Archives of Inequalities and Applications 2 (2004)1-12

LOWER BOUNDS FOR NATURAL PSEUDODISTANCES VIA SIZE FUNCTIONS

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ABSTRACT. Let us consider two C^1 closed homeomorphic manifolds $\mathcal{M}, \mathcal{N}$ and two C^1 functions $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$, called measuring functions. The natural pseudodistance d between the pairs $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ is defined as the infimum of $\Theta(f) \stackrel{def}{=} \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$, as f varies in the set of all homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. In this paper we show that size functions allow us to get a lower bound for d. Furthermore, we prove that this lower bound can be assumed equal either to $|c' - c''|$ or to $\frac{1}{2}|c' - c''|$, where c', c'' are two suitable critical values of the measuring functions.

AMS (MOS) Subject Classification. Primary 58C05; Secondary 53C15.

Keywords. Natural pseudodistance, measuring function, size function.

1. INTRODUCTION

Comparing two homeomorphic closed manifolds $\mathcal{U}, \mathcal{V}$ usually means quantifying the difference between the structures we are considering on them. These structures can often be described by two functions φ, ψ from two suitable manifolds $\mathcal{M}, \mathcal{N}$ to real numbers. As an example, a Riemannian structure on a smooth manifold $\mathcal{U}$ can be seen as a real-valued function defined on the Whitney sum $\mathcal{M} = T(\mathcal{U}) \oplus T(\mathcal{U})$.

One way of making our quantitative comparison is to compute the infimum of an operator Θ defined on a suitable set H of homeomorphisms. By assuming that two C^1 closed homeomorphic manifolds, $\mathcal{M}$, $\mathcal{N}$, are given, we are naturally led to study the value $d \stackrel{def}{=} \inf_{f \in H(\mathcal{M},\mathcal{N})} \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$ for every *arbitrary* pair of C^1 functions $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$, where $H(\mathcal{M},\mathcal{N})$ denotes the set of all homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. We shall call φ and ψ *measuring functions.*

This work was partially supported by MURST (Italy), ELSAG (Italy), INdAM (Italy) and NATO. Investigation supported by the University of Bologna. Funds for selected research topics.

Received April 5, 2002

If $d = 0$ then for every $\epsilon > 0$ a homeomorphism exists for which the difference between the values taken by the measuring functions at corresponding points is less than ϵ. On the other hand, if the infimum d is large, we know that every homeomorphism between the considered manifolds must substantially change the values taken by our measuring function.

This procedure resembles the ones used in defining the Fréchet distance (cf. [1]), the Lipschitz distance (cf. [9]) and, in some senses, the Teichmüller distance (cf. [14]). The same kind of approach is followed in [5].

We underline that the measuring functions can be arbitrarily chosen and there are many examples of functions whose extrema have been used extensively for studying and comparing manifolds (cf., e.g., [2, 9, 10, 11, 12]).

Because of these properties and their usefulness in modelling minimization problems in Geometry, the functional Θ and the value d seem to be worth studying from a mathematical point of view.

Furthermore, in recent years an applicative interest in this subject has developed. In particular, the purpose of comparing "shapes" of manifolds and topological spaces to solve Computer Vision problems has made the computation of d a useful task, together with its "twin" and strictly related concept of *size function*. For more theoretical details and examples of practical applications we refer to [3, 7, 13, 15, 16, 17, 18].

All these reasons, together with the challenging difficulty in computing d, have motivated our research.

In this paper we shall prove that size functions allow us to get a lower bound for d. This inequality is given in Theorem 1. Furthermore, we shall find the supremum s of such a lower bound by showing (Theorem 2) that s equals either $|c' - c''|$ or $\frac{1}{2}|c' - c''|$, where c', c'' are two suitable critical values of the involved measuring functions φ and ψ. These results allow us to get information about d indirectly and simply, avoiding having to study the set $H(\mathcal{M}, \mathcal{N})$, which is usually very difficult to manage. They should be compared with the results obtained in [4], where we prove that a suitable multiple of d by a positive integer k coincides with the distance between two critical values of the functions φ, ψ.

In the following Section 2 we formally set out the problem we are studying and introduce some definitions, while our results are given in Section 3.

