Forum Math. 16 (2004), 695–715

Forum Mathematicum

Natural pseudodistances between closed manifolds

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(Communicated by Héctor J. Sussmann)

Abstract. Let us consider two closed homeomorphic manifolds $\mathcal{M}, \mathcal{N}$ of class C^1 and two functions $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$ of class C^1. The natural pseudodistance d between the pairs $(\mathcal{M}, \varphi), (\mathcal{N}, \psi)$ is defined as the infimum of $\Theta(f) \stackrel{def}{=} \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$, as f varies in the set of all homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. In this paper we prove that a suitable multiple of d by a positive integer k coincides with the distance between two critical values of the functions φ, ψ.

2000 Mathematics Subject Classification: 58C05, 58K05; 53C15.

Introduction

The problem of comparing two manifolds $\mathcal{M}$ and $\mathcal{N}$ by computing the infimum of an operator Θ defined on a suitable set H of homeomorphisms is a classic object of study in many fields of Geometry. The Fréchet distance (cf. [1]), the Lipschitz distance (cf. [9]) and, in some senses, the Teichmüller distance (cf. [16]) are only a few examples showing the importance of such an approach to comparing manifolds.

A simple way of defining the operator Θ is the following one. For each manifold we choose a convenient real-valued function and compute how much each homeomorphism $f \in H$ "changes" such a function. This measurement is the value taken by our operator Θ at f. The usual task is to make $\Theta(f)$ as small as possible, and to take its infimum as a pseudodistance between the considered manifolds.

A structure on a manifold $\mathcal{A}$ can often be seen as a function φ from another manifold $\mathcal{M}$ to the real numbers (e.g., a Riemannian structure on a smooth manifold $\mathcal{M}$ can be seen as a real-valued function defined on the Whitney sum $T(\mathcal{M}) \oplus T(\mathcal{M})$). Hence the functional Θ also allows us to compare some kinds of structures on manifolds by using suitable real-valued functions.

On the other hand, there are many examples of functions whose extrema have been extensively used for studying and comparing manifolds (cf., e.g., [2, 9, 10, 11, 14]). So, by assuming that two closed homeomorphic manifolds $\mathcal{M}, \mathcal{N}$ of class C^1 are

This work was partially supported by MURST (Italy), ELSAG (Italy), INdAM (Italy) and NATO. Investigation supported by University of Bologna. Funds for selected research topics.

given, we are naturally led to study the value $d \stackrel{def}{=} \inf_{f \in H(\mathcal{M},\mathcal{N})} \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$ for every *arbitrary* pair of functions $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$ of class C^1, where the symbol $H(\mathcal{M},\mathcal{N})$ denotes the set of all homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. The functions φ and ψ are called *measuring functions*.

The closeness of d to zero means that there are homeomorphisms for which the difference between the values taken by the measuring functions at corresponding points is arbitrarily small. On the other hand, if such an infimum is large, we have that every homeomorphism between the considered manifolds must change the values taken by our measuring function considerably.

Finally, it is also interesting to highlight the strong similarity between the pseudodistance defined by the infimum of $\Theta(f)$ and the Fréchet distance between surfaces.

Apart from its generality, the interest in the approach we have just described is mainly due to its usefulness in modelling minimization problems in Applied Geometry. In particular, the purpose of comparing "shapes" of manifolds and topological spaces for solving Computer Vision problems has made the computation of d a useful task, together with the "twin" and strictly related concept of *size function*. For more theoretical details and examples of practical applications we refer to [4, 5, 7, 12, 17, 18, 19, 20, 21, 22].

All these reasons, together with the challenging difficulty in computing d, have motivated our research.

In this paper we investigate some properties of the infimum d of Θ. We prove that a suitable multiple of d by a positive integer k coincides with the distance between two critical values of the functions φ and ψ (Theorem 6.2). Previous results should be compared to those obtained in [3] for natural pseudodistances and in [6] for size functions.

In the following Section 1 we give the main definitions, while in Section 2 our problem is made clearer by showing some examples. The core of the paper begins in Section 3 where some key concepts for our proofs are given. Section 4 provides the technical results required in Section 5 for proving our main result, in a weaker form (Theorem 5.2). Section 6 allows us to weaken the hypotheses required in the previous theorem, in order to arrive at Theorem 6.2. In Section 7 some final remarks are given.

1 Setting the problem

Let us consider the collection *Size* of all pairs $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a closed manifold of class C^1 and $\varphi : \mathcal{M} \to \mathbb{R}$ is a function of class C^1. We shall call $(\mathcal{M}, \varphi)$ a *size pair* of class C^1 and φ a *measuring function*. However, from this section to Section 5 we shall also assume that $\mathcal{M}$ is a smooth manifold and φ is a smooth Morse function. These hypotheses will simplify our proofs from a technical point of view. In Section 6 we shall weaken our assumptions and come back to the case of class C^1.

