsilon\rangle\Big/\overset{D^k}{\underset{y+2\epsilon}{\approx}}$ is less than or equal to the cardinality of $\mathcal{M}_\Delta^{k+1}\langle x\rangle\Big/\overset{D^k}{\underset{y}{=}}$: since by definition of ℓ^{D^k} and $\ell_{\mathcal{P}}^{D^k}$ we have that $\ell^{D^k}(x,y)$ is equal to the number of equivalence classes of $\mathcal{M}_\Delta^{k+1}\langle x\rangle\Big/\overset{D^k}{\underset{y}{=}}$ and $\ell_{\mathcal{P}}^{D^k}(x-2\epsilon,y+2\epsilon)$ is equal to the number of equivalence classes of $\mathcal{P}_\Delta^{k+1}\langle x-2\epsilon\rangle\Big/\overset{D^k}{\underset{y+2\epsilon}{\approx}}$ the considered inequality will follow immediately. Now we construct the function F. For every equivalence class $C\varepsilon\mathcal{P}_\Delta^{k+1}\langle x-2\epsilon\rangle\Big/\overset{D^k}{\underset{y+2\epsilon}{\approx}}$ we fix arbitrarily a $(k+1)$ −tuple $\beta=(P_{i_0},P_{i_1},\ldots,P_{i_k})\varepsilon C$ and a $(k+1)$ −tuple $\alpha\varepsilon\mathcal{M}^{k+1}$ with $d_k(\alpha,\beta)<\epsilon$ and define $F(C)$ as the equivalence class of α in $\mathcal{M}_\Delta^{k+1}(x)\Big/\overset{D^k}{\underset{y}{=}}$. We observe that such an α exists because of the definition of ϵ−covering and that $\alpha\,\epsilon\,\mathcal{M}_\Delta^{k+1}(x)$ because of the inequality $\Delta(\beta)\le x-2\epsilon$ and of Lemma (2.1). Let us prove that F is injective. If $F(C_1)=F(C_2)$ then, calling β_1 and β_2 the representatives of C_1 and C_2 considered in the previous part of the proof and recalling that $x<y$ $\left(\text{in fact } x+2\epsilon\le y-2\epsilon\right)$, we have that there exists a $y-D^k$−homotopy between the $(k+1)$−tuples α_1 and α_2 corresponding to β_1 and β_2 with respect to the above-mentioned arbitrary choice. It follows from Lemma (2.2)(i) that $\beta_1\overset{D^k}{\underset{y+2\epsilon}{\approx}}\beta_2$ and therefore $C_1=C_2$. By the way we observe that this implies $\beta_1=\beta_2$. We have so proved that F is an injective function. Now we have to prove that $\ell^{D^k}(x,y)\le\ell_{\mathcal{P}}^{D^k}\left(x+2\epsilon,y-2\epsilon\right)$. In order to do it we shall proceed as above and construct an injective map G from $\mathcal{M}_\Delta^{k+1}(x)\Big/\overset{D^k}{\underset{y}{=}}$ to $\mathcal{P}_\Delta^{k+1}(x+2\epsilon)\Big/\overset{D^k}{\underset{y-2\epsilon}{\approx}}$ $\Big($we shall suppose that $\mathcal{M}_\Delta^{k+1}(x)\Big/\overset{D^k}{\underset{y}{=}}$ is not empty because otherwise we would have $\ell^{D^k}(x,y)=0$ and the thesis would be trivial$\Big)$. This fact will imply that the cardinality of $\mathcal{M}_\Delta^{k+1}(x)\Big/\overset{D^k}{\underset{y}{=}}$ is less than or equal to the cardinality of $\mathcal{P}_\Delta^{k+1}(x+2\epsilon)\Big/\overset{D^k}{\underset{y-2\epsilon}{\approx}}$ and therefore that our inequality holds. Now let us construct the function G. For every equivalence class $C\,\epsilon\,\mathcal{M}_\Delta^{k+1}(x)\Big/\overset{D^k}{\underset{y}{=}}$ we fix arbitrarily a $(k+1)$−tuple $\alpha=\left(Q_0,Q_1,\dots,Q_k\right)\epsilon\,C$ and define $G(C)$ as the equivalence class of $\nu(\alpha)$ in $\mathcal{P}_\Delta^{k+1}(x+2\epsilon)\Big/\overset{D^k}{\underset{y-2\epsilon}{\approx}}$. We observe that $\nu(\alpha)\,\epsilon\,\mathcal{P}_\Delta^{k+1}(x+2\epsilon)$ because of the definition of $\nu(\alpha)$, the inequality $\Delta(\alpha)\le x$ and Lemma (2.1). Let us prove that G is injective. If $G(C_1)=G(C_2)$ then, calling α_1 and α_2 the representatives of C_1 and C_2 considered in the previous part of the proof and keeping in mind the hypothesis $x+2\epsilon\le y-2\epsilon$, we have that there exists a $(y-2\epsilon)-D^k$−equivalence sequence from $\nu(\alpha_1)$ to $\nu(\alpha_2)$. Since $d_k\left(\nu(\alpha_1),\alpha_1\right)<\epsilon$ and $d_k\left(\nu(\alpha_2),\alpha_2\right)<\epsilon$ we can apply Lemma (2.2)(ii) and obtain that $\alpha_1\overset{D^k}{\underset{y}{=}}\alpha_2$ and therefore $C_1=C_2$. By the way we observe that this implies $\alpha_1=\alpha_2$. We have so proved that G is an injective function. So the double inequality is proved. □