2. SETTING THE PROBLEM

2.1. **The natural pseudodistance.** Let us consider the set $Size_n$ of all pairs $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a closed C^1 n-manifold and $\varphi : \mathcal{M} \to \mathbb{R}$ is a C^1 function. We shall call $(\mathcal{M}, \varphi)$ an (n-dimensional) *size pair* and φ a *measuring function*.

Assume $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ are two size pairs. We shall denote by $H(\mathcal{M}, \mathcal{N})$ the set of all homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$.

Definition 1. *If $H(\mathcal{M}, \mathcal{N}) \neq \emptyset$, the function $\Theta : H(\mathcal{M}, \mathcal{N}) \to \mathbb{R}$*

$$\Theta(f) = \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$$

is called natural measure *with respect to the measuring functions φ and ψ.*

In plain words, Θ measures how much f changes the values taken by the measuring functions at corresponding points.

Definition 2. *We shall call* natural pseudodistance *the pseudodistance $\delta : Size_n \times Size_n \to \mathbb{R} \cup \{+\infty\}$ so defined:*

$$\delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi)) = \begin{cases} \inf_{f \in H(\mathcal{M}, \mathcal{N})} \Theta(f) & \text{if } H(\mathcal{M}, \mathcal{N}) \neq \emptyset \\ +\infty & \text{otherwise.} \end{cases}$$

In the following, the symbol d will denote the value of the natural pseudodistance computed between the pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ we are considering. As we previously explained, such a pseudodistance gives a method for comparing two manifolds with respect to the measuring functions we have chosen.

We point out that δ is not a distance, since two size pairs can have a vanishing pseudodistance without being equal. On the other hand, the symmetry property and the triangle inequality can be trivially proved.

Now we give two simple examples in order to make our definitions clear.

Example 1. *In $\mathbb{R}^3$ consider the unit sphere $\mathcal{S}$ of equation $x^2 + y^2 + z^2 = 1$ and the ellipsoid $\mathcal{E}$ of equation $x^2 + 4y^2 + 9z^2 = 1$. On $\mathcal{S}$ and $\mathcal{E}$ consider respectively the measuring functions φ and ψ that take every point of $\mathcal{S}$ and $\mathcal{E}$ to the Gaussian curvature of the considered manifold at that point. We have $\delta((\mathcal{S}, \varphi), (\mathcal{E}, \psi)) = 35$. In fact, $\varphi(\mathcal{S}) = \{1\}$, while $\psi(\mathcal{E}) = [4/9, 36]$, and therefore for every $f \in H((\mathcal{S}, \varphi), (\mathcal{E}, \psi))$ it results that $\Theta(f) = 35$.*

Example 2. *Let us consider the size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$, where $\mathcal{M}$ and $\mathcal{N}$ are the smooth surfaces shown in Figure 1 and the measuring functions φ, ψ are the ordinate function.*

It is easy to build a sequence (f_n) of homeomorphisms such that $\Theta(f_n) \to \frac{1}{2}|\varphi(A) - \varphi(B)|$. This implies that the natural pseudodistance d between the size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ verifies the inequality

$$d \leq \frac{1}{2}|\varphi(A) - \varphi(B)|.$$

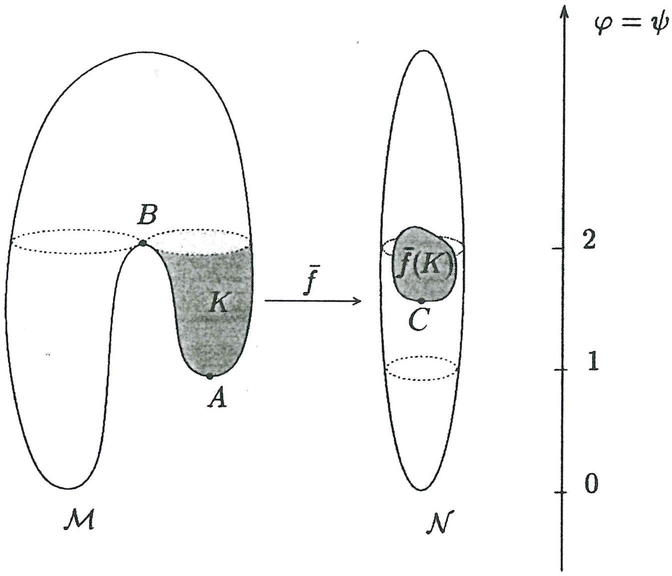

FIGURE 1. For the size pairs $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ no homeomorphism exists having natural measure equal to the natural pseudodistance. In this case $d = \frac{1}{2}|\varphi(A) - \varphi(B)|$.

