Counting components of multi-dimensional sub-level sets for shape comparison

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Outline

1. **A Metric Approach to Shape Comparison**

2. Size functions

3. Some new theoretical results

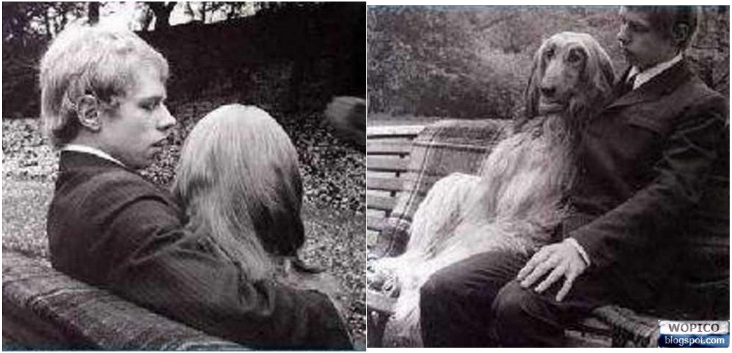
Informal position of the problem

Every comparison of properties involves the presence of

- an observer perceiving the properties
- a methodology to compare the properties

Informal position of the problem

The perception properties depend on the subjective interpretation of an observer:



Informal position of the problem

The perception properties depend on the subjective interpretation of an observer:

Informal position of the problem

The concept of shape is subjective and relative. It is based on the act of perceiving, depending on the chosen observer. Persistent perceptions are fundamental in order to approach this concept.

- “Science is nothing but perception.” *Plato*
- “Reality is merely an illusion, albeit a very persistent one.” *Albert Einstein*

Our formal setting

Our formal setting:

- Each perception is formalized by a pair $(X, \vec{\varphi})$, where X is a topological space and $\vec{\varphi}$ is a continuous function.
- X represents the set of observations made by the observer, while $\vec{\varphi}$ describes how each observation is interpreted by the observer.

Our formal setting

Example a Let us consider Computerized Axial Tomography, where for each unit vector v in the real plane a real number is obtained, representing the total amount of mass $\varphi(v)$ encountered by an X-ray beam directed like v. In this case the topological space X equals the set of all unit vectors in $\mathbb{R}^2$, i.e. S^1. The filtering function is $\varphi : S^1 \to \mathbb{R}$.

Example b Let us consider a rectangle R containing an image, represented by a function $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3) : R \to \mathbb{R}^3$ that describes the RGB components of the colour for each point in the image. The filtering function is $\vec{\varphi} : R \to \mathbb{R}^3$.

Our formal setting

- Persistence is quite important. Without persistence (in space, time, with respect to the analysis level...) perception could have little sense. This remark compels us to require that
  - X is a topological space and $\vec{\varphi}$ is a continuous function; this function $\vec{\varphi}$ describes X from the point of view of the observer. It is called a measuring function (or filtering function).
  - Persistent Topology is used to study the stable properties of the pair $(X, \vec{\varphi})$.

Our formal setting

We can now define the following (extended) pseudo-metric:

$$\delta\left((X,\vec{\varphi}),(Y,\vec{\psi})\right) = \inf_{h\in Hom(X,Y)} \max_i \max_{x\in X} |\varphi_i(x) - \psi_i \circ h(x)|$$

if the set $Hom(X, Y)$ of the homeomorphisms between X and Y is not empty, while δ takes the value $+\infty$ otherwise.
We shall call $\delta\left((X,\vec{\varphi}),(Y,\vec{\psi})\right)$ the natural pseudo-distance between $(X,\vec{\varphi})$ and $(Y,\vec{\psi})$.

The functional $\Theta(h) = \max_i \max_{x\in X} |\varphi_i(x) - \psi_i \circ h(x)|$ represents the "cost" of the matching between observations induced by h. The lower this cost, the better the matching between the two observations.

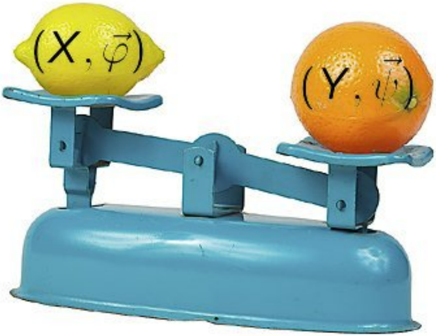
Our formal setting

- The natural pseudo-distance δ measures the dissimilarity between the perceptions expressed by the pairs $(X, \vec{\varphi})$, $(Y, \vec{\psi})$.
- The value δ is small if and only if we can find a homeomorphism between X and Y that induces a small change of the measuring function (i.e., of the shape property we are interested to study).