Theorem (2.2). *Let* $\bar{x},\bar{y},b,c\,\epsilon\,\mathbf{R}$ *with* $b,c\ge 0$ *and* $\bar{x}+2\epsilon\le\bar{y}-2\epsilon$. *If the function* $\ell_{\mathcal{P}}^{D^k}$ *takes the same value* v *in the two points* $\left(\bar{x}+2\epsilon,\bar{y}-2\epsilon\right)$ *and* $\left(\bar{x}-b-2\epsilon,\bar{y}+c+2\epsilon\right)$ *then it results* $\ell^{D^k}(x,y)=v$ *for every* (x,y) *in the rectangle* $\left\{(x,y)\epsilon\mathbf{R}^2:\bar{x}-b\le x\le\bar{x},\ \bar{y}\le y\le\bar{y}+c\right\}$.

Proof. Because of its definition the function $\ell_{\mathcal{P}}^{D^k}(x,y)$ is non-decreasing in the variable x and non-increasing in the variable y, so the hypothesis that the function $\ell_{\mathcal{P}}^{D^k}$ takes the same value v in the two points $\left(\bar{x}+2\epsilon,\bar{y}-2\epsilon\right)$ and $\left(\bar{x}-b-2\epsilon,\bar{y}+c+2\epsilon\right)$ implies that $\ell_{\mathcal{P}}^{D^k}$ has constant value v in the rectangle $\left\{(x,y)\epsilon\mathbf{R}^2:\ \bar{x}-b-2\epsilon\le x\le\bar{x}+2\epsilon,\ \bar{y}-2\epsilon\le y\le\bar{y}+c+2\epsilon\right\}$. This fact allows us to prove by Theorem (2.1) that for (x,y) in the rectangle $\left\{(x,y)\epsilon\mathbf{R}^2:\ \bar{x}-b\le x\le\bar{x},\ \bar{y}\le y\le\bar{y}+c\right\}$ it results $\ell^{D^k}(x,y)=v$. □

Remark (2.4). Theorem (2.2) leads naturally to an algorithm to compute ℓ^{D^k}. We can choose arbitrarily a real number $\lambda\ge 2\epsilon$ and compute the function $\ell_{\mathcal{P}}^{D^k}$ in the set $S_\lambda=\left\{(x,y)\epsilon\mathbf{R}^2:\ x=i\lambda,\ y=j\lambda,\ i,j\,\epsilon\,\mathbf{N},\ j\ge i\right\}$: every time that we find the same value v in two points $(i\lambda,j\lambda)$ and $\left((i-p)\lambda,(j+q)\lambda\right)$ of S_λ with $p,q\ge 2$ we can say that in the closed rectangle defined by the vertices $\left((i-1)\lambda,(j+1)\lambda\right)$ and $\left((i-p+1)\lambda,(j+q-1)\lambda\right)$ the value of ℓ^{D^k} is v. Obviously, if our manifold's "shape" is complicated and the size function ℓ^{D^k} has a lot of discontinuity points in a little space, in order to apply usefully our algorithm we shall have to choose an ϵ-covering constructed by using a very small ϵ and a value λ not far-away from 2ϵ.