Moreover, no homeomorphism between $\mathcal{M}$ and $\mathcal{N}$ exists for which $\Theta(f) \leq \frac{1}{2}|\varphi(A) - \varphi(B)|$. In order to prove that, we consider a closed set K, as in Figure 1 (K is the closure of the connected component of A in the open set $\varphi^{-1}((-\infty, 2))$).

Suppose a homeomorphism $\bar{f} : \mathcal{M} \to \mathcal{N}$ exists for which $\Theta(\bar{f}) \leq \frac{1}{2}|\varphi(A) - \varphi(B)|$. Call C a point of $\bar{f}(K)$ at which $\psi_{|\bar{f}(K)}$ takes its minimum. C cannot be a critical point for ψ since otherwise we should have $\Theta(\bar{f}) > \frac{1}{2}|\varphi(A) - \varphi(B)|$, because of the values that ψ takes at its two critical points. Hence $C \in \partial \bar{f}(K)$, $\bar{f}^{-1}(C) \in \partial K$ and $\varphi(\bar{f}^{-1}(C)) = \varphi(B)$. It follows that $\psi(C) \geq \frac{1}{2}(\varphi(A) + \varphi(B))$.

Since A is an interior point of K we get $\psi(\bar{f}(A)) > \psi(C)$ and therefore $\psi(\bar{f}(A)) > \frac{1}{2}(\varphi(A) + \varphi(B))$. Hence it should follow

$$|\varphi(A) - \psi(\bar{f}(A))| = \psi(\bar{f}(A)) - \varphi(A) > \frac{1}{2}(\varphi(B) - \varphi(A)) = \frac{1}{2}|\varphi(A) - \varphi(B)|,$$

against the hypothesis about $\Theta(\bar{f})$.

It follows that $\Theta(f) > \frac{1}{2}|\varphi(A) - \varphi(B)|$ for every $f \in H(\mathcal{M}, \mathcal{N})$.

Therefore

$$d \geq \frac{1}{2}|\varphi(A) - \varphi(B)|.$$

In Section 3.2 we shall get the same inequality in a simpler and more direct way, using the lower bound we are going to prove in this paper.

In conclusion, $\delta((\mathcal{M},\varphi),(\mathcal{N},\psi)) = \frac{1}{2}|\varphi(A)-\varphi(B)|$.

In general, the natural pseudodistance is far from being easily computable as in the previous Examples 1 and 2. In Example 1, for every homeomorphism $f \in H(\mathcal{S},\mathcal{E})$ we have that $\Theta(f)$ equals the Hausdorff distance $\delta_H(\varphi(\mathcal{S}),\psi(\mathcal{E}))$ between the sets $\varphi(\mathcal{S})$ and $\psi(\mathcal{E})$ in $\mathbb{R}$. It is clear that $\delta((\mathcal{M},\varphi),(\mathcal{N},\psi))$ is always greater than or equal to $\delta_H(\varphi(\mathcal{M}),\psi(\mathcal{N}))$ and therefore $\Theta(f)$ must be the natural pseudodistance we want to compute. We also point out that, in Example 1, the images of φ and ψ are different sets, so the natural pseudodistance is trivially positive.

In Example 2 the natural pseudodistance is strictly greater than the (vanishing) Hausdorff distance between the images of the two measuring functions.

Remark 1. *The term "natural" is used in order to distinguish the pseudodistance studied here from other pseudodistances we can define between submanifolds of the Euclidean space and between manifolds paired with measuring functions (cf.* [5]*).*

Further information about the natural pseudodistance can be found in [4].

2.2. **Basic definitions and results about size functions.** Assume a size pair $(\mathcal{M},\varphi)$ is given.