- For more information:
- P. Donatini, P. Frosini, *Natural pseudodistances between closed manifolds*, Forum Mathematicum, 16 (2004), n. 5, 695-715.
- P. Donatini, P. Frosini, *Natural pseudodistances between closed surfaces*, Journal of the European Mathematical Society, 9 (2007), 331-353.

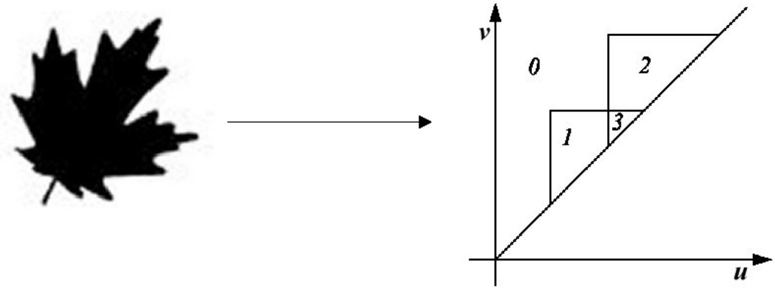
Our formal setting

In plain words, the natural pseudo-distance δ is obtained by trying to match the observations (taken in the topological spaces X and Y), in a way that minimizes the change of properties that the observer judges relevant (the filtering functions $\vec{\varphi}$ and $\vec{\psi}$).



Natural pseudo-distance and size functions

- The natural pseudo-distance is usually difficult to compute.
- Lower bounds for the natural pseudo-distance δ can be obtained by computing the size functions.



Main definitions:

Given a topological space X and a continuous function $\vec{\varphi} : X \to \mathbb{R}^k$,

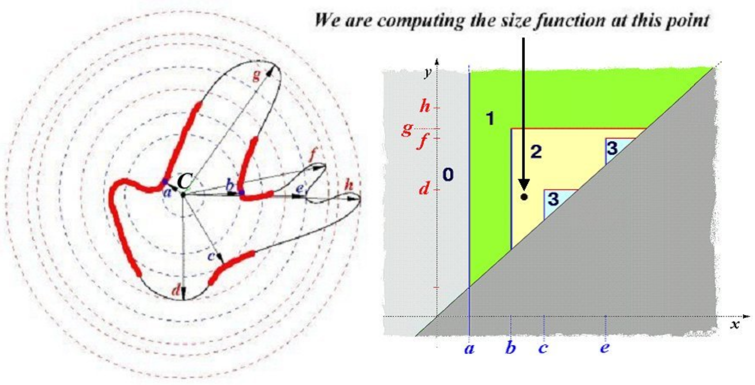
Lower level sets

For every $\vec{u} \in \mathbb{R}^k$, $X\langle \vec{\varphi} \preceq \vec{u} \rangle = \{x \in X : \vec{\varphi}(x) \preceq \vec{u}\}$.
($(u_1, \ldots, u_k) \preceq (v_1, \ldots, v_k)$ means $u_j \leq v_j$ for every index j.)

Definition (F. 1991)

The Size Function of $(X, \vec{\varphi})$ is the function ℓ that takes each pair $(\vec{u}, \vec{v})$ with $\vec{u} \prec \vec{v}$ to the number $\ell(\vec{u}, \vec{v})$ of connected components of the set $X\langle \vec{\varphi} \preceq \vec{v} \rangle$ that contain at least one point of the set $X\langle \vec{\varphi} \preceq \vec{u} \rangle$.

Example of a size function, in the case that the filtering function φ has only one component



We observe that each size function can be described by giving a set of points (vertices of triangles in figure). sizeshow.jar+cerchio.avi

Persistent homology groups and size homotopy groups

Size functions have been generalized by Edelsbrunner and al. to homology in higher degree (i.e., counting the number of holes instead of the number of connected components). This theory is called Persistent Homology:

H. Edelsbrunner, D. Letscher, A. Zomorodian, *Topological persistence and simplification*, Discrete & Computational Geometry, vol. 28, no. 4, 511–533 (2002).