Remark (2.5). In conclusion of this Section we point out that the concept of ϵ−covering has been given in Definition (2.1) by using balls only for sake of simplicity. In fact the same results could be obtained by substituting the concept of open ball $B^o(P_i,\epsilon)$ of center P_i and radius ϵ for the one of open connected subset of $B^o(P_i,\epsilon)$ in defining the concept of ϵ-covering.

3. Size functions in V–homotopic theory.

The technique used to compute the function $\ell^{D^k}(x,y)$ can be adapted to compute also the function $\ell^V(x,y)$. In order to do it we start by setting Definition (3.1), analogous in V–homotopic theory to Definition (2.2) given in D^k–homotopic theory. Also in this Section $\mathcal{M}$ will be a piecewise C^∞ and closed n–submanifold of the Euclidean space $\mathbb{E}^m$ and we shall suppose that an ϵ–covering $\mathcal{B}_\epsilon$ of $\mathcal{M}$ is given together with the set $\mathcal{P}=\{P_0,P_1,\ldots,P_h\}$ of its centers.

Definition (3.1). For every $x,y \in \mathbb{R}$ we shall denote by $\mathcal{P}_{vol}^{m+1}\langle x\rangle$ the set of the elements of $\mathcal{P}^{m+1}$ on which the function *vol* takes a value not greater than x and by $\overset{V}{\underset{y}{\approx}}$ the equivalence relation on $\mathcal{P}_{vol}^{m+1}\langle x\rangle$ defined this way: if $\alpha,\beta \in \mathcal{P}_{vol}^{m+1}\langle x\rangle$ it results $\alpha \overset{V}{\underset{y}{\approx}} \beta$ if either $\alpha=\beta$ or there exists a finite sequence $\left(\gamma(i)\right)_{i=0,\ldots,r}$ of (m+1)–tuples in $\mathcal{P}_{vol}^{m+1}\langle y\rangle$ such that $\gamma(0)=\alpha$, $\gamma(r)=\beta$ and for every index i with $0\le i\le r-1$ $\gamma(i)$ and $\gamma(i+1)$ are adjacent (m+1)–tuples. This equivalence relation will be called y–V–*equivalence* and the sequence $\left(\gamma(i)\right)_{i=0,\ldots,r}$ will be said to be a *y–V–equivalence sequence from α to β of length r with respect to* $\mathcal{B}_\epsilon$. We shall denote by $\ell_{\mathcal{P}}^V(x,y)$ the number of equivalence classes in which $\mathcal{P}_{vol}^{m+1}\langle x\rangle$ is divided by y–V–equivalence.

Before going on with the exposition we need two lemmas and a corollary of the former lemma.

Lemma (3.1). *If* $\alpha=(Q_0,\ldots,Q_{p-1},Q_p,Q_{p+1},\ldots,Q_m)$, $\beta=(Q_0,\ldots,Q_{p-1},R,Q_{p+1},\ldots,Q_m)\in \mathbb{E}^{m(m+1)}$ *with* $\|R-Q_p\|<\epsilon$ *and* V_p *is the* $(m-1)$*–dimensional (and therefore always non-negative) volume of the convex hull of the set* $I_p=\{Q_0,\ldots,Q_{p-1},Q_{p+1},\ldots,Q_m\}$ *then* $|vol(\alpha)-vol(\beta)|\le\frac{V_p\cdot\epsilon}{m}$.

Proof. If $V_p=0$ then $vol(\alpha)=vol(\beta)=0$ and so $|vol(\alpha)-vol(\beta)|=\frac{V_p\cdot\epsilon}{m}$. In the case $V_p\neq 0$ let $\vec{N}$ be one of the two versors applied in Q_p and orthogonal to the linear hull of I_p. The maximum of $\left|vol(\alpha)-vol\big(\beta(R)\big)\right|$ with R varying in the closed ball of center Q_p and radius ϵ is obtained only in the two boundary points $Q_p+\epsilon\vec{N}$ and $Q_p-\epsilon\vec{N}$. In such points we have $\left|vol(\alpha)-vol\big(\beta(R)\big)\right|=\frac{V_p\cdot\epsilon}{m}$. So for $\|R-Q_p\|<\epsilon$ we have $|vol(\alpha)-vol(\beta)|<\frac{V_p\cdot\epsilon}{m}$. Therefore in any case the thesis is true. □