Definition 3. For every real number y, we shall say that two points $P, Q \in \mathcal{M}$ are $\langle\varphi \le y\rangle$*-homotopic* if and only if either $P = Q$ or a continuous path $\gamma : [0,1] \to \mathcal{M}$ exists in $\mathcal{M}$ joining P and Q such that $\varphi(\gamma(t)) \le y$ for every $t \in [0,1]$. In case it exists, γ will be called a $\langle\varphi \le y\rangle$*-homotopy* from P to Q. If P and Q are $\langle\varphi \le y\rangle$-homotopic we shall write $P \cong_{\varphi\le y} Q$.

It is easy to see that the relation of $\langle\varphi \le y\rangle$-homotopy is an equivalence relation on $\mathcal{M}$ and all its subsets for every $y \in \mathbb{R}$.

Definition 4. For every $x \in \mathbb{R}$ let $\mathcal{M}\langle\varphi \le x\rangle$ denote the set $\{P \in \mathcal{M} : \varphi(P) \le x\}$. Consider the function $\ell_{(\mathcal{M},\varphi)} : \mathbb{R}^2 \to \mathbb{N}\cup\{+\infty\}$ defined by setting $\ell_{(\mathcal{M},\varphi)}(x,y)$ equal to the number of equivalence classes into which $\mathcal{M}\langle\varphi \le x\rangle$ is divided by the equivalence relation of $\langle\varphi \le y\rangle$-homotopy. We shall call $\ell_{(\mathcal{M},\varphi)}$ the *size function associated with the pair* $(\mathcal{M},\varphi)$.

An example of two simple size functions is given in Figure 2. Here $\mathcal{M}$ and $\mathcal{N}$ are the surfaces we spoke about in Example 2 and each measuring function is the ordinate of the considered point. The number displayed in each region denotes the value taken by the size function in that region. For the main properties of size functions we refer to [6] and [7].

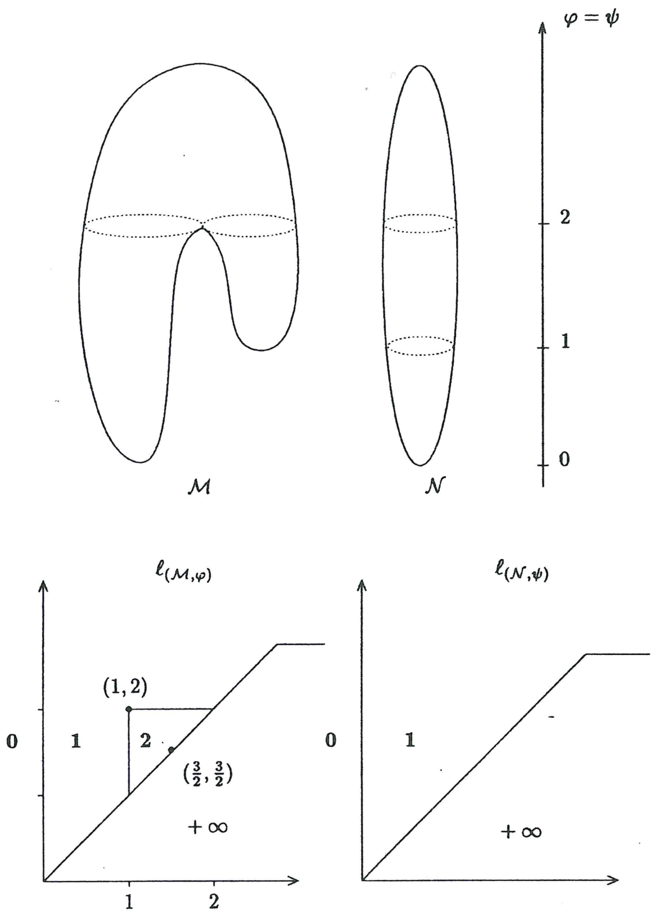

FIGURE 2. Two simple examples of size functions.

