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Multidimensional Size Functions for Shape Comparison

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Abstract Size Theory has proven to be a useful framework for shape analysis in the context of pattern recognition. Its main tool is a shape descriptor called size function. Size Theory has been mostly developed in the 1-dimensional setting, meaning that shapes are studied with respect to functions, defined on the studied objects, with values in $\mathbb{R}$. The potentialities of the k-dimensional setting, that is using functions with values in $\mathbb{R}^k$, were not explored until now for lack of an efficient computational approach. In this paper we provide the theoretical results leading to a concise and complete shape descriptor also in the multidimensional case. This is possible because we prove that in Size Theory the comparison of multidimensional size functions can be reduced to the 1-dimensional case by a suitable change of variables. Indeed, a foliation in half-planes can be given, such that the restriction of a multidimensional size function to each of these half-planes turns out to be a classical size function in two scalar variables. This leads to the definition of a new distance between multidimensional size functions, and to the proof of their stability with respect to that distance. Experiments are carried out to show the feasibility of the method.

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1 Introduction

In the shape analysis literature many methods have been developed, which describe shapes making use of properties of real-valued functions defined on the studied object. Usually, the role of these functions is to quantitatively measure the geometric properties of the shape while taking into account its topology. Because of the topological information conveyed by these descriptors, they often are better suited to compare shapes non-rigidly related to each other. On the other hand, the metrical information provided by the function reveals specific instances of features.

The use of different functions defined on the same object can give insights of its shape from different perspectives. Although many approaches rely on a fixed function for describing shapes (e.g., the height function for contour trees [25], and the distance transform for the medial axis [4, 37, 38]), the possibility of adopting different functions characterizes an increasing number of methods [3]. This strategy is currently being investigated also in graphics for the quite different purpose of detecting generators of the first homology group (cf. [39]).

Since the early 90s, Size Theory was proposed as a geometrical/topological approach to shape description and comparison based on the use of classes of functions (cf., e.g., the papers [26, 27, 43, 44], and also the survey [29] and

Sect. 8.4 of the textbook [34]), finding its roots in the classical Morse Theory ([28]). Particular classes of functions have been singled out as better suited than others to deal with specific problems, such as obtaining invariance under groups of transformations (cf., e.g., [16, 36, 42]), or working with particular classes of objects (cf., e.g., [7, 33, 40, 41]). Nevertheless, the choice of the most appropriate functions for a particular application is not fixed a priori but can be changed up to the problem at hand.

Most of the research in this field was developed for functions with values in $\mathbb{R}$. Retrospectively, we call this setting 1-dimensional. The possibility of constructing an analogous theory based on functions with values in $\mathbb{R}^k$, therefore called k-dimensional or multidimensional, was investigated for the first time in [31]. The advantage of working with k-valued functions is that shapes can be simultaneously investigated by k different 1-valued functions. In other words, k different functions cooperate to produce a single shape descriptor.

In this multidimensional framework, the comparison of two objects in a dataset (e.g. 3D-models, images or sounds) is translated into the comparison of two suitable topological spaces $\mathcal{M}$ and $\mathcal{N}$, endowed with two continuous functions $\vec{\varphi}: \mathcal{M} \to \mathbb{R}^k$, $\vec{\psi}: \mathcal{N} \to \mathbb{R}^k$. These functions are called *k-dimensional measuring functions* and are chosen according to the application. In other words, they can be seen as descriptors of the properties considered relevant for the comparison. In the same paper [31] the natural pseudo-distance d between the pairs $(\mathcal{M}, \vec{\varphi})$, $(\mathcal{N}, \vec{\psi})$ was introduced, setting $d((\mathcal{M}, \vec{\varphi}), (\mathcal{N}, \vec{\psi}))$ equal to the infimum of the change of the measuring function, induced by composition with all the homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$ (if any). The natural pseudo-distance d is a measure of the dissimilarity between the studied shapes. Unfortunately, the direct computation of d is quite difficult, even for $k = 1$, although strong properties can be proved in this case (cf. [17, 19, 20]).

To overcome this difficulty, the strategy proposed in [31] for any k consists in obtaining lower bounds for d using *size homotopy groups*. This strategy has been broadly developed in the case $k = 1$, using *size functions* instead of size homotopy groups ([14, 15, 18]). Size functions are shape descriptors that analyze the variation of the number of connected components in the lower-level sets of the studied space with respect to the chosen measuring function. When $k = 1$, size functions can be efficiently employed in applications since they admit a combinatorial representation as formal series of points and lines in $\mathbb{R}^2$ ([21, 30, 35]). Based on this concise representation, the *matching distance* between size functions was introduced in [35] and further studied in [14, 15]. The first useful property of the matching distance is its stability under perturbations of measuring functions (with respect to the max-norm). Furthermore, the matching distance between two size functions furnishes a lower bound for the natural pseudo-distance between the corresponding size pairs.

The use of the multidimensional size theory in real applications needs answers to the following questions: How to combinatorially represent multidimensional size functions? How to compare multidimensional size functions in a way that is resistant to perturbations? How to obtain a lower bound for the natural pseudo-distance based on multidimensional size functions? Is the method computationally affordable? The main aim of this paper is to answer these questions.

Outline All the basic results of the 1-dimensional Size Theory that we need also in the k-dimensional case are recalled in Sect. 2. Before introducing the concept of k-dimensional size function in Sect. 3, we discuss the motivations for this extension.

Our first result is the proof that in Size Theory the comparison of multidimensional size functions can be reduced to the 1-dimensional case by a suitable change of variables (Theorem 3). The key idea is to show that a foliation in half-planes can be given, such that the restriction of a multidimensional size function to these half-planes turns out to be a classical size function in two scalar variables. This reduction scheme is presented in Sect. 4.

This approach implies that each size function, with respect to a k-dimensional measuring function, can be completely and compactly described by a parameterized family of discrete descriptors (Remark 3). This follows by applying to each plane in our foliation the representation of classical size functions by means of formal series of points and lines. An important consequence is the proof of the stability of this new descriptor (and hence of the corresponding k-dimensional size function) with respect to perturbations of the measuring functions, also in higher dimensions (Proposition 2), by using the stability result proved for 1-dimensional size functions. Moreover, we prove stability of this descriptor also with respect to perturbations of the foliation leaves.

As a further contribution, we show that a matching distance between size functions, with respect to measuring functions taking values in $\mathbb{R}^k$, can easily be introduced (Definition 8). This matching distance provides a lower bound for the natural pseudo-distance, also in the multidimensional case (Theorem 4). The higher discriminatory power of multidimensional size functions in comparison to 1-dimensional ones is proved in Proposition 4 and illustrated by a simple example. All these results, presented in Sect. 5, motivate the introduction of Multidimensional Size Theory in real applications and open the way to their effective use.

The main issues related to the passage from the theoretical to the computational model are discussed in Sect. 6, where we also present experiments on small datasets. In this way we demonstrate that all the properties proved in the theoretical setting actually hold in concrete applications.

Before concluding the paper, we formally explore some links existing between multidimensional size functions and the concept of *vineyard*, recently introduced in [12].

2 Background on 1-dimensional Size Theory

In the 1-dimensional case, Size Theory aims to study the shape of objects based on capturing quantitative topological properties provided by some real continuous function φ defined on the space $\mathcal{M}$ representing the object. Actually, the term "1-dimensional" refers to the fact that φ takes values in $\mathbb{R}$.

In this section we review the basic results in 1-dimensional Size Theory. We confine ourselves to recall those results that in the following sections will be extended to the k-dimensional setting, where shapes will be studied with respect to functions $\vec{\varphi}:\mathcal{M}\to\mathbb{R}^k$. When not explicitly stated, the different notations φ and $\vec{\varphi}$ will make self-evident if we are referring to the 1- or to the k-dimensional case.

Two main tools have been introduced in Size Theory: *size functions* and the *natural pseudo-distance*.

The 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$ is a shape descriptor of the pair $(\mathcal{M},\varphi)$, called *size pair*, where $\mathcal{M}$ is a topological space representing the studied object, and $\varphi:\mathcal{M}\to\mathbb{R}$ is a continuous function, called *measuring function*. Here we assume that $\mathcal{M}$ is a non-empty, compact and locally connected Hausdorff space. Intuitively, the size function associated with the size pair $(\mathcal{M},\varphi)$ captures the topological changes occurring in the lower level sets $\mathcal{M}\langle\varphi\le x\rangle=\{P\in\mathcal{M}:\varphi(P)\le x\}$ as x varies in $\mathbb{R}$. More formally, we have the following definitions (cf. [14, 27, 44]).

Definition 1 For every $y\in\mathbb{R}$, two points $P,Q\in\mathcal{M}$ are said to be $\langle\varphi\le y\rangle$*-connected* if and only if a connected subset of $\mathcal{M}\langle\varphi\le y\rangle$ exists, containing P and Q.

Definition 2 The 1*-dimensional size function* associated with the size pair $(\mathcal{M},\varphi)$ is the function $\ell_{(\mathcal{M},\varphi)}:\{(x,y)\in\mathbb{R}^2:x<y\}\to\mathbb{N}$, defined by setting $\ell_{(\mathcal{M},\varphi)}(x,y)$ equal to the number of equivalence classes in which the set $\mathcal{M}\langle\varphi\le x\rangle$ is divided by the $\langle\varphi\le y\rangle$-connectedness relation.

In order to make this definition clear, we refer the reader to Fig. 1. The object to be studied is the curve $\mathcal{M}$ depicted by a solid line, and the measuring function φ is distance from the point P. The size function $\ell_{(\mathcal{M},\varphi)}$ is illustrated on the right. As can be seen, the domain $\{(x,y)\in\mathbb{R}^2:x<y\}$ is divided into triangular regions (that may be bounded or unbounded), in which the value of $\ell_{(\mathcal{M},\varphi)}$ is constant. The displayed numbers correspond to the values taken by the size function in each region. For example, for $a\le x<y<b$, the set $\mathcal{M}\langle\varphi\le x\rangle$ consists of two connected components

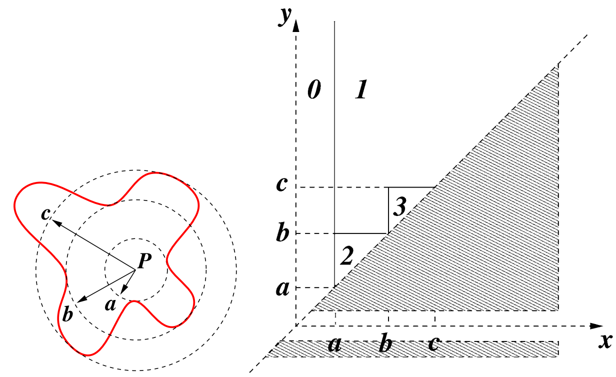

Fig. 1 (**a**) The object to be studied is the curve $\mathcal{M}$ depicted by a *solid line*, and its features are investigated through the measuring function φ distance from the point P. (**b**) The size function $\ell_{(\mathcal{M},\varphi)}$

that cannot be joined under y. So $\ell_{(\mathcal{M},\varphi)}(x,y)=2$ for $a\le x<y<b$. For $a\le x<b$ and $y\ge b$, the two connected components of $\mathcal{M}\langle\varphi\le x\rangle$ can be joined under y, so $\ell_{(\mathcal{M},\varphi)}(x,y)=1$.

It is not difficult to see that $\ell_{(\mathcal{M},\varphi)}(x,y)$ counts the connected components in $\mathcal{M}\langle\varphi\le y\rangle$ containing at least one point of $\mathcal{M}\langle\varphi\le x\rangle$.

The shape descriptions furnished by size functions can be compared using an appropriate distance, called *matching distance* [14, 15]. In this way, assessing the similarity between objects is achieved by computing the matching distance between the corresponding size functions.