Size functions have been also generalized to size homotopy groups:

P. Frosini, M. Mulazzani, *Size homotopy groups for computation of natural size distances*, Bulletin of the Belgian Mathematical Society, vol. 6, no. 3, 455–464 (1999).

Some important theoretical facts:

- The theory of size functions for filtering functions taking values in $\mathbb{R}^k$ can be reduced to the case of size functions taking values in $\mathbb{R}$, by a suitable foliation of their domain;
- On each leaf of the foliation, size functions are described by a collection of points (the vertices of the triangles seen previously);
- Size functions can be compared by measuring the difference between these collections of points, by a matching distance;
- Size functions (and persistent homology groups) are stable with respect to perturbations of the filtering functions (measured via the max-norm). More precisely, the matching distance between two size functions is a lower bound for the corresponding natural pseudo-distance.

Stability with respect to domain perturbation

Size functions have been proven stable with respect to domain perturbation, in a suitable sense. This stability can be formalized in the multidimensional setting.
The idea is adding to the filtering function a component that describes the "*belonging level*" of each point in the domain. Indeed, the following result holds, with respect to the Hausdorff distance δ_H:

Theorem

Let K_1, K_2 be non-empty closed subsets of a triangulable subspace X of $\mathbb{R}^n$. Let $d_{K_1}, d_{K_2} : X \to \mathbb{R}$ be their respective distance functions. Moreover, let $\vec{\varphi}_1, \vec{\varphi}_2 : X \to \mathbb{R}^k$ be vector-valued continuous functions. Then, defining $\vec{\Phi}_1, \vec{\Phi}_2 : X \to \mathbb{R}^{k+1}$ by $\vec{\Phi}_1 = (d_{K_1}, \vec{\varphi}_1)$ and $\vec{\Phi}_2 = (d_{K_2}, \vec{\varphi}_2)$, the following inequality holds:

$$D_{match}\left(\ell_{\vec{\Phi}_1}, \ell_{\vec{\Phi}_2}\right) \leq \max\left\{\delta_H(K_1, K_2), \|\vec{\varphi}_1 - \vec{\varphi}_2\|_\infty\right\}.$$

Stability with respect to domain perturbation

The next result shows that the size function of $\vec{\Phi}$ still provides a shape descriptor for K as seen through $\vec{\varphi}_{|K}$.

Theorem

Let K be a non-empty triangulable subset of a triangulable subspace X of $\mathbb{R}^n$. Moreover, let $\vec{\varphi}: X \to \mathbb{R}^k$ be a continuous function. Setting $\vec{\Phi}: X \to \mathbb{R}^{k+1}$, $\vec{\Phi} = (d_K, \vec{\varphi})$, for every $\vec{u}, \vec{v} \in \mathbb{R}^k$ with $\vec{u} \prec \vec{v}$, there exists a real number $\hat{b} > 0$ such that, for any $b \in \mathbb{R}$ with $0 < b \leq \hat{b}$, there exists a real number $\hat{a} = \hat{a}(b)$, with $0 < \hat{a} < b$, for which

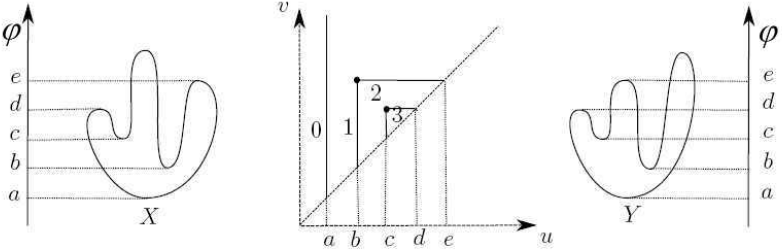
$$\ell_{\vec{\varphi}_{|K}}(\vec{u}, \vec{v}) = \ell_{\vec{\Phi}}\left((a, \vec{u}), (b, \vec{v})\right),$$

for every $a \in \mathbb{R}$ with $0 \leq a \leq \hat{a}$. In particular,

$$\ell_{\vec{\varphi}_{|K}}(\vec{u}, \vec{v}) = \lim_{b \to 0^+} \ell_{\vec{\Phi}}\left((0, \vec{u}), (b, \vec{v})\right).$$

Uniqueness of models in persistent homology: the case of curves

What can be said about the pair (X, φ), if we know its size function?