Remark 2. *For $x \leq y$, computing $\ell_{(\mathcal{M},\varphi)}(x, y)$ is equivalent to counting the arcwise connected components of the set $M\langle\varphi \leq y\rangle$ that contain at least one point P, such that $\varphi(P) \leq x$.*

We observe that every size function is locally constant at each point where it is continuous. The next result is an immediate consequence of Corollary 2.3 in [6]:

Proposition 1. *Assume that x and y are not critical values for the measuring function φ and that $x < y$. Then $\ell_{(\mathcal{M},\varphi)}$ is locally constant at (x, y).*

This statement constrains the position of discontinuities in size functions. The example in Figure 2 shows that, for each discontinuity point (x, y) with $x < y$, at least one of the values x, y is critical for the measuring function.

For more details about size functions we refer to [5, 6, 7, 8].

3. A LOWER BOUND FOR δ

3.1. **The main result.** Now we shall prove the following result, allowing us to find a lower bound for the natural pseudodistance.

Theorem 1. *Let $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ be two size pairs. If $\ell_{(\mathcal{M},\varphi)}(x,y) > \ell_{(\mathcal{N},\psi)}(\xi,\eta)$ then $\delta((\mathcal{M},\varphi),(\mathcal{N},\psi)) \geq \min\{\xi - x, y-\eta\}$.*

Proof. First of all we observe that $\ell_{(\mathcal{M},\varphi)}(x,y) > 0$ and hence $\mathcal{M}\langle\varphi \leq x\rangle \neq \emptyset$. If it were true that $d < \min\{\xi - x, y - \eta\}$ then a homeomorphism $f : \mathcal{M} \to \mathcal{N}$ would exist such that

$$\Theta(f) < \min\{\xi - x, y - \eta\}.$$

This would allow us to construct an injective function $f^* : \mathcal{M}\langle\varphi \leq x\rangle/_{\cong_{\varphi\leq y}} \to \mathcal{N}\langle\psi \leq \xi\rangle/_{\cong_{\psi\leq\eta}}$ in the following way: for every $C \in \mathcal{M}\langle\varphi \leq x\rangle/_{\cong_{\varphi\leq y}}$ we set $f^*(C)$ equal to the equivalence class of $f(P_C)$ in $\mathcal{N}\langle\psi \leq \xi\rangle/_{\cong_{\psi\leq\eta}}$, where $P_C \in \mathcal{M}$ is an arbitrarily fixed point in the class C (the function f^* will depend on the point we have chosen in each class).

This function is well defined, since $P_C \in \mathcal{M}\langle\varphi \leq x\rangle$ implies

$$\psi(f(P_C)) \leq \varphi(P_C) + \Theta(f) < x + (\xi - x) = \xi$$

and hence $f(P_C) \in \mathcal{N}\langle\psi \leq \xi\rangle$.

In order to prove injectivity, let us suppose $f^*(C) = f^*(C')$. Then either the relation $f(P_C) \cong_{\psi\leq\eta} f(P_{C'})$ holds with $f(P_C) \neq f(P_{C'})$, or $f(P_C) = f(P_{C'})$. In the former case a path $\gamma : [0,1] \to \mathcal{N}$ exists, joining $f(P_C)$ to $f(P_{C'})$, along which the function ψ takes values smaller than or equal to η. By observing that

$$\varphi(f^{-1}\circ\gamma(t)) \leq \psi(\gamma(t)) + \Theta(f^{-1}) = \psi(\gamma(t)) + \Theta(f) < \eta + (y-\eta) = y,$$

we obtain that $f^{-1}\circ\gamma$ is a path joining P_C to $P_{C'}$ along which the measuring function φ takes values smaller than y. Therefore $P_C \cong_{\varphi\leq y} P_{C'}$ and hence $C = C'$. In the latter case $P_C = P_{C'}$ and so, once more, $C = C'$.