The core of the definition of matching distance is the observation that the information contained in a size function can be combinatorially stored in a formal series of lines and points of the plane, called respectively *cornerlines* (or cornerpoints at infinity) and *cornerpoints* (cf. [21, 30, 35]). Precisely, we have the following definitions.

Definition 3 For every vertical line r, with equation $x=k$, we define the number $\mu(r)$ as the minimum, over all the positive real numbers ϵ with $k+\epsilon<1/\epsilon$, of

$$\ell_{(\mathcal{M},\varphi)}(k+\epsilon,1/\epsilon)-\ell_{(\mathcal{M},\varphi)}(k-\epsilon,1/\epsilon).$$

When this finite number, called *multiplicity of r* for $\ell_{(\mathcal{M},\varphi)}$, is strictly positive, we call the line r a *cornerline* (or *cornerpoint at infinity*) for the size function.

Definition 4 For every point $p=(x,y)$ with $x<y$, the number $\mu(p)$ is the minimum over all the positive real numbers ϵ, with $x+\epsilon<y-\epsilon$, of

$$\ell_{(\mathcal{M},\varphi)}(x+\epsilon,y-\epsilon)-\ell_{(\mathcal{M},\varphi)}(x-\epsilon,y-\epsilon)$$
$$-\ell_{(\mathcal{M},\varphi)}(x+\epsilon,y+\epsilon)+\ell_{(\mathcal{M},\varphi)}(x-\epsilon,y+\epsilon).$$

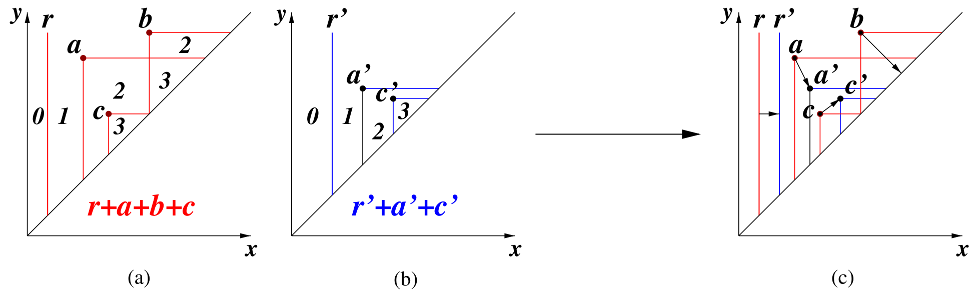

Fig. 2 (**a**) Size function corresponding to the formal series $r+a+b+c$. (**b**) Size function corresponding to the formal series $r'+a'+c'$. (**c**) The matching between the two formal series, realizing the matching distance between the two size functions

The finite number $\mu(p)$ is called *multiplicity of* p for $\ell_{(\mathcal{M},\varphi)}$. Moreover, we call *cornerpoint* for $\ell_{(\mathcal{M},\varphi)}$ any point p such that the number $\mu(p)$ is strictly positive.

Cornerpoints and cornerlines allow us to compactly represent size functions as formal series (cf. Fig. 2(a) and (b)), in virtue of the following theorem, asserting that the multiplicities of cornerpoints and cornerlines are sufficient to compute size functions' values (cf. [15, 30]). For the sake of simplicity, in this statement, lines with equation $x=k$ are identified to points "at infinity" with coordinates (k,∞).

Theorem 1 *For every* $(\bar{x},\bar{y})$ *with* $\bar{x}<\bar{y}<\infty$ *we have*

$$\ell_{(\mathcal{M},\varphi)}(\bar{x},\bar{y})=\sum_{\substack{(x,y):\ x<y\le\infty\\ x\le\bar{x},y>\bar{y}}}\mu\big((x,y)\big).$$

Given two size functions ℓ_1 and ℓ_2, their comparison can be translated to the problem of comparing their multisets of cornerpoints. Hence, let us consider the multiset C_1 (resp. C_2) of all cornerpoints for ℓ_1 (resp. ℓ_2) counted with their multiplicities, augmented by adding a countable infinity of points of the diagonal $\{(x,y)\in\mathbb{R}^2 : x=y\}$. We can compare ℓ_1 and ℓ_2 using the matching distance

$$d_{match}(\ell_1,\ell_2)=\min_{\sigma}\max_{p\in C_1}\delta(p,\sigma(p))$$

where σ varies among all the bijections between C_1 and C_2 and

$$\delta((x,y),(x',y'))=\min\left\{\max\{|x-x'|,|y-y'|\},\ \max\left\{\frac{y-x}{2},\frac{y'-x'}{2}\right\}\right\}.$$

Roughly speaking, the matching distance d_{match} between two size functions is the minimum, over all the matchings between the cornerpoints of the two size functions, of the maximum of the L_∞-distances between two matched cornerpoints. Since two size functions can have a different number of cornerpoints, these can be also matched to points of the diagonal, as illustrated in Fig. 2(c). Notice that the definition of δ implies that matching two points of the diagonal has no cost. For more details on this distance we refer the reader to [14, 15].

A key property of the matching distance is its stability with respect to perturbations of the measuring functions, as stated by the following theorem ([14, 15]).

Theorem 2 *If* $(\mathcal{M},\varphi)$, $(\mathcal{M},\chi)$ *are size pairs and* $\max_{P\in\mathcal{M}}|\varphi(P)-\chi(P)|\le\eta$, *then it holds that* $d_{match}(\ell_{(\mathcal{M},\varphi)},\ell_{(\mathcal{M},\chi)})\le\eta$.

The second tool of Size Theory is the natural pseudo-distance (cf. [17–20, 31]). It is a measure of the similarity between two size pairs not mediated by any shape descriptor. The idea underlying the definition of the natural pseudo-distance between two size pairs $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$, with $\mathcal{M}$ and $\mathcal{N}$ homeomorphic, is to measure the maximum jump between the values taken by the measuring functions φ and ψ when $\mathcal{M}$ is homeomorphically deformed into $\mathcal{N}$:

$$d((\mathcal{M},\varphi),(\mathcal{N},\psi))=\inf_f\max_{P\in\mathcal{M}}|\varphi(P)-\psi(f(P))|,$$

where f varies among all the homeomorphisms between $\mathcal{M}$ and $\mathcal{N}$.

The matching distance turns out to be less informative about the dissimilarity between the studied shapes than the natural pseudo-distance. Indeed, it holds that

$$d_{match}(\ell_{(\mathcal{M},\varphi)},\ell_{(\mathcal{N},\psi)})\le d((\mathcal{M},\varphi),(\mathcal{N},\psi)) \tag{1}$$

(cf. [14, 15]).

Nevertheless, the computation of the natural pseudo-distance can be accomplished only in a few cases. Therefore inequality (1) turns out to be very useful as an estimation from below of the natural pseudo-distance. Moreover, in [14] we have proved that d_{match} gives the best lower bound for the natural pseudo-distance, in the sense that any other distance between size functions would furnish a worse bound.

It is worth mentioning here that, using tools of Algebraic Topology, other lower bounds for the natural pseudo-distance are obtained in [31] by means of *size homotopy groups*. These are shape descriptors that generalize size functions by taking into account classes of loops in the lower level sets instead of connected components. However, size homotopy groups are not computationally feasible. Always from the perspective of Algebraic Topology, yet another shape descriptor, the *Size Functor*, has been developed in [5] for 1-dimensional measuring functions, based on homology.

More recently, similar ideas have independently led Edelsbrunner et al. to the definition of *Persistent Homology* (cf. [23, 24]), and Allili et al. to the definition of the *Morse Homology Descriptor* (cf. [2]).

In particular, the concepts of size function and cornerpoint have been recently rediscovered in the framework of Persistent Homology (cf. [23, 24]). In the terminology of Persistent Homology, size functions correspond to the dimension zero persistent Betti number while formal series of cornerpoints correspond to persistence diagrams. Also the theorem stating the stability of the matching distance has a counterpart in Persistent Homology (cf. [10, 11]). Moreover, the first persistent homology group equals the Abelianization of the first size homotopy group.

3 k-dimensional Size Functions

Multidimensional size functions are the extension of the notion of size function to the case when the measuring function is multivalued, that is it takes values in $\mathbb{R}^k$ instead of $\mathbb{R}$. In other words, a k-dimensional measuring function $\vec{\varphi}$ is a k-tuple $(\varphi_1, \ldots, \varphi_k)$ of 1-dimensional measuring functions.

3.1 Motivation

The idea of using k-dimensional measuring functions arises from the observation that the shape of an object can be more thoroughly characterized by means of a set of measuring functions, each investigating specific features of the shape under study, e.g., scientific data from physical or medical studies that typically consists of a large number of measurements taken within a domain of interest. This observation has quite early led to the definition, in [31], of multidimensional measuring functions and, consequently, of multidimensional size functions and natural pseudo-distance. However, this research line has not been exploited in concrete applications because it was not clear how to develop efficient algorithms for the computation and comparison of multidimensional size functions. Therefore, most efforts have been paid to the search for batteries of 1-dimensional measuring functions to be used separately to produce 1-dimensional size functions, whose information would be merged a posteriori (cf., e.g., [16, 33, 36, 41, 42]). Although a statistical method is studied in [9], no general strategy is available to carry out the blending of 1-dimensional size functions in order to produce a single shape descriptor. On the contrary, the possibility of working from the beginning with k-dimensional measuring functions allows us to produce just one shape descriptor containing the information of k different measuring functions at the same time.

A natural question arises, whether the results obtained using k-dimensional measuring functions were not obtainable from k 1-dimensional measuring functions. Actually, a result of this paper is not only the introduction of a procedure to carry out computations in k-dimensional Size Theory, but also furnishing the proof, illustrated by examples, that k-dimensional size functions do not simply contain the sum of the information contained in 1-dimensional size functions.

3.2 Definitions

For the present paper, $\mathcal{M}$, $\mathcal{N}$ denote two non-empty compact and locally connected Hausdorff spaces.

In Multidimensional Size Theory [31], any pair $(\mathcal{M}, \vec{\varphi})$, where $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k) : \mathcal{M} \to \mathbb{R}^k$ is a continuous function, is called a *size pair*. The function $\vec{\varphi}$ is called a *k-dimensional measuring function*. The following relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^k$: for $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$, we say $\vec{x} \preceq \vec{y}$ (resp. $\vec{x} \prec \vec{y}$) if and only if $x_i \le y_i$ (resp. $x_i < y_i$) for every index $i = 1, \ldots, k$. Moreover, $\mathbb{R}^k$ is endowed with the usual max-norm: $\|(x_1, x_2, \ldots, x_k)\|_\infty = \max_{1 \le i \le k} |x_i|$. In this framework, if $\mathcal{M}$ and $\mathcal{N}$ are homeomorphic, the size pairs $(\mathcal{M}, \vec{\varphi})$ and $(\mathcal{N}, \vec{\psi})$ can be compared by means of the *natural pseudo-distance d*, defined as

$$d((\mathcal{M}, \vec{\varphi}), (\mathcal{N}, \vec{\psi})) = \inf_f \max_{P \in \mathcal{M}} \|\vec{\varphi}(P) - \vec{\psi}(f(P))\|_\infty ,$$

where f varies among all the homeomorphisms between $\mathcal{M}$ and $\mathcal{N}$. The term pseudo-distance means that d can vanish even if the size pairs do not coincide.

Now we introduce the k-dimensional analogue of size function for a size pair $(\mathcal{M}, \vec{\varphi})$. We shall use the following notations: Δ^+ will be the open set $\{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$, while $\Delta = \partial \Delta^+$. Here, and in what follows, $\mathbb{R}^k \times \mathbb{R}^k$ and $\mathbb{R}^{2k}$ are identified.

For every k-tuple $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, let $\mathcal{M}\langle \vec{\varphi} \preceq \vec{x} \rangle$ be the set $\{P \in \mathcal{M} : \varphi_i(P) \le x_i,\ i = 1, \ldots, k\}$.