Sometimes we shall speak about dilations of subsets of a smooth manifold $\mathcal{M}$ and use the norm $\|P - Q\|$ for $P, Q \in \mathcal{M}$. In these cases we shall implicitly assume that an embedding of $\mathcal{M}$ into a Euclidean space has been arbitrarily chosen, so that both previous concepts make sense. Obviously, Whitney's Theorem assures that such an embedding does exist.

Assume $(\mathscr{M}, \varphi), (\mathscr{N}, \psi)$ are two size pairs. The symbol $H(\mathscr{M}, \mathscr{N})$ will denote the set of all homeomorphisms from $\mathscr{M}$ to $\mathscr{N}$.

Definition 1.1. If $H(\mathscr{M}, \mathscr{N}) \neq \emptyset$, the function $\Theta : H(\mathscr{M}, \mathscr{N}) \to \mathbb{R}$ given by

$$\Theta(f) = \max_{P \in \mathscr{M}} |\varphi(P) - \psi(f(P))|$$

is called the *natural size measure* with respect to the measuring functions φ and ψ.

In plain words, Θ measures how much f changes the values taken by the measuring functions at corresponding points.

Definition 1.2. We shall call *natural size pseudodistance* the pseudodistance δ : *Size* $\times$ *Size* $\to \mathbb{R} \cup \{+\infty\}$ defined by the formula:

$$\delta((\mathscr{M}, \varphi), (\mathscr{N}, \psi)) = \begin{cases} \inf_{f \in H(\mathscr{M}, \mathscr{N})} \Theta(f) & \text{if } H(\mathscr{M}, \mathscr{N}) \neq \emptyset \\ +\infty & \text{otherwise.} \end{cases}$$

In the following the symbol d will denote the value of the natural size pseudodistance computed between the pairs $(\mathscr{M}, \varphi)$ and $(\mathscr{N}, \psi)$ we are considering. As we previously explained, such a pseudodistance gives a method for comparing two manifolds with respect to the measuring functions we have chosen.

We point out that δ is not a distance, since two size pairs can have a vanishing pseudodistance without being equal (see Figure 5 for a non-trivial example).

In the following section we shall show that the infimum of $\Theta(f)$ varying f in $H(\mathscr{M}, \mathscr{N})$ is not always a minimum. When such an infimum is also a minimum, we shall say that each homeomorphism $f \in H(\mathscr{M}, \mathscr{N})$ with $d = \Theta(f)$ is an *optimal homeomorphism*.

In order to simplify our notations, we shall assume that the manifolds $\mathscr{M}$ and $\mathscr{N}$ do not meet and that the corresponding measuring functions are obtained by restriction of a function $\omega : \mathscr{M} \cup \mathscr{N} \to \mathbb{R}$, so that $\varphi = \omega_{|\mathscr{M}}$ and $\psi = \omega_{|\mathscr{N}}$. Therefore we shall be allowed to use just one symbol to denote both the measuring functions. These hypotheses are not restrictive, since we can always replace the size pair $(\mathscr{N}, \psi)$ with a new size pair $(\overline{\mathscr{N}}, \bar{\psi})$ having vanishing pseudodistance from the previous one and such that $\mathscr{M} \cap \overline{\mathscr{N}} = \emptyset$. Sometimes, when not confusing, we shall use the symbol ω to denote both $\omega_{|\mathscr{M}}$ and $\omega_{|\mathscr{N}}$.

Example 1.3. In $\mathbb{R}^3$ consider the unit sphere $\mathscr{S}$ of equation $x^2 + y^2 + z^2 = 1$ and the ellipsoid $\mathscr{E}$ of equation $x^2 + 4y^2 + 9z^2 = 1$. On $\mathscr{S}$ and $\mathscr{E}$ consider respectively the measuring functions φ and ψ that take every point of $\mathscr{S}$ and $\mathscr{E}$ to the Gaussian curvature of the considered manifold at that point. We have $\delta((\mathscr{S}, \varphi), (\mathscr{E}, \psi)) = 35$. In fact $\varphi(\mathscr{S}) = \{1\}$ while $\psi(\mathscr{E}) = [4/9, 36]$, and therefore for every $f \in H(\mathscr{S}, \mathscr{E})$ it results that $\Theta(f) = 35$.