The injectivity of f^* contradicts the hypothesis $\ell_{(\mathcal{M},\varphi)}(x,y) > \ell_{(\mathcal{N},\psi)}(\xi,\eta)$. □

3.2. **The optimal lower bound for δ.** As an example, let us apply the previous Theorem 1 to the surfaces $\mathcal{M}$, $\mathcal{N}$ in Example 2, displayed in Figure 2. We observe that the inequalities $\ell_{(\mathcal{M},\varphi)}(x,y) > \ell_{(\mathcal{N},\psi)}(\xi,\eta)$ and $\min\{\xi - x, y-\eta\} \geq 0$ hold if and only if $\xi \geq x, \eta \leq y$ and (x,y) and (ξ,η) belong to the triangle $T = \{(a,b) \in \mathbb{R}^2 \mid a \leq b, a \geq 1, b < 2\}$, where the size functions $\ell_{(\mathcal{M},\varphi)}$ and $\ell_{(\mathcal{N},\psi)}$ take the value 2 and 1, respectively. Hence we are interested in finding the supremum of $\min\{\xi - x, y - \eta\}$ for $(x,y),(\xi,\eta) \in T$, $\xi \geq x$ and $\eta \leq y$, so that the lower bound given in Theorem 1 is as good as possible.

In our case the supremum s we are looking for is not a maximum. In fact the set T is not closed and it can easily be seen that the supremum s coincides with the value $\min\left\{\bar{\xi}-1, 2-\bar{\eta}\right\}$ where $\bar{\xi}=\bar{\eta}=3/2$. Hence we immediately have

$$d \geq s = \frac{1}{2},$$

that is half the distance between the two critical values 1 and 2 of φ.

Now we shall show that the fact the critical values of the measuring functions are involved in the computation of s is not accidental.

Definition 5. *Let*

$$A = \left\{((x,y),(\xi,\eta)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \xi \geq x, \eta \leq y, \ell_{(\mathcal{M},\varphi)}(x,y) > \ell_{(\mathcal{N},\psi)}(\xi,\eta)\right\}$$

and, in case $A \neq \emptyset$, let

$$s = \sup_{((x,y),(\xi,\eta))\in A} \left\{\min\left\{\xi - x, y - \eta\right\}\right\}.$$

In other words, s is the best nonnegative lower bound we can get for δ by applying Theorem 1. Moreover, that theorem implies that if $\mathcal{M}$ and $\mathcal{N}$ are homeomorphic then the value s is finite, since it is upperly bounded by the natural pseudodistance.

Remark 3. *If the measuring functions φ and ψ are Morse functions, the corresponding size functions are locally right-constant in the half-plane $\{(x,y) \in \mathbb{R}^2 | x \leq y\}$ in both the variables (cf. [6], Corollary 2.1), and the number of the critical values of φ and ψ is finite. Therefore Proposition 1 implies that, if a point (x,y) exists such that $x < y$ and $\ell_{(\mathcal{M},\varphi)}(x,y) \neq \ell_{(\mathcal{N},\psi)}(x,y)$, then an open ball B exists such that $\ell_{(\mathcal{M},\varphi)}$ and $\ell_{(\mathcal{N},\psi)}$ take two different constant values in B. It follows that, possibly by exchanging the order of the two considered size pairs, $A \neq \emptyset$ and $s > 0$. Hence Theorem 1 can always produce a positive lower bound when the size functions do not coincide in the open half-plane $\{(x,y) \in \mathbb{R}^2 | x < y\}$, provided that the size pairs are conveniently ordered and the measuring functions are Morse functions.*

The next theorem relates s to the critical values of the measuring functions.

Theorem 2. *Assume $A \neq \emptyset$ and $s < +\infty$. The value s is equal either to the distance between a critical value of φ and a critical value of ψ or to half the distance between two critical values of φ.*

Proof. If $s = 0$ our thesis holds trivially by taking two coinciding critical values of φ. Hence we can assume that $s > 0$. Incidentally, we observe that Remark 3 does not