Definition 5 For every k-tuple $\vec{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, we shall say that two points $P, Q \in \mathcal{M}$ are $\langle \vec{\varphi} \preceq \vec{y} \rangle$*-connected* if and only if a connected subset of $\mathcal{M}\langle \vec{\varphi} \preceq \vec{y} \rangle$ exists, containing P and Q.

Definition 6 We shall call *(k-dimensional) size function* associated with the size pair $(\mathcal{M}, \vec{\varphi})$ the function $\ell_{(\mathcal{M}, \vec{\varphi})} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(\mathcal{M}, \vec{\varphi})}(\vec{x}, \vec{y})$ equal to the number of equivalence classes in which the set $\mathcal{M}\langle \vec{\varphi} \preceq \vec{x} \rangle$ is divided by the $\langle \vec{\varphi} \preceq \vec{y} \rangle$-connectedness relation.

Remark 1 In other words, $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})$ counts the connected components in $\mathcal{M}\langle\vec{\varphi}\preceq\vec{y}\rangle$ containing at least one point of $\mathcal{M}\langle\vec{\varphi}\preceq\vec{x}\rangle$.

The direct representation of multidimensional size functions from their definition involves working in domains of $\mathbb{R}^{2k}$, so that it becomes rapidly inaccessible as k increases. Moreover, a direct generalization of cornerpoints from the 1-dimensional to the multidimensional case is not straightforward.

These obstacles in treating the multidimensional case appear also in the related theory of Multidimensional Persistent Homology (see [6]). Indeed, the authors conclude that paper claiming that multidimensional persistence has an essentially different character from its 1-dimensional version, since their approach does not seem to lead to a concise, complete and stable descriptor in the multidimensional case, whereas it does in classical Persistent Homology (see [10]). Probably for this reason, in Persistent Homology a different strategy has been developed to study the persistent homology groups associated with two 1-dimensional measuring functions at the same time (see [12]). It is based on the idea of interpolating the two measuring functions and considering parametric families of persistent homology groups, producing the so-called *vineyards*. We shall explore the relations between multidimensional size functions and vineyards in Sect. 7.

A combinatorial representation of size functions by formal series of cornerpoints is central in order to define a good distance between size functions and obtain lower bounds for the natural pseudo-distance, as we have seen in the 1-dimensional case. In the following sections we shall show that this can be achieved by foliating the domain Δ^+ of $\mathbb{R}^{2k}$ into half-planes so that the restriction of multidimensional size functions to each leaf of this foliation actually is a 1-dimensional size function. Therefore, the theory developed for $k=1$ applies to each leaf of the foliation, yielding a combinatorial treatment also in the multidimensional case.

4 Reduction to the 1-dimensional Case

In this section, we will show that there exists a parameterized family of half-planes in $\mathbb{R}^k\times\mathbb{R}^k$ such that the restriction of $\ell_{(\mathcal{M},\vec{\varphi})}$ to each of these planes can be seen as a particular 1-dimensional size function.

Definition 7 For every unit vector $\vec{l}=(l_1,\dots,l_k)$ of $\mathbb{R}^k$ such that $l_i>0$ for $i=1,\dots,k$, and for every vector $\vec{b}=(b_1,\dots,b_k)$ of $\mathbb{R}^k$ such that $\sum_{i=1}^k b_i=0$, we shall say that the pair $(\vec{l},\vec{b})$ is *admissible*. We shall denote the set of all admissible pairs in $\mathbb{R}^k\times\mathbb{R}^k$ by Adm_k. Given an admissible pair $(\vec{l},\vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ of $\mathbb{R}^k\times\mathbb{R}^k$ by the following parametric equations:

$$\begin{cases}\vec{x}=s\vec{l}+\vec{b},\\ \vec{y}=t\vec{l}+\vec{b}\end{cases}$$

for $s,t\in\mathbb{R}$, with $s<t$.

Proposition 1 *For every $(\vec{x},\vec{y})\in\Delta^+$ there exists one and only one admissible pair $(\vec{l},\vec{b})$ such that $(\vec{x},\vec{y})\in\pi_{(\vec{l},\vec{b})}$.*

Proof The claim immediately follows by taking, for $i=1,\dots,k$,

$$l_i=\frac{y_i-x_i}{\sqrt{\sum_{j=1}^k(y_j-x_j)^2}},$$

$$b_i=\frac{x_i\sum_{j=1}^k y_j-y_i\sum_{j=1}^k x_j}{\sum_{j=1}^k(y_j-x_j)}.$$

Then, $\vec{x}=s\vec{l}+\vec{b}$, $\vec{y}=t\vec{l}+\vec{b}$, with

$$s=\frac{\sum_{j=1}^k x_j}{\sum_{j=1}^k l_j}=\frac{\sum_{j=1}^k x_j\sqrt{\sum_{j=1}^k(y_j-x_j)^2}}{\sum_{j=1}^k(y_j-x_j)},$$

$$t=\frac{\sum_{j=1}^k y_j}{\sum_{j=1}^k l_j}=\frac{\sum_{j=1}^k y_j\sqrt{\sum_{j=1}^k(y_j-x_j)^2}}{\sum_{j=1}^k(y_j-x_j)}.$$

□

Now we can prove the reduction to the 1-dimensional case.

Theorem 3 *Let $(\vec{l},\vec{b})$ be an admissible pair, and $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}:\mathcal{M}\to\mathbb{R}$ be defined by setting*

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P)=\max_{i=1,\dots,k}\left\{\frac{\varphi_i(P)-b_i}{l_i}\right\}.$$

Then, for every $(\vec{x},\vec{y})=(s\vec{l}+\vec{b},t\vec{l}+\vec{b})\in\pi_{(\vec{l},\vec{b})}$ the following equality holds:

$$\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})=\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s,t).$$

Proof For every $\vec{x}=(x_1,\dots,x_k)\in\mathbb{R}^k$, with $x_i=sl_i+b_i$, $i=1,\dots,k$, it holds that $\mathcal{M}\langle\vec{\varphi}\preceq\vec{x}\rangle=\mathcal{M}\langle F^{\vec{\varphi}}_{(\vec{l},\vec{b})}\le s\rangle$. This is true because

$$\begin{aligned}\mathcal{M}\langle\vec{\varphi}\preceq\vec{x}\rangle&=\{P\in\mathcal{M}:\varphi_i(P)\le x_i,\ i=1,\dots,k\}\\&=\{P\in\mathcal{M}:\varphi_i(P)\le sl_i+b_i,\ i=1,\dots,k\}\\&=\left\{P\in\mathcal{M}:\frac{\varphi_i(P)-b_i}{l_i}\le s,\ i=1,\dots,k\right\}\\&=\mathcal{M}\langle F^{\vec{\varphi}}_{(\vec{l},\vec{b})}\le s\rangle.\end{aligned}$$

Analogously, for every $\vec{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, with $y_i = tl_i + b_i$, $i = 1, \ldots, k$, it holds that $\mathcal{M}\langle \vec{\varphi} \preceq \vec{y} \rangle = \mathcal{M}\langle F^{\vec{\varphi}}_{(\vec{l},\vec{b})} \leq t \rangle$. Therefore Remark 1 implies the claim. □

In the following, we shall use the symbol $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ in the sense of Theorem 3.

Remark 2 By applying the parameterization described in Definition 7, we can change the problem of computing and describing k-dimensional size functions into the one of computing and describing families of 1-dimensional size functions. We point out that the same technique can be straightforwardly applied to the ranks of size homotopy groups and to the ranks of multidimensional persistent homology groups, since our approach allows us to exploit the theory developed in dimension 1 essentially without any change.

Remark 3 Theorem 3 allows us to represent each multidimensional size function as a parameterized family of formal series of points and lines, on the basis of the description introduced in [21, 30, 35] for the 1-dimensional case and recalled in Sect. 2. Indeed, we can associate a formal series $\sigma_{(\vec{l},\vec{b})}$ with each admissible pair $(\vec{l}, \vec{b})$, with $\sigma_{(\vec{l},\vec{b})}$ describing the 1-dimensional size function $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$. The family $\{\sigma_{(\vec{l},\vec{b})} : (\vec{l}, \vec{b}) \in Adm_k\}$ is a new complete descriptor for $\ell_{(\mathcal{M},\vec{\varphi})}$, in the sense that two multidimensional size functions coincide if and only if the corresponding parameterized families of formal series coincide.

5 Lower Bounds for the k-dimensional Natural Pseudo-Distance

As recalled in Sect. 2, 1-dimensional size functions can be compared by means of a distance, called *matching distance*. This distance is based on the observation that each 1-dimensional size function can be represented by a formal series of cornerpoints. The matching distance is computed by finding an optimal matching between the multisets of cornerpoints describing two size functions. Key properties of the matching distance are its stability under perturbations of the measuring functions and the possibility of computing lower bounds for the natural pseudo-distance. In this section we show that these results are still valid also in the multidimensional case. Moreover, we prove that the multidimensional matching distance provides a better lower bound for the natural pseudo-distance than the 1-dimensional matching distance. We conclude the section with an example showing that the information contained in size functions corresponding to measuring functions $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k)$ is not simply the sum of that contained in the 1-dimensional size functions corresponding to $\varphi_1, \ldots, \varphi_k$, considered independently. In Sect. 6 we shall check this point on a dataset of shapes.

5.1 Main Results

In the sequel, we shall denote by $d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})$ the matching distance between the 1-dimensional size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ and $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$.

As an easy corollary of Theorem 3 and Remark 3 we have that the set of 1-dimensional size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$, as $(\vec{l}, \vec{b})$ varies in Adm_k, completely characterizes $\ell_{(\mathcal{M},\vec{\varphi})}$.

Corollary 1 *Let us consider the size pairs $(\mathcal{M}, \vec{\varphi})$, $(\mathcal{N}, \vec{\psi})$. Then, the identity $\ell_{(\mathcal{M},\vec{\varphi})} \equiv \ell_{(\mathcal{N},\vec{\psi})}$ holds if and only if $d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}) = 0$, for every admissible pair $(\vec{l}, \vec{b})$.*

The next result proves the stability of d_{match} with respect to the chosen measuring function, i.e. that small enough changes in $\vec{\varphi}$ with respect to the max-norm induce small changes of $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ with respect to the matching distance.