Corollary (3.1). *Let* $\mathcal{R}$ *be a subset of* $\mathbb{E}^m$ *and let* $M_{\mathcal{R}}$ *be the supremum of the* $(m-1)$*–dimensional volumes of the convex hulls of the subsets of* $\mathcal{R}$ *containing exactly* m *points. If* $\alpha,\beta\in\mathcal{R}^{m+1}$ *and* $d_m(\alpha,\beta)<\epsilon$ *then* $|vol(\alpha)-vol(\beta)|\le\frac{m+1}{m}\cdot M_{\mathcal{R}}\cdot\epsilon$.

Proof. It follows immediately by applying Lemma (3.1) m+1 times. □

Note: for sake of conciseness in the following we shall denote by ω the value $\frac{m+1}{m}\cdot M_{\mathcal{M}\cup\mathcal{P}}\cdot\epsilon$.

Lemma (3.2). *Let* $y\in\mathbb{R}$, $\alpha_1,\alpha_2\in\mathcal{M}^{m+1}$, $\beta_1,\beta_2\in\mathcal{P}^{m+1}$, $d_m(\alpha_1,\beta_1)<\epsilon$ *and* $d_m(\alpha_2,\beta_2)<\epsilon$. *The following statements hold:*

i) If $\alpha_1,\alpha_2\in\mathcal{M}^{m+1}\langle y\rangle$ *and* $\alpha_1\overset{V}{\underset{y}{=}}\alpha_2$ *then* $\beta_1,\beta_2\in\mathcal{P}_{vol}^{m+1}\langle y+\omega\rangle$ *and* $\beta_1 \overset{V}{\underset{y+\omega}{\approx}} \beta_2$.

ii) If $\beta_1,\beta_2 \in \mathcal{P}^{m+1}_{vol}\langle y\rangle$ *and* $\beta_1 \overset{V}{\underset{y}{\approx}} \beta_2$ *then* $\alpha_1,\alpha_2 \in \mathcal{M}^{m+1}\langle y+\omega\rangle$ *and* $\alpha_1 \overset{V}{\underset{y+\omega}{=}} \alpha_2$.

Proof. Analogous to the one of Lemma (2.2) by using Corollary (3.1) instead of Lemma (2.1). □

Now we can prove the main results in this Section. They will allow us to actually compute the values of the function $\ell^V(x,y)$.

Theorem (3.1). *For every* $x,y \in \mathbf{R}$ *with* $x+\omega \leq y-\omega$ *we have* $\ell^V_{\mathcal{P}}\left(x-\omega,y+\omega\right) \leq \ell^V(x,y) \leq \ell^V_{\mathcal{P}}\left(x+\omega,y-\omega\right)$.

Proof. Analogous to the one of Theorem (2.1) by using Corollary (3.1) and Lemma (3.2) instead of Lemma (2.1) and Lemma (2.2). □

Theorem (3.2). *Let* $\bar{x},\bar{y},b,c \in \mathbf{R}$ *with* $b,c \geq 0$ *and* $\bar{x}+\omega \leq \bar{y}-\omega$. *If the function* $\ell^V_{\mathcal{P}}$ *takes the same value v in the two points* $\left(\bar{x}+\omega,\bar{y}-\omega\right)$ *and* $\left(\bar{x}-b-\omega,\bar{y}+c+\omega\right)$ *then it results* $\ell^V(x,y)=v$ *for every* (x,y) *in the rectangle* $\left\{(x,y)\in \mathbf{R}^2 : \bar{x}-b \leq x \leq \bar{x},\ \bar{y} \leq y \leq \bar{y}+c\right\}$.