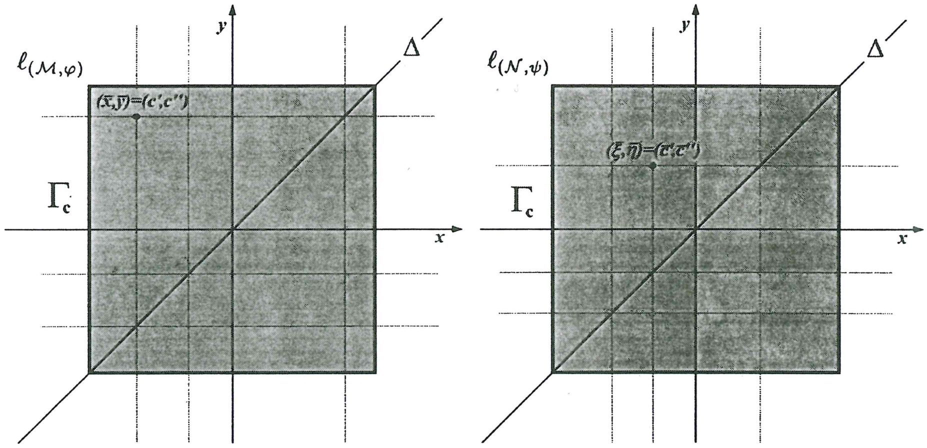

FIGURE 3. The case $\bar{x}, \bar{y} \in C_\varphi \cup \{c\}$ and $\bar{\xi}, \bar{\eta} \in C_\psi \cup \{c\}$. The dotted lines represent the points with at least one coordinate equal to a critical value of the measuring function.

imply that $s > 0$, since φ and ψ are not assumed to be Morse. For every $c \in \mathbb{R}$ that is not a critical value either for φ or for ψ, let us consider the square

$$\Gamma_c = \left\{(x, y) \in \mathbb{R}^2 \mid |x| \leq c, |y| \leq c\right\}$$

and the compact set

$$A_c = \overline{A \cap (\Gamma_c \times \Gamma_c)}.$$

For every large enough c we have that $A_c \neq \emptyset$ and the value

$$s_c = \max_{((x,y),(\xi,\eta)) \in A_c} \{\min\{\xi - x, y - \eta\}\}$$

is defined. In plain words, we want to study s by approximating its value by s_c which is easier to study, since it is a maximum on a compact set instead of a supremum on A.

Choose a pair $((\bar{x}, \bar{y}), (\bar{\xi}, \bar{\eta})) \in A_c$ such that $s_c = \min\left\{\bar{\xi} - \bar{x}, \bar{y} - \bar{\eta}\right\}$ (it exists since A_c is compact). In the following the symbols C_φ and C_ψ will denote the sets of critical values of φ and ψ, respectively.

We shall prove that it is not restrictive to assume both the following properties:

1) $\bar{x}$ and $\bar{y}$ belong to $C_\varphi \cup \{c\}$;
2) either $\bar{\xi}$ and $\bar{\eta}$ belong to $C_\psi \cup \{c\}$ or $\bar{\xi} = \bar{\eta} = \dfrac{\bar{x} + \bar{y}}{2}$.

Define L_φ^c as the set of all points of $\mathbb{R}^2$ having at least one coordinate in the set $C_\varphi \cup \{c\}$. Let L_ψ^c be the analogous set for ψ. Consider the line $\Delta = \{(x, y) \in \mathbb{R}^2 | x = y\}$. Proposition 1 implies that, in every connected component of the set $\Gamma_c - (L_\varphi^c \cup \Delta)$, the function $\ell_{(\mathcal{M},\varphi)}$ is constant. Similarly, in every connected component

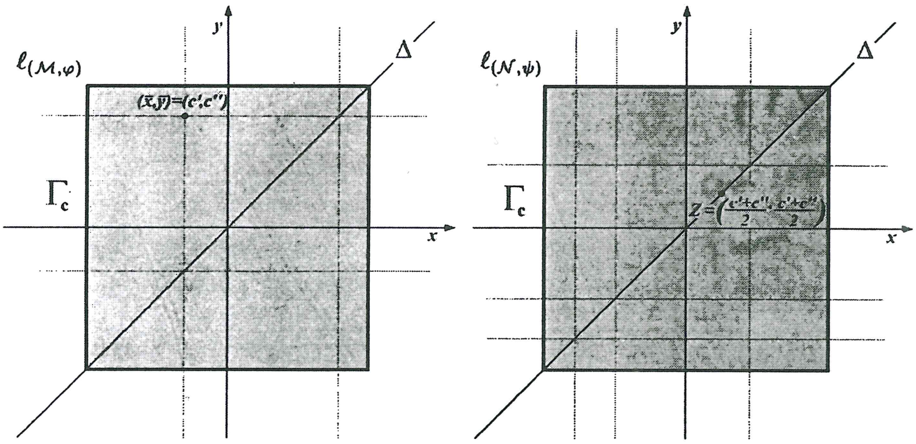

FIGURE 4. The case $\bar{x},\bar{y} \in C_\varphi \cup \{c\}$ and $\bar{\xi} = \bar{\eta} = \dfrac{\bar{x}+\bar{y}}{2}$. The dotted lines represent the points with at least one coordinate equal to a critical value of the measuring function.