Proposition 2 *If $(\mathcal{M}, \vec{\varphi})$, $(\mathcal{M}, \vec{\chi})$ are size pairs and $\max_{P\in\mathcal{M}} \|\vec{\varphi}(P) - \vec{\chi}(P)\|_\infty \leq \epsilon$, then for each admissible pair $(\vec{l}, \vec{b})$, it holds that*

$$d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{M},F^{\vec{\chi}}_{(\vec{l},\vec{b})})}) \leq \frac{\epsilon}{\min_{i=1,\ldots,k} l_i}.$$

Proof From Theorem 2, taking $\eta = \max_{P\in\mathcal{M}} |F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\chi}}_{(\vec{l},\vec{b})}(P)|$, we obtain that

$$d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{M},F^{\vec{\chi}}_{(\vec{l},\vec{b})})}) \leq \max_{P\in\mathcal{M}} |F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\chi}}_{(\vec{l},\vec{b})}(P)|.$$

Let us now fix $P \in \mathcal{M}$. Then, denoting by $\hat{\imath}$ the index for which $\max_i \frac{\varphi_i(P)-b_i}{l_i}$ is attained, by the definition of $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ and $F^{\vec{\chi}}_{(\vec{l},\vec{b})}$ we have that

$$\begin{aligned} F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) - F^{\vec{\chi}}_{(\vec{l},\vec{b})}(P) &= \max_i \frac{\varphi_i(P) - b_i}{l_i} - \max_i \frac{\chi_i(P) - b_i}{l_i} \\ &= \frac{\varphi_{\hat{\imath}}(P) - b_{\hat{\imath}}}{l_{\hat{\imath}}} - \max_i \frac{\chi_i(P) - b_i}{l_i} \end{aligned}$$

$$\le \frac{\varphi_{\hat\imath}(P)-b_{\hat\imath}}{l_{\hat\imath}} - \frac{\chi_{\hat\imath}(P)-b_{\hat\imath}}{l_{\hat\imath}}$$
$$= \frac{\varphi_{\hat\imath}(P)-\chi_{\hat\imath}(P)}{l_{\hat\imath}} \le \frac{\|\vec\varphi(P)-\vec\chi(P)\|_\infty}{\min_{i=1,\dots,k} l_i}.$$

In the same way, we obtain $F^{\vec\chi}_{(\vec l,\vec b)}(P) - F^{\vec\varphi}_{(\vec l,\vec b)}(P) \le \frac{\|\vec\varphi(P)-\vec\chi(P)\|_\infty}{\min_{i=1,\dots,k} l_i}$. Therefore, if $\max_{P\in\mathcal M}\|\vec\varphi(P)-\vec\chi(P)\|_\infty \le \epsilon$,

$$\max_{P\in\mathcal M} |F^{\vec\varphi}_{(\vec l,\vec b)}(P) - F^{\vec\chi}_{(\vec l,\vec b)}(P)| \le \max_{P\in\mathcal M} \frac{\|\vec\varphi(P)-\vec\chi(P)\|_\infty}{\min_{i=1,\dots,k} l_i} \le \frac{\epsilon}{\min_{i=1,\dots,k} l_i}. \qquad \square$$

Analogously, it is possible to prove that d_{match} is stable with respect to the choice of the half-planes in the foliation. Indeed, the next proposition states that small enough changes in $(\vec l,\vec b)$ with respect to the max-norm induce small changes of $\ell_{(\mathcal M, F^{\vec\varphi}_{(\vec l,\vec b)})}$ with respect to the matching distance.

Proposition 3 *If $(\mathcal M,\vec\varphi)$ is a size pair, $(\vec l,\vec b)\in Adm_k$ and ϵ is a real number with $0\le\epsilon<\min_{i=1,\dots,k} l_i$, then for every admissible pair $(\vec l',\vec b')$ with $\|(\vec l,\vec b)-(\vec l',\vec b')\|_\infty\le\epsilon$, it holds that*

$$d_{match}(\ell_{(\mathcal M, F^{\vec\varphi}_{(\vec l,\vec b)})}, \ell_{(\mathcal M, F^{\vec\varphi}_{(\vec l',\vec b')})}) \le \epsilon\cdot\frac{\max_{P\in\mathcal M}\|\vec\varphi(P)\|_\infty + \|\vec l\|_\infty + \|\vec b\|_\infty}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}.$$

Proof From Theorem 2, taking $\eta = \max_{P\in\mathcal M}|F^{\vec\varphi}_{(\vec l,\vec b)}(P) - F^{\vec\varphi}_{(\vec l',\vec b')}(P)|$, we obtain that

$$d_{match}(\ell_{(\mathcal M, F^{\vec\varphi}_{(\vec l,\vec b)})}, \ell_{(\mathcal M, F^{\vec\varphi}_{(\vec l',\vec b')})}) \le \max_{P\in\mathcal M}|F^{\vec\varphi}_{(\vec l,\vec b)}(P) - F^{\vec\varphi}_{(\vec l',\vec b')}(P)|.$$

Let us now fix $P\in\mathcal M$. Then, denoting by $\hat\imath$ the index for which $\max_i \frac{\varphi_i(P)-b_i}{l_i}$ is attained, by the definition of $F^{\vec\varphi}_{(\vec l,\vec b)}$ and $F^{\vec\varphi}_{(\vec l',\vec b')}$ we have that

$$F^{\vec\varphi}_{(\vec l,\vec b)}(P) - F^{\vec\varphi}_{(\vec l',\vec b')}(P)$$
$$= \max_i \frac{\varphi_i(P)-b_i}{l_i} - \max_i \frac{\varphi_i(P)-b'_i}{l'_i}$$
$$= \frac{\varphi_{\hat\imath}(P)-b_{\hat\imath}}{l_{\hat\imath}} - \max_i \frac{\varphi_i(P)-b'_i}{l'_i}$$
$$\le \frac{\varphi_{\hat\imath}(P)-b_{\hat\imath}}{l_{\hat\imath}} - \frac{\varphi_{\hat\imath}(P)-b'_{\hat\imath}}{l'_{\hat\imath}}$$
$$= \frac{(l'_{\hat\imath}-l_{\hat\imath})\varphi_{\hat\imath}(P) - l'_{\hat\imath}b_{\hat\imath} + l_{\hat\imath}b'_{\hat\imath}}{l_{\hat\imath}l'_{\hat\imath}}$$
$$= \frac{(l'_{\hat\imath}-l_{\hat\imath})\varphi_{\hat\imath}(P) + l_{\hat\imath}(b'_{\hat\imath}-b_{\hat\imath}) + b_{\hat\imath}(l_{\hat\imath}-l'_{\hat\imath})}{l_{\hat\imath}l'_{\hat\imath}}$$
$$\le \frac{|l'_{\hat\imath}-l_{\hat\imath}||\varphi_{\hat\imath}(P)| + l_{\hat\imath}|b'_{\hat\imath}-b_{\hat\imath}| + |b_{\hat\imath}||l_{\hat\imath}-l'_{\hat\imath}|}{l_{\hat\imath}l'_{\hat\imath}}$$
$$\le \frac{\epsilon(\|\vec\varphi(P)\|_\infty + \|\vec l\|_\infty + \|\vec b\|_\infty)}{l_{\hat\imath}(l_{\hat\imath}-\epsilon)}$$
$$\le \frac{\epsilon(\|\vec\varphi(P)\|_\infty + \|\vec l\|_\infty + \|\vec b\|_\infty)}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}.$$

Analogously, we can prove that $F^{\vec\varphi}_{(\vec l',\vec b')}(P) - F^{\vec\varphi}_{(\vec l,\vec b)}(P) \le \frac{\epsilon(\|\vec\varphi(P)\|_\infty + \|\vec l\|_\infty + \|\vec b\|_\infty)}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}$. Therefore,

$$\max_{P\in\mathcal M}|F^{\vec\varphi}_{(\vec l,\vec b)}(P) - F^{\vec\varphi}_{(\vec l',\vec b')}(P)| \le \epsilon\cdot\frac{\max_{P\in\mathcal M}\|\vec\varphi(P)\|_\infty + \|\vec l\|_\infty + \|\vec b\|_\infty}{\min_{i=1,\dots,k}\{l_i(l_i-\epsilon)\}}.$$

$\square$

We underline that Proposition 2 and Proposition 3 prove the stability of our computational approach.

Now we are able to present our next result, showing that a lower bound exists for the multidimensional natural pseudo-distance.

Theorem 4 *Let $(\mathcal M,\vec\varphi)$ and $(\mathcal N,\vec\psi)$ be two size pairs, with $\mathcal M$, $\mathcal N$ homeomorphic. Setting $d((\mathcal M,\vec\varphi),(\mathcal N,\vec\psi)) = \inf_f \max_{P\in\mathcal M}\|\vec\varphi(P)-\vec\psi(f(P))\|_\infty$, where f varies among all the homeomorphisms between $\mathcal M$ and $\mathcal N$, it holds that*

$$\sup_{(\vec l,\vec b)\in Adm_k}\min_{i=1,\dots,k} l_i\cdot d_{match}(\ell_{(\mathcal M, F^{\vec\varphi}_{(\vec l,\vec b)})}, \ell_{(\mathcal N, F^{\vec\psi}_{(\vec l,\vec b)})}) \le d((\mathcal M,\vec\varphi),(\mathcal N,\vec\psi)).$$

Proof For any homeomorphism f between $\mathcal M$ and $\mathcal N$, it holds that $\ell_{(\mathcal N, F^{\vec\psi}_{(\vec l,\vec b)})} \equiv \ell_{(\mathcal M, F^{\vec\psi}_{(\vec l,\vec b)}\circ f)}$. Moreover, by applying Proposition 2 with $\epsilon = \max_{P\in\mathcal M}\|\vec\varphi(P)-\vec\psi(f(P))\|_\infty$ and $\vec\chi = \vec\psi\circ f$, and observing that $F^{\vec\psi}_{(\vec l,\vec b)}\circ f \equiv F^{\vec\psi\circ f}_{(\vec l,\vec b)}$, we have

$$\min_{i=1,\dots,k} l_i\cdot d_{match}(\ell_{(\mathcal M, F^{\vec\varphi}_{(\vec l,\vec b)})}, \ell_{(\mathcal N, F^{\vec\psi}_{(\vec l,\vec b)})}) \le \max_{P\in\mathcal M}\|\vec\varphi(P)-\vec\psi(f(P))\|_\infty$$

for every admissible $(\vec l,\vec b)$. Since this is true for each homeomorphism f between $\mathcal M$ and $\mathcal N$, the claim immediately follows. $\square$

Remark 4 We observe that the left side of the inequality in Theorem 4 defines a distance between multidimensional size functions associated with homeomorphic spaces. When the spaces are not assumed to be homeomorphic, it still verifies all the properties of a distance, except for the fact that it may take the value $+\infty$. In other words, it defines an extended distance.

Definition 8 Let $(\mathcal{M},\vec{\varphi})$ and $(\mathcal{N},\vec{\psi})$ be two size pairs. We shall call *multidimensional matching distance* the extended distance defined by setting

$$D_{match}(\ell_{(\mathcal{M},\vec{\varphi})},\ell_{(\mathcal{N},\vec{\psi})}) = \sup_{(\vec{l},\vec{b})\in Adm_k}\min_{i=1,\dots,k} l_i\cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}).$$

Theorem 4 states that $D_{match}(\ell_{(\mathcal{M},\vec{\varphi})},\ell_{(\mathcal{N},\vec{\psi})})$ is a lower bound for the natural pseudo-distance $d((\mathcal{M},\vec{\varphi}),(\mathcal{N},\vec{\psi}))$.

Remark 5 If we choose a non-empty subset $A\subseteq Adm_k$ and we substitute $\sup_{(\vec{l},\vec{b})\in Adm_k}$ with $\sup_{(\vec{l},\vec{b})\in A}$ in Definition 8, we obtain an (extended) pseudo-distance between k-dimensional size functions. If A is finite, this pseudo-distance appears to be particularly suitable for applications, from a computational point of view. The key point here is that Proposition 3 ensures the stability with respect to the choice of the considered half-planes of the foliation. Moreover, this pseudo-distance furnishes a lower bound for the natural pseudo-distance, since it results to be less than or equal to the multidimensional matching distance $D_{match}(\ell_{(\mathcal{M},\vec{\varphi})},\ell_{(\mathcal{N},\vec{\psi})})$.

Remark 6 Another interesting choice for applications could be the weighted mean pseudo-distance computed on the finite subset $A=\{(\vec{l^j},\vec{b^j}) : j=1,\dots,h\}\subseteq Adm_k$, and defined as $\sum_{j=1}^h w^j\cdot\min_{i=1,\dots,k} l_i^j\cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l^j},\vec{b^j})})},\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l^j},\vec{b^j})})})$ (assuming that w^j are real numbers with $w^j>0$, for every $j=1,\dots,h$, and $\sum_{j=1}^h w^j=1$): Indeed, it takes into account the information conveyed from each leaf $(\vec{l^j},\vec{b^j})\in A$. This weighted mean pseudo-distance between size functions gives a lower bound for the natural pseudo-distance that is at most as good as that of Remark 5.

The following result proves that the lower bound for the natural pseudo-distance provided by $D_{match}(\ell_{(\mathcal{M},\vec{\varphi})},\ell_{(\mathcal{N},\vec{\psi})})$ is better than the ones obtained from the 1-dimensional matching distances $d_{match}(\ell_{(\mathcal{M},\varphi_i)},\ell_{(\mathcal{N},\psi_i)})$, for $i=1,\dots,k$.