Proof. Analogous to the one of Theorem (2.2) by using Theorem (3.1) instead of Theorem (2.1). □

Remark (3.1). What we have pointed out in Remark (2.4) regarding the function ℓ^{D^k} is true also for the function l^V if we substitute Theorem (2.2) by Theorem (3.2), the value 2ε by the value ω and $\mathbb{N}$ by $\mathbb{Z}$. It is easy to prove that $M_{\mathcal{M}\cup\mathcal{P}}$ is less than or equal to $\sigma_{m-1}\cdot\Delta(\mathcal{M}\cup\mathcal{P})^{m-1}$ where σ_{m-1} denotes the non-oriented volume of the standard $(m-1)$–simplex of edge 1 and $\Delta(\mathcal{M}\cup\mathcal{P})$ is the diameter of the set

$\mathcal{M}\cup\mathcal{P}$. Therefore as an upper bound of ω we can use the value $\frac{m+1}{m}\cdot\sigma_{m-1}\cdot\Delta(\mathcal{M}\cup\mathcal{P})^{m-1}\cdot\epsilon$, which is easier to compute than ω.

4. Explicit computation: two examples in $\mathbf{D}^1$–homotopic theory.

As pointed out in Remarks (2.4) and (3.1), Theorem (2.2) and (3.2) give a method to actually compute the size function in $\mathbf{D}^k$– and $\mathbf{V}$–homotopic theory. In this Section we shall give some values of the size function ℓ^{D^1} (computed via Theorem (2.2)) for two 1–dimensional submanifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathbf{E}^2$ as an example of the usefulness of this technique. The use of this method is not conditioned by mathematical difficulties but only by the efficiency and speed in calculation of the used hardware and software. To obtain the following two examples we have used a personal computer: its limitations in speed and memory together with the artlessness of the used algorithm have prevented us from employing ϵ–coverings with a large number of balls and studying more complicated manifolds.

Example (4.1). Let $\mathcal{M}_1$ be the 1–submanifold of $\mathbf{E}^2$ defined by the plane curve of equation $\rho=2+2\cos^2(2\vartheta)$ (represented in polar coordinates). The figures 4.1 and 4.2 show respectively the

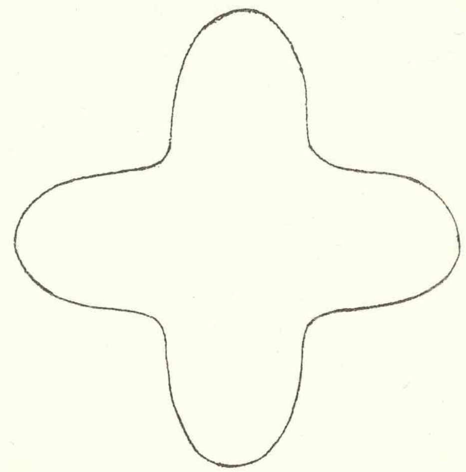

Fig. 4.1

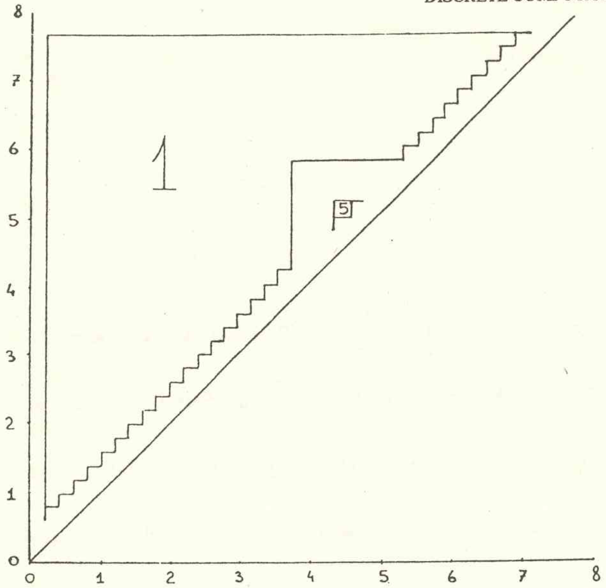

Fig. 4.2

manifold $\mathcal{M}_1$ and the values taken by the corresponding size function ℓD^1 in two connected subsets of the real parameter plane. In the computation we have used a 0.1−covering of $\mathcal{M}_1$ constituted by 99 balls and have considered the domain $\left\{(x,y)\varepsilon\mathbb{R}^2\text{: } 0\leq x\leq y\leq 8\right\}$. We have not displayed the values of $\ell D^1(x,y)$ for x>y because they are trivial.