Example 1.4. Consider the two tori $\mathscr{T}, \mathscr{T}' \subset \mathbb{R}^3$ generated by the rotation around the y-axis of the circles lying in the plane yz and with centres $A = (0, 0, 3)$ and $B = (0, 0, 4)$, and radii 2 and 1, respectively (see Figure 1). As measuring function φ (resp. φ') on $\mathscr{T}$ (resp. on $\mathscr{T}'$) we take the restriction to $\mathscr{T}$ (resp. to $\mathscr{T}'$) of the function $\zeta : \mathbb{R}^3 \to \mathbb{R}$, $\zeta(x, y, z) = z$. We point out that, for both $\mathscr{T}$ and $\mathscr{T}'$, the image of the measuring function is the closed interval $[-5, 5]$. We can easily prove that the natural size pseudodistance between $(\mathscr{T}, \varphi)$ and $(\mathscr{T}', \varphi')$ is 2 (for a proof see [8]). Moreover, the homeomorphism f, taking each point of $\mathscr{T}$ to the point having the same toroidal coordinates in $\mathscr{T}'$, has natural size measure $\Theta(f) = 2$.

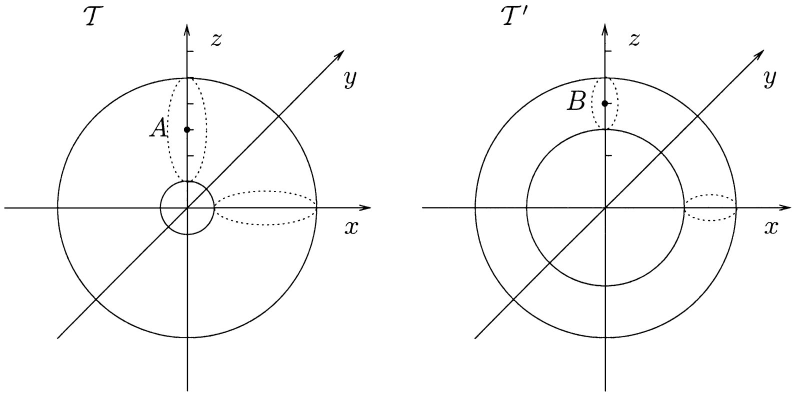

Figure 1. In this case an optimal (i.e. minimizing Θ) homeomorphism exists and $d = 2$; d equals the distance between a critical value of φ and a critical value of φ'.

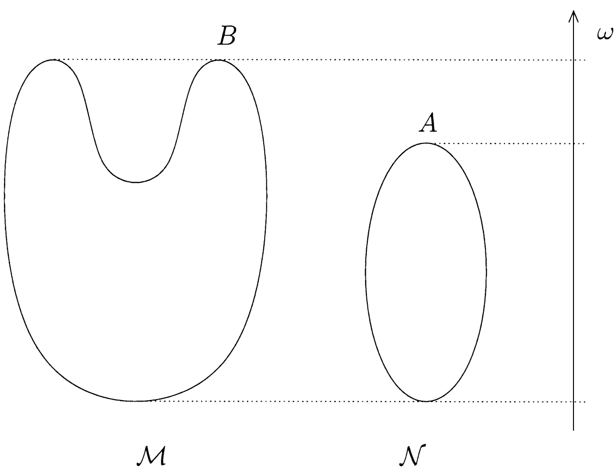

Figure 2. The natural pseudodistance between the size pairs $(\mathscr{M}, \omega)$ and $(\mathscr{N}, \omega)$ is $\omega(B) - \omega(A)$.

In general, d is far from being easily computable as in previous Examples 1.3 and 1.4. In Example 1.3, for every homeomorphism $f \in H(\mathscr{S}, \mathscr{E})$ we have that $\Theta(f)$ equals the Hausdorff distance $\delta_H(\varphi(\mathscr{S}), \psi(\mathscr{E}))$ between the sets $\varphi(\mathscr{S})$ and $\psi(\mathscr{E})$ in $\mathbb{R}$. Now it is clear that the natural size pseudodistance $\delta((\mathscr{M}, \varphi), (\mathscr{N}, \psi))$ is always greater than or equal to $\delta_H(\varphi(\mathscr{M}), \psi(\mathscr{N}))$ and therefore $\Theta(f)$ must be the natural size pseudodistance we want to compute. We also point out that, in Example 1.3, the images of φ and ψ are different sets and so the natural size pseudodistance is trivially positive.

In Example 1.4 the natural size pseudodistance is strictly greater than the (vanishing) Hausdorff distance between the images of the two measuring functions.

Computing natural size pseudodistances is usually very difficult. For this reason the concepts of *size function* and *size homotopy group* have been developed, making it easier to compute the value d, using some lower-bound theorems. Anyway, here we cannot illustrate these strongly correlated concepts, and hence we refer to [5, 6, 7, 8] for more details.