Proposition 4 *Let $(\mathcal{M},\vec{\varphi})$ and $(\mathcal{N},\vec{\psi})$ be two size pairs, with $\vec{\varphi}=(\varphi_1,\dots,\varphi_k)$, $\vec{\psi}=(\psi_1,\dots,\psi_k)$. Then, for $j=1,\dots,k$, it holds that*

$$d_{match}(\ell_{(\mathcal{M},\varphi_j)},\ell_{(\mathcal{N},\psi_j)})\le D_{match}(\ell_{(\mathcal{M},\vec{\varphi})},\ell_{(\mathcal{N},\vec{\psi})}).$$

Proof Let $\mu=\max_{P\in\mathcal{M}}\|\vec{\varphi}(P)\|_\infty$ and $\nu=\max_{Q\in\mathcal{N}}\|\vec{\psi}(Q)\|_\infty$. For $j=1,\dots,k$, consider the admissible pair $(\vec{l^j},\vec{b^j})$, where $\vec{l^j}=(l_1^j,\dots,l_k^j)$ and $\vec{b^j}=(b_1^j,\dots,b_k^j)$ are defined by the following relations:

$$l_i^j=\frac{1}{\sqrt{k}},\quad \text{for } i=1,\dots,k,$$

$$b_i^j=\begin{cases}-\frac{2(k-1)}{k}\cdot\max\{\mu,\nu\}, & \text{if } i=j;\\ \frac{2}{k}\cdot\max\{\mu,\nu\}, & \text{if } i\neq j.\end{cases}$$

From Theorem 3, for every $(\vec{x},\vec{y})=(s\vec{l^j}+\vec{b^j},t\vec{l^j}+\vec{b^j})\in\pi_{(\vec{l^j},\vec{b^j})}$ it follows that $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})=\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l^j},\vec{b^j})})}(s,t)$, $\ell_{(\mathcal{N},\vec{\psi})}(\vec{x},\vec{y})=\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l^j},\vec{b^j})})}(s,t)$ with $F^{\vec{\varphi}}_{(\vec{l^j},\vec{b^j})}(P)=\max_{i=1,\dots,k}\{\frac{\varphi_i(P)-b_i^j}{l_i^j}\}=\sqrt{k}(\varphi_j(P)-b_j^j)$ and $F^{\vec{\psi}}_{(\vec{l^j},\vec{b^j})}(Q)=\max_{i=1,\dots,k}\{\frac{\psi_i(Q)-b_i^j}{l_i^j}\}=\sqrt{k}(\psi_j(Q)-b_j^j)$, for every $P\in\mathcal{M}$ and $Q\in\mathcal{N}$. By the definition of d_{match} we have

$$\begin{aligned}&d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l^j},\vec{b^j})})},\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l^j},\vec{b^j})})})\\&=d_{match}(\ell_{(\mathcal{M},\sqrt{k}(\varphi_j-b_j^j))},\ell_{(\mathcal{N},\sqrt{k}(\psi_j-b_j^j))})\\&=\sqrt{k}\cdot d_{match}(\ell_{(\mathcal{M},\varphi_j-b_j^j)},\ell_{(\mathcal{N},\psi_j-b_j^j)})\\&=\sqrt{k}\cdot d_{match}(\ell_{(\mathcal{M},\varphi_j)},\ell_{(\mathcal{N},\psi_j)}),\end{aligned}$$

and hence $\min_{i=1,\dots,k} l_i^j\cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l^j},\vec{b^j})})},\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l^j},\vec{b^j})})})=\frac{1}{\sqrt{k}}\cdot d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l^j},\vec{b^j})})},\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l^j},\vec{b^j})})})=d_{match}(\ell_{(\mathcal{M},\varphi_j)},\ell_{(\mathcal{N},\psi_j)})$. From the definition of $D_{match}(\ell_{(\mathcal{M},\vec{\varphi})},\ell_{(\mathcal{N},\vec{\psi})})$, the claim immediately follows. □

5.2 An Example

As a consequence of Proposition 4, the multidimensional size function associated with $\vec{\varphi}$ contains at least as much information about the studied shape as the whole set of the 1-dimensional size functions associated with $\varphi_1,\dots,\varphi_k$. Actually, the size function of $\vec{\varphi}$ can be strictly more informative than the set of the size functions of $\varphi_1,\dots,\varphi_k$. This fact will be evident from the experiments illustrated in Sect. 6, but can also be easily checked in the following example.

In $\mathbb{R}^3$ consider the set $\mathcal{Q}=[-1,1]\times[-1,1]\times[-1,1]$ and the sphere $\mathcal{S}$ of equation $x^2+y^2+z^2=1$. Let also $\vec{\Phi}=$

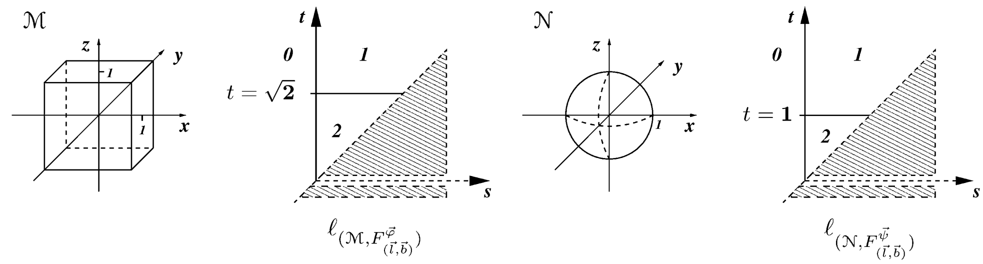

Fig. 3 The topological spaces $\mathcal{M}$ and $\mathcal{N}$ and the size functions $\ell_{(\mathcal{M},F^{\vec\varphi}_{(\vec l,\vec b)})}$, $\ell_{(\mathcal{N},F^{\vec\psi}_{(\vec l,\vec b)})}$ associated with the half-plane $\pi_{(\vec l,\vec b)}$, for $\vec l=(\frac{\sqrt2}{2},\frac{\sqrt2}{2})$ and $\vec b=(0,0)$

$(\Phi_1,\Phi_2):\mathbb{R}^3\to\mathbb{R}^2$ be the continuous function, defined as $\vec\Phi(x,y,z)=(|x|,|z|)$. In this setting, consider the size pairs $(\mathcal{M},\vec\varphi)$ and $(\mathcal{N},\vec\psi)$ where $\mathcal{M}=\partial\mathcal{Q}$, $\mathcal{N}=\mathcal{S}$, and $\vec\varphi$ and $\vec\psi$ are respectively the restrictions of $\vec\Phi$ to $\mathcal{M}$ and $\mathcal{N}$. In order to compare the size functions $\ell_{(\mathcal{M},\vec\varphi)}$ and $\ell_{(\mathcal{N},\vec\psi)}$, we are interested in studying the foliation in half-planes $\pi_{(\vec l,\vec b)}$, where $\vec l=(\cos\theta,\sin\theta)$ with $\theta\in(0,\frac{\pi}{2})$, and $\vec b=(a,-a)$ with $a\in\mathbb{R}$. Any such half-plane is represented by

$$\begin{cases} x_1=s\cos\theta+a,\\ x_2=s\sin\theta-a,\\ y_1=t\cos\theta+a,\\ y_2=t\sin\theta-a\end{cases}$$

with $s,t\in\mathbb{R}$, $s<t$. Figure 3 shows the size functions $\ell_{(\mathcal{M},F^{\vec\varphi}_{(\vec l,\vec b)})}$ and $\ell_{(\mathcal{N},F^{\vec\psi}_{(\vec l,\vec b)})}$, for $\theta=\frac{\pi}{4}$ and $a=0$, i.e. $\vec l=(\frac{\sqrt2}{2},\frac{\sqrt2}{2})$ and $\vec b=(0,0)$. With this choice, we obtain that $F^{\vec\varphi}_{(\vec l,\vec b)}=\sqrt2\max\{\varphi_1,\varphi_2\}=\sqrt2\max\{|x|,|z|\}$ and $F^{\vec\psi}_{(\vec l,\vec b)}=\sqrt2\max\{\psi_1,\psi_2\}=\sqrt2\max\{|x|,|z|\}$. Therefore, Theorem 3 implies that, for every $(x_1,x_2,y_1,y_2)\in\pi_{(\vec l,\vec b)}$

$$\begin{aligned}\ell_{(\mathcal{M},\vec\varphi)}(x_1,x_2,y_1,y_2)&=\ell_{(\mathcal{M},\vec\varphi)}\left(\frac{s}{\sqrt2},\frac{s}{\sqrt2},\frac{t}{\sqrt2},\frac{t}{\sqrt2}\right)\\&=\ell_{(\mathcal{M},F^{\vec\varphi}_{(\vec l,\vec b)})}(s,t),\\ \ell_{(\mathcal{N},\vec\psi)}(x_1,x_2,y_1,y_2)&=\ell_{(\mathcal{N},\vec\psi)}\left(\frac{s}{\sqrt2},\frac{s}{\sqrt2},\frac{t}{\sqrt2},\frac{t}{\sqrt2}\right)\\&=\ell_{(\mathcal{N},F^{\vec\psi}_{(\vec l,\vec b)})}(s,t).\end{aligned}$$

In this case, by Theorem 4 and Remark 5 (applied for A containing just the admissible pair that we have chosen), a lower bound for the natural pseudo-distance $d((\mathcal{M},\vec\varphi),(\mathcal{N},\vec\psi))$ is given by

$$\begin{aligned}\frac{\sqrt2}{2}d_{match}(\ell_{(\mathcal{M},F^{\vec\varphi}_{(\vec l,\vec b)})},\ell_{(\mathcal{N},F^{\vec\psi}_{(\vec l,\vec b)})})&=\frac{\sqrt2}{2}(\sqrt2-1)\\&=1-\frac{\sqrt2}{2}.\end{aligned}$$

Indeed, the matching distance $d_{match}(\ell_{(\mathcal{M},F^{\vec\varphi}_{(\vec l,\vec b)})},\ell_{(\mathcal{N},F^{\vec\psi}_{(\vec l,\vec b)})})$ is equal to the cost of moving the point of coordinates $(0,\sqrt2)$ onto the point of coordinates $(0,1)$, computed with respect to the max-norm. The points $(0,\sqrt2)$ and $(0,1)$ are cornerpoints for the size functions $\ell_{(\mathcal{M},F^{\vec\varphi}_{(\vec l,\vec b)})}$ and $\ell_{(\mathcal{N},F^{\vec\psi}_{(\vec l,\vec b)})}$, respectively.

We conclude by observing that $\ell_{(\mathcal{M},\varphi_1)}\equiv\ell_{(\mathcal{N},\psi_1)}$ and $\ell_{(\mathcal{M},\varphi_2)}\equiv\ell_{(\mathcal{N},\psi_2)}$. In other words, the multidimensional size functions, with respect to $\vec\varphi$, $\vec\psi$, are able to discriminate the cube and the sphere, while both the 1-dimensional size functions, with respect to φ_1,φ_2 and ψ_1,ψ_2, cannot do that. This higher discriminatory power of multidimensional size functions motivates their definition and use.

6 Computational Issues and Experiments

We now discuss some issues arising when applying in a discrete setting the concepts defined in the previous sections for the continuum. First, a computational scheme coherent with the mathematical model is illustrated. Next, this scheme is applied to different datasets of discrete models.

6.1 Computational Aspects: PL and Digital Implementation

When dealing with Computer Graphics and Computer Vision applications, a relevant problem is to find a suitable discretization of both the space $\mathcal{M}$ and the function $\vec\varphi$. As classical discrete models we mention the simplicial complexes and the digital spaces. In this Section we highlight how our theory is able to go well in both model representations.