of the set $\Gamma_c - (L^c_\psi \cup \Delta)$ the function $\ell_{(\mathcal{N},\psi)}$ is constant. Moreover, we observe that the connected components of those sets are either triangles or rectangles. Therefore, it is not restrictive to assume that $(\bar{x},\bar{y})$ is a vertex of one of the rectangles that are connected components of the set $\Gamma_c - (L^c_\varphi \cup \Delta)$ (see Figure 3). In fact, we can move the original point to a vertex without decreasing the value $\min\left\{\bar{\xi}-\bar{x}, \bar{y}-\bar{\eta}\right\}$ or leaving the closure of the considered connected component. It follows that Property 1) holds.

Furthermore, the point $(\bar{\xi},\bar{\eta})$ can be chosen in the following way.

Call $Z = ((\bar{x}+\bar{y})/2, (\bar{x}+\bar{y})/2)$ the orthogonal projection of $(\bar{x},\bar{y})$ onto the line Δ. Also in this case it is not difficult to prove that the point $(\bar{\xi},\bar{\eta})$, maximizing the value $\min\{\xi-\bar{x},\bar{y}-\eta\}$ can be assumed to be either the vertex of one of the triangles and rectangles that are connected components of $\Gamma_c - (L^c_\psi \cup \Delta)$ or the point Z (see Figure 3 and Figure 4). Therefore Property 2) holds.

Properties 1) and 2) imply that $s_c \in \left\{|c'-c''| : c' \in C_\varphi \cup \{c\}, c'' \in C_\psi \cup \{c\}\right\} \cup \left\{\frac{1}{2}|c'-c''| : c', c'' \in C_\varphi \cup \{c\}\right\}$. Since $s_c \le s < +\infty$ and $C_\varphi \cup C_\psi$ is bounded, it must be that $s_c \notin \left\{|c'-c|, |c''-c|, \frac{1}{2}|c'-c| : c' \in C_\varphi, c'' \in C_\psi\right\}$ when c is large enough. It follows that $s_c \in \left\{|c'-c''| : c' \in C_\varphi, c'' \in C_\psi\right\} \cup \left\{\frac{1}{2}|c'-c''| : c', c'' \in C_\varphi\right\}$ for every large enough c. Therefore we can find a divergent sequence (c_i) for which (s_{c_i}) is defined, and two sequences (c'_i), (c''_i) such that, for every index i, either $s_{c_i} = |c'_i - c''_i|$ with $c'_i \in C_\varphi, c''_i \in C_\psi$ or $s_{c_i} = \frac{1}{2}|c'_i - c''_i|$ with $c'_i, c''_i \in C_\varphi$. Incidentally, we observe that the hypothesis $s < +\infty$ must be assumed, since the manifolds $\mathcal{M}$ and $\mathcal{N}$ could be non-homeomorphic.

Since C_φ and C_ψ are compact sets, if the former condition holds for an infinite number of values of the index i, we can assume (c'_i) and (c''_i) converging to two values c'_φ, c''_ψ in C_φ and C_ψ, respectively (possibly by extracting two suitable subsequences). Otherwise, the latter condition holds for an infinite number of values of the index i, and we can assume that (c'_i) and (c''_i) converge to two values c'_φ, c''_φ in C_φ, respectively.

In conclusion, $s = \sup_c s_c = \lim_{i\to\infty} s_{c_i}$ equals either $|c'_\varphi - c''_\psi|$ or $\frac{1}{2}|c'_\varphi - c''_\varphi|$, and hence our thesis is proved. □

Acknowledgment. We wish to thank M. Ferri for our helpful discussions. Anyway, the authors are solely responsible for any possible errors.

We are grateful to P. Ciampi and J. Brel for their indispensable support.

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