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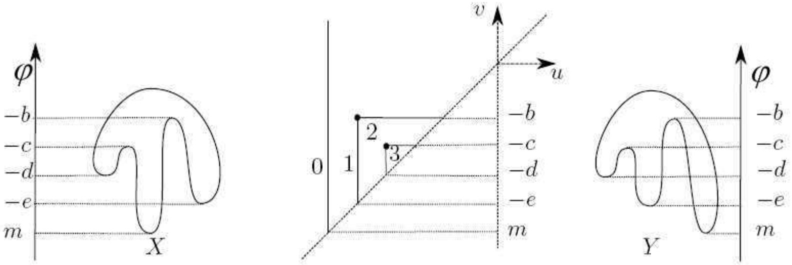

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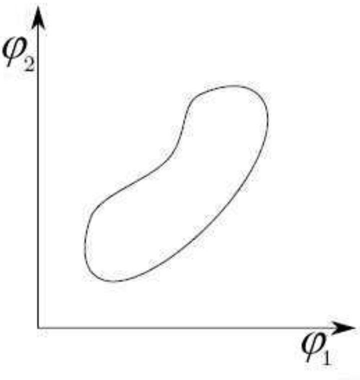
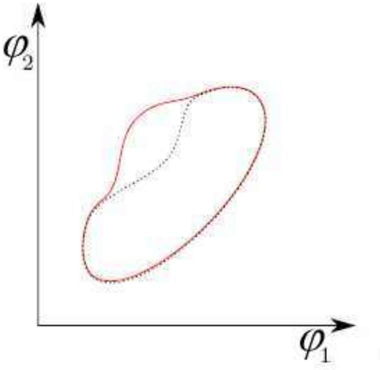




Uniqueness of models in persistent homology: the case of curves

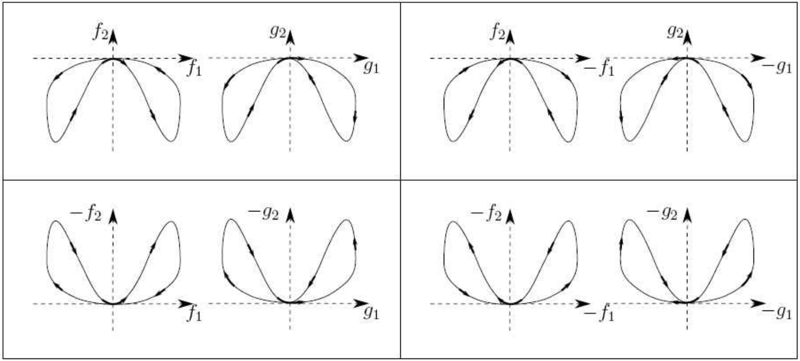
We have recently proven the following result:

Theorem

Let $f, g : S^1 \to \mathbb{R}^2$ be "generic" functions from S^1 to $\mathbb{R}^2$. If the size functions of the four pairs of filtering functions $(\pm f_1, \pm f_2), (\pm g_1, \pm g_2)$ (with corresponding signs) coincide, then there exists a C^1-diffeomorphism $h : S^1 \to S^1$ such that $g \circ h = f$. Moreover, it is unique.

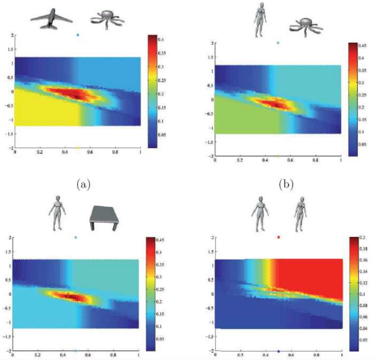
Uniqueness of models in persistent homology: the case of curves

The assumption that f and g are generic is important, as the following example shows.
Let us consider the cases $\vec{\varphi} = (\pm f_1, \pm f_2)$ and $\vec{\varphi} = (\pm g_1, \pm g_2)$.



Computation

Each multi-dimensional size function can be reduced to an infinite family of 1-dimensional size functions, through a foliation. A recently proven error bound has allowed us to extend the algorithm for computing the matching distance between 1-dimensional size functions to the multi-dimensional case.



Conclusions

- We have illustrated the concept of multi-dimensional size function, seen as a mathematical tool to compare shape properties;
- Some recent theoretical results about multi-dimensional size functions have been presented, concerning their stability with respect to domain perturbation and the solution of the inverse problem in the case of curves.