Given a model $|X|$ represented by the geometric realization of a simplicial complex X of arbitrary dimension n, and a piecewise linear map $\vec\varphi:|X|\to\mathbb{R}^k$, first defined on the vertices of X and then linearly extended over all the other simplices by using barycentric coordinates, we consider the size pair $(|X|,\vec\varphi)$. Triangulations and tetrahedralizations are examples of geometric realizations of simplicial complexes for $n=2$ and $n=3$, respectively.

Digital spaces are mainly relevant in image processing. A digital space is often represented by a graph structure based on the local adjacency relations of the digital points (e.g., pixels, voxels, etc.) [22]. In our experiments on digital spaces, the shape of interest is represented by the non-zero points of a binary 3D image, but our reasonings are immediately applicable to 2D and higher dimensions. Moreover, the procedure would be analogous if the shape were represented by all the points of the image (zero and non-zero). To obtain a size pair from this data, we take a geometric realization $|I|$ of a graph that encodes the non-zero elements of the image as nodes and the neighborhood adjacency among the digital points as edges. Depending on the number of neighbors that may be adjacent to a point, the connectivity (i.e. the number of edges) of the graph depends on the number of neighbors that are admitted to be adjacent to a point in a 2D image (i.e., 8-, 6- or 4-neighborhoods) or 3D image (e.g., 6-, 18- or 26-neighborhoods). Then, the size pair is the pair $(|I|, \vec{\varphi})$, where $\vec{\varphi} : |I| \to \mathbb{R}^k$ is a piecewise linear function first defined on the nodes of the graph and then linearly extended to the edges.

Once the size pair has been identified, the proposed reduction of k-dimensional size functions to the 1-dimensional case allows us to use the existing framework for computing 1-dimensional size functions [13], based on a discrete structure.

The algorithm in [13] takes as input a *size graph*, i.e. a pair (G, f) where G is a graph and $f : V(G) \to \mathbb{R}$ is a function labeling each vertex of G by a real number. For the simplicial case, our input is given by the 1-skeleton of the complex X, with the nodes labeled by the values of the restriction of $\vec{\varphi}$ on the vertices of the complex. For the case of digital spaces, the size graph corresponds to the graph used to encode the binary image. Similarly to the simplicial case, the node labels given in input correspond to the restriction of $\vec{\varphi}$ to the vertices of the graph.

The output of the algorithm is the multiset of cornerpoints and cornerlines that completely determines the *discrete size function* of the size graph. We recall the definition of discrete size function of a size graph ([32]): For every $a < b$, $\ell_{(G,f)}(a, b)$ is the number of components containing at least one vertex labeled by a value not greater than a, of the subgraph obtained by cutting away the vertices labeled by a value strictly greater than b (and the connecting edges, too). It is important to notice that, due to the piecewise linear nature of $\vec{\varphi}$, for both the simplicial and the digital model, the discrete size function of the size graph coincides with the size function of the original size pair.

The algorithm in [13] directly computes the multiset of cornerpoints and cornerlines that completely determines a 1-dimensional size function in $O(n \log n + m \cdot \alpha(2m + n, n))$ operations, where n and m are the number of vertices and edges in the size graph, respectively, and α is the inverse of the Ackermann function [1]. Following Remark 5, we evaluate the k-dimensional size function with measuring function $\vec{\varphi}$ over a subset $A \subseteq Adm_k$ of half-planes whose cardinality is h. Hence, we extract h 1-dimensional size functions, i.e., one for each half-plane of A. Therefore, the computational complexity for evaluating the k-dimensional size function over h half-planes is $O(h \cdot (n \log n + m \cdot \alpha(2m + n, n)))$.

In the same way, as stated in Definition 8, the computation of the k-dimensional matching distance easily follows from the computation of the 1-dimensional matching distance d_{match} together with the computation of the minimum of the k factors l_i, for $i = 1, \dots, k$, over the set A of half-planes. Since the computational complexity for computing the 1-dimensional matching distance in a single half-plane is $O(p^{2.5})$, with p the number of cornerpoints taken into account for the comparison, it follows that computing the k-dimensional matching distance between two k-dimensional size functions requires $O(h \cdot (p^{2.5} + k))$ operations.

6.2 Experimental Results

We present some experimental results on a set of 8 human models represented by triangle meshes and 6 different objects represented by voxelized digital models. Our goal is to check the stability of the proposed framework and to validate its potential for shape comparison.

To this aim, we have to define a multidimensional measuring function to describe the shapes, and to choose and discretize a foliation, for both the cases of meshes and digital objects.

Let us begin with triangle meshes. We define a 2-dimensional measuring function extending to triangle meshes and to the multidimensional case the reasonings in [36], where complete families of invariant 1-dimensional measuring functions are introduced. For a given triangle mesh of vertices $\{P_1, \dots, P_n\}$ we compute the barycenter $B = \frac{1}{n}\sum_{i=1}^{n} P_i$, and normalize the model so that it is contained in a unit sphere. We then define a vector

$$\vec{v} = \frac{\sum_{i=1}^{n}(P_i - B)||P_i - B||}{\sum_{i=1}^{n}||P_i - B||^2}.$$

A parametric family of real-valued functions can be defined by setting, for each point P_i and for each $\alpha \in \mathbb{R}$

$$\varphi_\alpha(P_i) = 1 - \frac{||P_i - (B + \alpha\vec{v})||}{\max_j ||P_j - (B + \alpha\vec{v})||}.$$

We can now set

$$\vec{\varphi}_{(\alpha_1,\alpha_2)}(P_i) = (\varphi_{\alpha_1}(P_i), \varphi_{\alpha_2}(P_i))$$

with $\alpha_1, \alpha_2 \in \mathbb{R}$, $i = 1, \dots, n$. The 2-dimensional measuring function we used in our experiments is $\vec{\varphi} = \vec{\varphi}_{(1,-1)}$.

It is worth noticing that the 2-dimensional measuring functions $\vec{\varphi}_{(\alpha_1,\alpha_2)}$ are invariant with respect to translation and rotation. Moreover, $\vec{\varphi}_{(\alpha_1,\alpha_2)}$ is well defined also if, e.g., for symmetry reasons, $\vec{v}$ is the null vector. The invariance with respect to scale comes from the a priori normalization of the model.

We write the foliation of $\mathbb{R}^2 \times \mathbb{R}^2$ as proposed in Sect. 5.2, that is to say we consider the foliation in half-planes $\pi_{(\vec{l},\vec{b})}$, where $\vec{l} = (\cos\theta, \sin\theta)$ with $\theta \in (0, \frac{\pi}{2})$, and $\vec{b} = (a, -a)$ with $a \in \mathbb{R}$.

In the following examples, we consider the set $A = \{(\vec{l}_i, \vec{b}), i = 1, \dots, 17\}$ of admissible pairs (cf. Remark 5), where $\vec{l}_i = (\cos\theta_i, \sin\theta_i)$ with $\theta_i = \frac{\pi}{36} i$, $i = 1, \dots, 17$, and $\vec{b} = (0, 0)$. For each half-plane π_i identified by $(\vec{l}_i, \vec{b}) \in A$, we compute the 1-dimensional measuring function $F^{\vec{\varphi}}_{(\vec{l}_i,\vec{b})}$ defined in Theorem 3.

For each model $\mathcal{K}$, Figs. 4 and 5 show, from left to right, the behavior of $\ell_{(\mathcal{K},\vec{\varphi})}(\vec{x}, \vec{y}) = \ell_{(\mathcal{K}, F^{\vec{\varphi}}_{(\vec{l}_i,\vec{b})})}(s, t)$ on the half-planes π_i corresponding to $\theta_i = \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$. Notice that, for space reasons, only a subset of A is selected for displaying, while the computations and results presented in the following refer to all the half-planes in the subset A.

Looking at the whole set of results in Figs. 4 and 5 suggests that multidimensional size functions have a high potential for comparing shapes. Indeed, it can be seen that the similarity between the models is reflected in the similarity between the corresponding 2-dimensional size functions. Moreover, the information is distributed all over the half-planes of the foliation, with a recognizable trend in similar models.

The behavior, in particular, of the 2-dimensional size functions computed on the first two models shown in Fig. 4 confirms the stability property suggested by Proposition 2. Indeed, the second model $\mathcal{K}_2$ is a simplified version of the first model $\mathcal{K}_1$, that is to say that $\mathcal{K}_2$ has 10% of the vertices of the original model $\mathcal{K}_1$. It can be seen that the 2-dimensional size functions of these two models are almost identical when restricted to the half-planes of the foliation: The perturbation of the pair $(\mathcal{K}_1, \vec{\varphi})$ produces small changes in the resulting size function, which are located near the diagonal. Indeed, cornerpoints that appear near the diagonal correspond to small details or noisy features, while cornerpoints that appear far away from the diagonal are representative of more significant shape features [29].

The computation on our set of models of the multidimensional matching distance in Definition 8 also furnishes interesting insights. In particular, the values reported in Table 1 confirm the higher discriminatory power of 2-dimensional size functions with respect to the 1-dimensional case. Each cell of the table provides two values. The first one is the value of the 2-dimensional matching distance between the corresponding models. The second one is obtained computing the 1-dimensional matching distance between the size functions associated to the 1-dimensional measuring functions φ_1 and φ_{-1}, and taking the maximum value. For all the models in the dataset, the 2-dimensional matching distance produces an (at least 35 times) better lower bound for the 2-dimensional natural pseudo-distance (cf. Proposition 4). In other words, this example confirms that comparing 2-dimensional size functions furnishes a better approximation of the 2-dimensional natural pseudo-distance, with respect to comparing the single 1-dimensional measuring functions corresponding to the components of $\vec{\varphi}$.

The experiments on digital spaces are performed over 3D binary images. In our settings, the nodes of the size graph correspond to the non-zero voxels and the edges are yielded by the 26- neighbor relation.

The models adopted in our experiments are taken from the McGill 3D Shape Benchmark.[1] To analyze these digital models we define a trivariate function $\vec{\varphi} = (\varphi_x, \varphi_y, \varphi_z)$, with $\varphi_x(v) = V_x - |v_x - B_x|$, $\varphi_y(v) = V_y - |v_y - B_y|$ and $\varphi_z(v) = V_z - |v_z - B_z|$; here $v = (v_x, v_y, v_z)$ denotes a voxel, $B = (B_x, B_y, B_z)$ is the barycenter of the model, and $V_x = \max_v |v_x - B_x|$ (analogous definition for V_y and V_z). In other words, we discriminate the models with respect to their spatial extent.

We choose the foliation of $\mathbb{R}^3 \times \mathbb{R}^3$ where a half-plane is represented by:

$$\begin{cases} x_1 = s\cos\theta\sin\phi + a, \\ x_2 = s\sin\theta\sin\phi + b, \\ x_3 = s\cos\phi - (a+b) \\ y_1 = t\cos\theta\sin\phi + a, \\ y_2 = t\sin\theta\sin\phi + b, \\ y_3 = t\cos\phi - (a+b) \end{cases}$$

with $s, t \in \mathbb{R}$, $s < t$, $a, b \in \mathbb{R}$, and $0 < \theta, \phi < \frac{\pi}{2}$. In our experiments, we consider the half-planes identified by $a = b = 0$ and the following pairs of angles (θ, ϕ): $(\frac{\pi}{12}, \frac{\pi}{4}), (\frac{\pi}{6}, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{\pi}{12}), (\frac{\pi}{4}, \frac{\pi}{6}), (\frac{\pi}{4}, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{\pi}{3}), (\frac{\pi}{12}, \frac{5\pi}{12}), (\frac{\pi}{3}, \frac{\pi}{4}), (\frac{5\pi}{12}, \frac{\pi}{4})$.