Example (4.2). Let $\mathcal{M}_2$ be the 1−submanifold of $\mathbb{E}^2$ defined by the plane curve of equation $\rho=5+4\sin(4\vartheta-\pi/2)+3\cos^2(\vartheta)$ (represented in polar coordinates). The figures 4.3 and 4.4 show

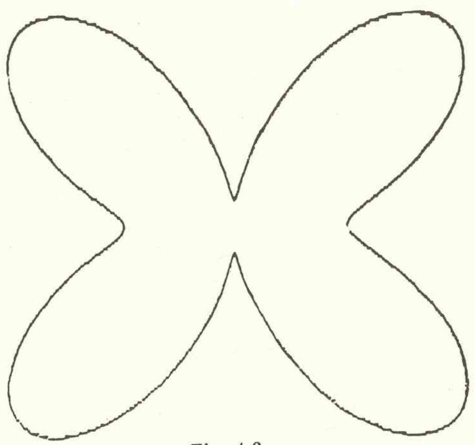

Fig. 4.3

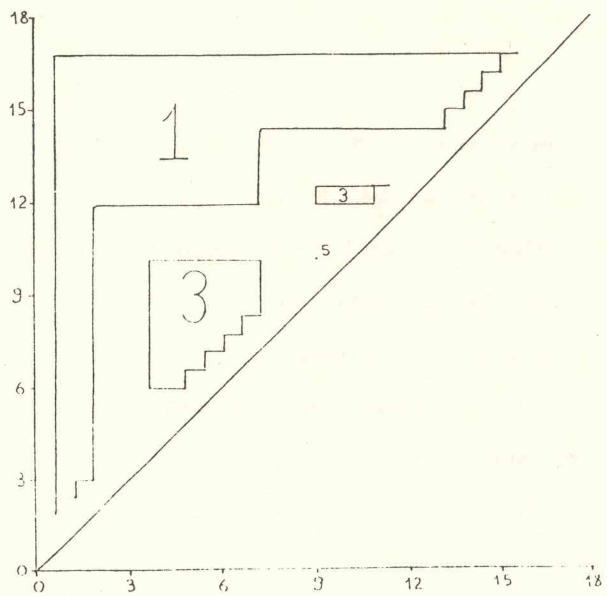

Fig. 4.4

respectively the manifold $\mathcal{M}_2$ and the values taken by the corresponding size function ℓD^1 in four connected subsets of the real parameter plane (one of these is a point). In the computation we have used a 0.3−covering of $\mathcal{M}_2$ constituted by 137 balls and have considered the domain $\left\{(x,y)\varepsilon\mathbb{R}^2\text{: } 0\leq x\leq y\leq 18\right\}$. We have not displayed the values of $\ell D^1(x,y)$ for x>y because they are trivial.

Remark (4.1). It is interesting to notice that the study of the size function usually becomes more and more difficult as we approach the line y=x. In its neighbourhood a lot of nearer and nearer discontinuity points can thicken corrispondingly to the existence of smaller and smaller protuberances on the considered manifold. The more detailed the analysis of our manifold's "shape" is, the more we are interested in studying the size function in the neighbourhood of the line y=x and therefore to use ϵ−covering with ϵ small.

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LINEAR MAXIMIN OBJECTIVE FUNCTION PROGRAMMING

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The following linear maximin programming problem:

$$\text{Max } z = \text{Min}(c_1'x, c_2'x, \ldots, c_n'x), \quad C = (c_1, \ldots, c_n)',$$

$$\text{subject to } Ax \geq v, \ x \geq 0,$$

where $A(q \times n)$ of rank $q < n$, $C(n \times n)$, and v are specified, has been studied by Gupta and Arora (1978). They solved the problem by a certain n-dimensional geometrical method, assuming C^{-1} to have nonpositive elements. In this paper we solve this problem without this restriction on C. A generalization of this problem to the vector case is also presented.

1. GUPTA AND ARORA'S METHOD

To solve the above maximin problem, Gupta and Arora (1978) set

$$x = C^{-1}y, \ \text{Max } z = \text{Min}(y_1, \ldots, y_n), \tag{1}$$

$$By = AC^{-1}y \geq v, \ C^{-1}y \geq 0. \tag{2}$$

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