The results proposed in Table 2 describe the dimension of our digital models, the average time taken by our algorithm to extract the 1-dimensional size function on a half-plane of the foliation, the total time required to compute the size function on the 9 half-planes considered, and the average and the total number of cornerpoints of the size function on the 9 half-planes.

Finally, Table 3 highlights how also for 3D images the matching distance obtained over the half-planes of the foliations using more than one function improves the lower bound approximation of the natural pseudo-distance. We notice that the results contained in the fifth row of Table 3 provide the best approximation, over 9 half-planes selected in the foliation, of the k-dimensional matching distance.

[1] http://www.cim.mcgill.ca/~shape/benchMark/.

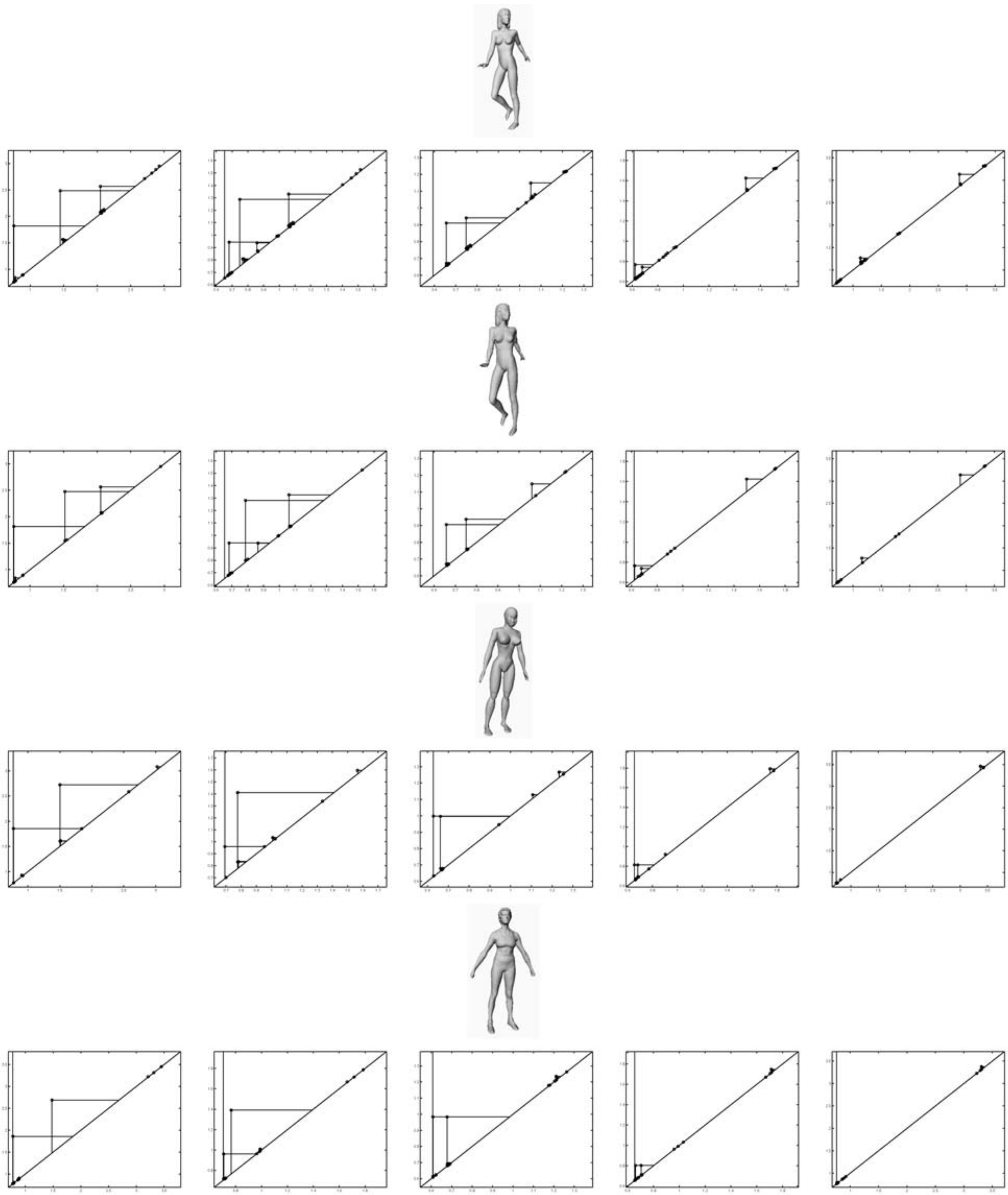

Fig. 4 2-dimensional size functions restricted to five leaves of the foliation, for the first four models

7 Links Between Dimension 0 Vineyards and Multidimensional Size Functions

In a recent paper [12], Cohen-Steiner et al. have introduced the concept of *vineyard*, that is a 1-parameter family of persistence diagrams associated with the homotopy f_t, interpolating between f_0 and f_1. These authors assume that the topological space is homeomorphic to the body of a simplicial complex, and that the measuring functions f_t are *tame*. We shall do the same in this section. We recall that dimension p persistence diagrams are a concise representation of

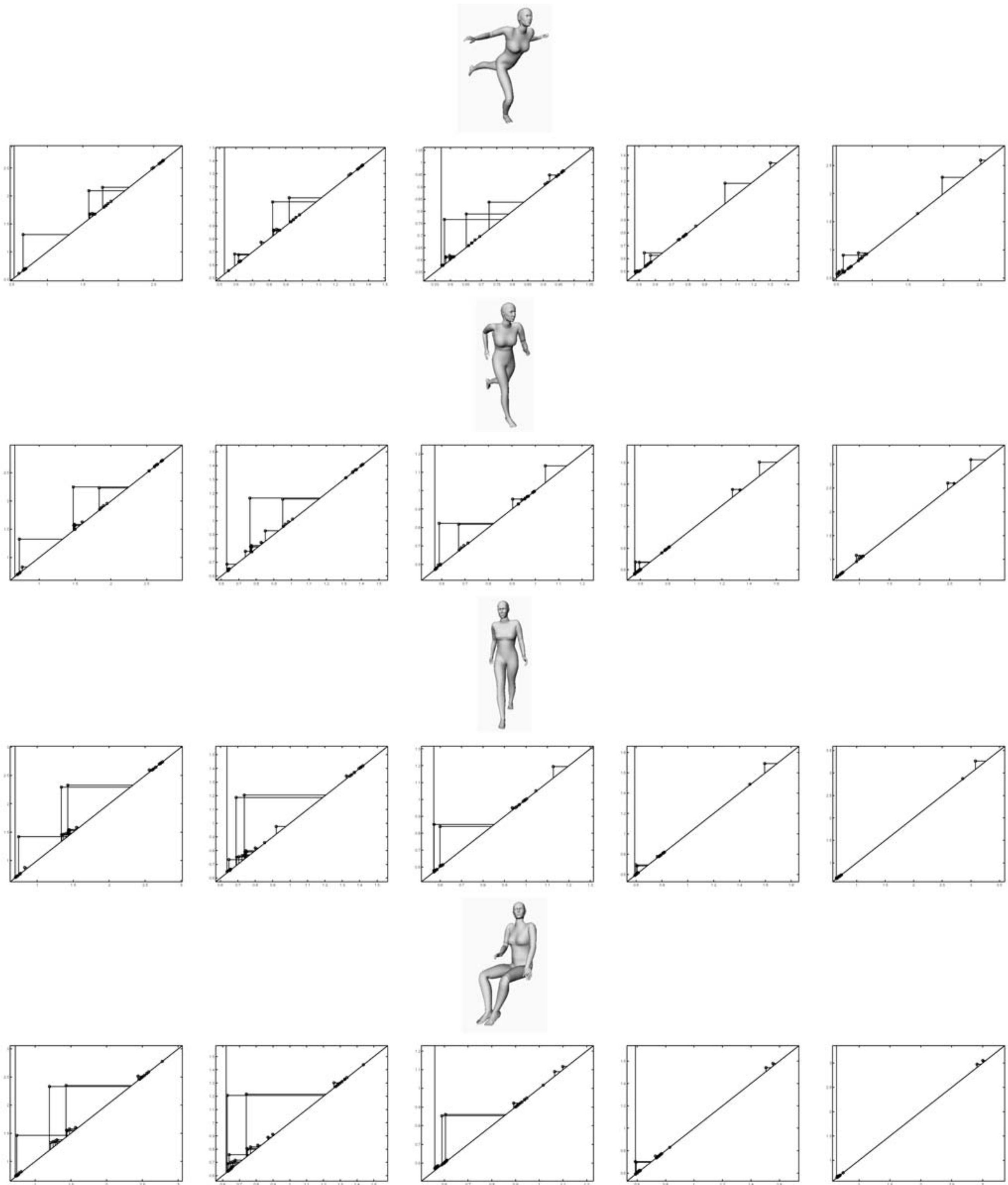

Fig. 5 2-dimensional size functions restricted to five leaves of the foliation, for other four models

the function *rank* $H_p^{x,y}$, where $H_p^{x,y}$ denotes the dimension p persistent homology group computed at point (x, y) (cf. [12]). Therefore, the information described by vineyards is equivalent to the knowledge of the function *rank* $H_p^{x,y}$, computed with respect to the function f_t. We are interested in the case $p = 0$. Since, by definition, for $x < y$, *rank* $H_0^{x,y}$ coin-

Table 1 Distances between models: the 2-dimensional matching distance between the 2-dimensional size functions associated with $\vec{\varphi}_{(1,-1)}$ (top value) and the maximum between the 1-dimensional matching distances associated to the 1-dimensional measuring functions φ_1 and φ_{-1} (bottom). The 2-dimensional matching distance provides a better lower bound for the natural pseudo-distance. Notice also that the values related to the 1-dimensional matching distance are each other very close, up to the considered number of digits, thus revealing a lower discriminatory power than that provided by the 2-dimensional matching distance

								
	0.0000 0.0000	0.0181 0.0003	0.1411 0.0025	0.1470 0.0026	0.1325 0.0023	0.1287 0.0022	0.1171 0.0020	0.1187 0.0021
	0.0181 0.0003	0.0000 0.0000	0.1419 0.0026	0.1478 0.0026	0.1304 0.0023	0.1265 0.0022	0.1171 0.0020	0.1187 0.0021
	0.1411 0.0025	0.1419 0.0025	0.0000 0.0000	0.0137 0.0002	0.1583 0.0028	0.1370 0.0024	0.1127 0.0020	0.1017 0.0018
	0.1470 0.0026	0.1478 0.0026	0.0137 0.0002	0.0000 0.0000	0.1533 0.0027	0.1381 0.0024	0.1137 0.0020	0.1021 0.0018
	0.1325 0.0023	0.1304 0.0023	0.1583 0.0028	0.1533 0.0027	0.0000 0.0000	0.0921 0.0014	0.0588 0.0016	0.1000 0.0017
	0.1287 0.0022	0.1265 0.0022	0.1370 0.0024	0.1381 0.0024	0.0921 0.0014	0.0000 0.0000	0.1069 0.0019	0.1048 0.0018
	0.1171 0.0020	0.1171 0.0020	0.1127 0.0020	0.1137 0.0020	0.0588 0.0016	0.1069 0.0019	0.0000 0.0000	0.0350 0.0006
	0.1187 0.0021	0.1187 0.0021	0.1017 0.0018	0.1021 0.0018	0.1000 0.0017	0.1048 0.0018	0.0350 0.0006	0.0000 0.0000

cides with the value taken by the size function $\ell_{(\mathcal{M},f_t)}(x,y)$, it follows that, for $x<y$, dimension 0 vineyards contain the same information as the 1-parameter family of size functions $\{\ell_{(\mathcal{M},f_t)}\}_{t\in[0,1]}$. Anyway, another interesting link exists between dimension 0 vineyards and multidimensional size functions. This link is expressed by the following theorem. In order to prove it, we need the next two lemmas. The former states that the relation of $\langle\vec{\varphi}\preceq\vec{y}\rangle$-connectedness passes to the limit.

Lemma 1 *Assume that $(\mathcal{M},\vec{\varphi})$ is a size pair and $\vec{y}=(y_1,\dots,y_k)\in\mathbb{R}^k$. If, for every $\varepsilon>0$, P and Q are $\langle\vec{\varphi}\preceq(y_1+\varepsilon,\dots,y_k+\varepsilon)\rangle$-connected in $\mathcal{M}$, then they are also $\langle\vec{\varphi}\preceq\vec{y}\rangle$-connected.*

Proof For every positive integer number n, let K_n be the connected component of $\mathcal{M}\langle\vec{\varphi}\preceq(y_1+\frac{1}{n},\dots,y_k+\frac{1}{n})\rangle$ containing P and Q. Since connected components are closed sets and $\mathcal{M}$ is compact, each K_n is compact. The set $\bigcap_n K_n$ is the intersection of a family of connected compact Hausdorff subspaces with the property that $K_{n+1}\subseteq K_n$ for every n, and hence it is connected (cf. Theorem 28.2 in [45], p. 203). Moreover, $\bigcap_n K_n$ is a subset of $\mathcal{M}\langle\vec{\varphi}\preceq\vec{y}\rangle$ and contains both P and Q. Therefore, P and Q are $\langle\vec{\varphi}\preceq\vec{y}\rangle$-connected. □

The following lemma allows us to study the behavior of multidimensional size functions near Δ (on Δ they have not been defined because of instability problems when the measuring functions are not assumed to be tame).

Lemma 2 *Let $(\mathcal{M},\vec{\varphi})$ be a size pair. If $\vec{x}\preceq\vec{y}$ then $\lim_{\varepsilon\to0^+}\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},(y_1+\varepsilon,\dots,y_k+\varepsilon))$ is equal to the number $L(\vec{x},\vec{y})$ of equivalence classes of $\mathcal{M}\langle\vec{\varphi}\preceq\vec{x}\rangle$ quotiented with respect to the $\langle\vec{\varphi}\preceq\vec{y}\rangle$-connectedness relation.*

Note that, for $\vec{x}\prec\vec{y}$, $L(\vec{x},\vec{y})$ simply coincides with $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})$.

Proof of Lemma 2 First of all we observe that the function $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},(y_1+\varepsilon,\dots,y_k+\varepsilon))$ is nonincreasing in the variable ε, and hence the value $\lim_{\varepsilon\to0^+}\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},(y_1+\varepsilon,\dots,y_k+\varepsilon))$ is defined. The statement of the lemma is trivial if $\lim_{\varepsilon\to0^+}\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},(y_1+\varepsilon,\dots,y_k+\varepsilon))=+\infty$,

Table 2 Space and time requirements for the computation of the size function of some 3D images of different dimensions. $|V|$ and $|E|$ represent the number of vertices and edges of the size graphs of the models. Avg. time is the average time required to compute the size function on a single half-plane of the foliation, while Total time refers to the computation of the size functions on 9 half-planes. Analogously, Avg. $|C|$ is the average number of cornerpoints of the size function on a single half-plane of the foliation, and Total $|C|$ is the sum of the number of cornerpoints of the size functions on 9 half-planes. These results are obtained using a AMD Athlon 3500, with 2 GB RAM

Model	$\|V\|$	$\|E\|$	**Avg. time**	**Total time**	**Avg.** $\|C\|$	**Total** $\|C\|$
	19 538	187 005	0.035 s	0.315 s	180	1623
	18 779	193 181	0.041 s	0.369 s	84	756
	11 504	97 757	0.015 s	0.135 s	16	142
	29 061	289 461	0.056 s	0.504 s	15	133
	114 277	1 233 063	0.212 s	1.908 s	28	254
	5472	45 689	0.006 s	0.054 s	14	122

since, for every $\varepsilon > 0$, the inequality $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, (y_1 + \varepsilon, \dots, y_k + \varepsilon)) \leq L(\vec{x}, \vec{y})$ holds by definition, and hence the equality $L(\vec{x}, \vec{y}) = +\infty$ immediately follows. Let us now assume that $\lim_{\varepsilon\to 0^+} \ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, (y_1+\varepsilon, \dots, y_k+\varepsilon)) = r < +\infty$. In this case a finite set $\mathcal{P} = \{P_1, \dots, P_r\}$ of points in $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\rangle$ exists such that, for every small enough $\varepsilon > 0$, every $P \in \mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\rangle$ is $\langle\vec{\varphi} \preceq (y_1 + \varepsilon, \dots, y_k + \varepsilon)\rangle$-connected to a point $P_j \in \mathcal{P}$ in $\mathcal{M}$. Furthermore, for $i \neq j$ the points $P_i, P_j \in \mathcal{P}$ are not $\langle\vec{\varphi} \preceq (y_1 + \varepsilon, \dots, y_k + \varepsilon)\rangle$-connected and hence not $\langle\vec{\varphi} \preceq \vec{y}\rangle$-connected either. From Lemma 1 it follows that every $P \in \mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\rangle$ is $\langle\vec{\varphi} \preceq \vec{y}\rangle$)-connected to a point $P_j \in \mathcal{P}$ in $\mathcal{M}$. Therefore $L(\vec{x}, \vec{y}) = r = \lim_{\varepsilon\to 0^+} \ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, (y_1+\varepsilon, \dots, y_k+\varepsilon))$. □

Theorem 5 *For $t \in I = [0, 1]$, consider the family of size pairs $(\mathcal{M}, f_t)$ where f_t is a homotopy between $f_0 : \mathcal{M} \to \mathbb{R}$ and $f_1 : \mathcal{M} \to \mathbb{R}$. Define $\vec{\chi} : \mathcal{M} \times I \to \mathbb{R}^3$ by $\vec{\chi}(P, t) = (f_t(P), t, -t)$. Then, for every $\bar{t} \in I$ and $\bar{x}, \bar{y} \in \mathbb{R}$ with $\bar{x} \leq \bar{y}$, it holds that*

$$\text{rank}\, H_0^{\bar{x},\bar{y}}(\bar{t}) = \lim_{\varepsilon\to 0^+} \ell_{(\mathcal{M}\times I,\vec{\chi})}(\bar{x}, \bar{t}, -\bar{t}, \bar{y}+\varepsilon, \bar{t}+\varepsilon, -\bar{t}+\varepsilon),$$

where $H_0^{\bar{x},\bar{y}}(\bar{t})$ denotes the dimension 0 persistent homology group computed at point $(\bar{x}, \bar{y})$ with respect to $f_{\bar{t}}$.

Proof We know that *rank* $H_0^{\bar{x},\bar{y}}(\bar{t})$ is equal to the number of equivalence classes of $\mathcal{M}\langle f_{\bar{t}} \leq \bar{x}\rangle$ quotiented with respect to the $\langle f_{\bar{t}} \leq \bar{y}\rangle$-connectedness relation. On the other hand, Lemma 2 states that $\lim_{\varepsilon\to 0^+} \ell_{(\mathcal{M}\times I,\vec{\chi})}(\bar{x}, \bar{t}, -\bar{t}, \bar{y}+\varepsilon, \bar{t}+\varepsilon, -\bar{t}+\varepsilon)$ is equal to the number of equivalence classes of $\mathcal{M} \times I\langle\vec{\chi} \preceq (\bar{x}, \bar{t}, -\bar{t})\rangle$ quotiented, with respect to the $\langle\vec{\chi} \preceq (\bar{y}, \bar{t}, -\bar{t})\rangle$-connectedness relation. By definition of $\vec{\chi}$, this last number equals the number of equivalence classes of $\mathcal{M}\langle f_{\bar{t}} \leq \bar{x}\rangle$ quotiented, with respect to the $\langle f_{\bar{t}} \leq \bar{y}\rangle$-connectedness relation. This concludes our proof. □

However, although these two links exist, the concept of multidimensional size function has quite different purposes than that of vineyard. First of all, vineyards are based on a 1-parameter parallel foliation of $\mathbb{R}^3$, while the study of multidimensional size functions depends on a $(2k-2)$-parameter non-parallel foliation of $\Delta^+ \subseteq \mathbb{R}^k \times \mathbb{R}^k$. In fact, multidimensional size functions are associated with k-dimensional measuring functions, instead of with a homo-

Table 3 Multidimensional and 1-dimensional matching distance between an airplane and another airplane (first column) or a chair model (second column). The first three rows refer to the multidimensional matching distance using 3 different half-planes of the foliation, the fourth and fifth row show the average and the maximum value of the multidimensional matching distance using 9 planes, and the last three rows represent the 1-dimensional matching distance using the single components of $\vec{\varphi}$

		
$\theta = \frac{\pi}{4}, \phi = \frac{\pi}{4}$	0.1337	0.7593
$\theta = \frac{\pi}{4}, \phi = \frac{\pi}{12}$	0.1336	0.7592
$\theta = \frac{\pi}{12}, \phi = \frac{\pi}{4}$	0.2198	0.4741
Average over 9 half-planes	0.1359	0.5449
Maximum over 9 half-planes	0.2198	0.7593
φ_x	0.0782	0.2543
φ_y	0.1151	0.3481
φ_z	0.0712	0.0963

topy between 1-dimensional measuring functions. Furthermore, [12] does not aim to identify distances for the comparison of vineyards, while we are interested in quantitative methods for comparing multidimensional size functions.

8 Conclusions and Future Work

We believe that this paper settles most of the problems connected to multidimensionality in Size Theory. Indeed, by reducing the theory of multidimensional size functions to the 1-dimensional case by a suitable change of variables, we have proved that multidimensional size functions are stable, with respect to the new distance D_{match}. Moreover they admit a simple and concise description. Our experiments show how to use them in concrete applications building on the existing computational techniques.

However some questions could be further investigated. Among them we list a few here.

- **Choice of the foliation.** Other foliations, different from the one we propose are possible. In general, we can choose a family Γ of continuous curves $\vec{\gamma}_{\vec{\alpha}} : \mathbb{R} \to \mathbb{R}^k$ such that (i) for $s < t$, $\vec{\gamma}_{\vec{\alpha}}(s) \prec \vec{\gamma}_{\vec{\alpha}}(t)$, (ii) for every $(\vec{x}, \vec{y}) \in \Delta^+$ there is one and only one $\vec{\gamma}_{\vec{\alpha}} \in \Gamma$ through $\vec{x}$, $\vec{y}$ and each component of $\vec{\gamma}_{\vec{\alpha}}$ is surjective (iii) the curve $\gamma_{\vec{\alpha}}$ depends continuously on the parameter $\vec{\alpha}$ (this last hypothesis is important in computation for stability reasons). In this case, the leaves of the foliation are given by the surfaces $(\vec{\gamma}_{\vec{\alpha}}(s), \vec{\gamma}_{\vec{\alpha}}(t))$, with $s < t$, parameterized by s, t. It would be interesting to study the dependence of our results on the choice of the foliation.
- **Choice of the planes inside the foliation.** The comparison technique expressed by Remark 5 requires the choice of a finite set of foliation leaves, on which we compute the reduction from multidimensional to 1-dimensional size functions. It would be interesting to determine a method to make this choice optimal.
- **Existence of size pairs having assigned k-dimensional size functions.** At this time we do not know if any link exists between the 1-dimensional size functions associated with the planes $\pi_{(\vec{l},\vec{b})}$, apart from continuity. A question naturally arises about the conditions of existence of size pairs having an assigned continuous family of size functions on the planes of our foliation.
- **Reduction moves for the multidimensional size graph.** In the discrete framework, the computation of 1-dimensional size functions is usually speeded up by applying some reduction moves to the size graph. Similar moves can be introduced in the multidimensional setting ([8]) but so far we do not know when this reduction is worth its computational cost.

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This paper is dedicated to the memory of Marco Gori.

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