Remark 1.5. The presence of the word "size" in our definitions is due to the link existing between the pseudodistance δ, size functions and size homotopy groups. However, for the sake of simplicity, we shall often drop the word "size" in the expressions "natural size measure" and "natural size pseudodistance". The term "natural" is used in order to distinguish the pseudodistance studied here from other pseudodistances we can define between submanifolds of the Euclidean space and between manifolds paired with measuring functions (cf. [5]).

We observe that in the previous Examples 1.3 and 1.4 there is an optimal homeomorphism (in particular, all homeomorphisms in $H(\mathscr{S}, \mathscr{E})$ are optimal). It is important to point out that optimal homeomorphisms do not generally exist, as we shall see in the next section.

2 Some examples about curves and surfaces

Example 2.1. The first example we give is shown in Figure 2. $\mathscr{M}$ and $\mathscr{N}$ are planar smooth curves and ω is the ordinate function. It is clear that the natural pseudodistance d between the size pairs $(\mathscr{M}, \omega)$ and $(\mathscr{N}, \omega)$ equals $\omega(B) - \omega(A)$, that is, the distance between a critical value of $\omega_{|\mathscr{M}}$ and a critical value of $\omega_{|\mathscr{N}}$.

In this example no optimal homeomorphism exists, since it ought to take both the maximum points for $\omega_{|\mathscr{M}}$ to A, against injectivity.

Example 2.2. Let us consider the smooth curves $\mathscr{M}$ and $\mathscr{N}$ in Figure 3. The points A and B are critical points of the function ω and $\omega(C) = \frac{1}{2}(\omega(A) + \omega(B)) = \omega(G)$. We want to prove that the natural pseudodistance between the size pairs $(\mathscr{M}, \omega)$ and $(\mathscr{N}, \omega)$ takes the value

$$d = \frac{1}{2}(\omega(A) - \omega(B))$$

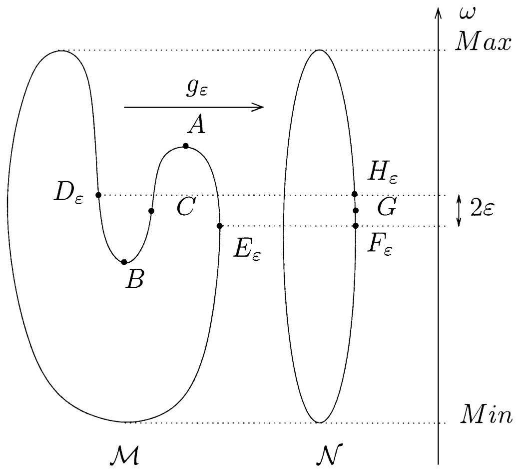

Figure 3. Construction of the homeomorphism g_ε for which $\Theta(g_\varepsilon) \leq d + \varepsilon$.

and that no optimal homeomorphism exists. In order to do that we shall construct a sequence of homeomorphisms (f_n) for which $\lim_n \Theta(f_n) = \frac{1}{2}(\omega(A) - \omega(B))$ and show that $\Theta(f) > \frac{1}{2}(\omega(A) - \omega(B))$ for every homeomorphism $f \in H(\mathcal{M}, \mathcal{N})$.

Let us start by proving that for every $\varepsilon > 0$ a homeomorphism $g_\varepsilon : \mathcal{M} \to \mathcal{N}$ exists, such that $\Theta(g_\varepsilon) \leq \frac{1}{2}(\omega(A) - \omega(B)) + 2\varepsilon$. Consider the points $D_\varepsilon, E_\varepsilon, H_\varepsilon$ and F_ε in Figure 3, verifying $\omega(D_\varepsilon) = \omega(H_\varepsilon) = \omega(C) + \varepsilon$ and $\omega(E_\varepsilon) = \omega(F_\varepsilon) = \omega(C) - \varepsilon$. We choose a homeomorphism g_ε taking the arc $D_\varepsilon C E_\varepsilon$ to the arc $H_\varepsilon G F_\varepsilon$ in such a way that $g_\varepsilon(D_\varepsilon) = H_\varepsilon$ and $g_\varepsilon(E_\varepsilon) = F_\varepsilon$. Outside the arc $D_\varepsilon C E_\varepsilon$ in $\mathcal{M}$ we define g_ε by taking every point P to a point $g_\varepsilon(P)$, verifying $\omega(P) = \omega(g_\varepsilon(P))$.

For every $n \in \mathbb{N} - \{0\}$ we set $f_n = g_{1/n}$. It is easy to prove that

$$\lim_n \Theta(f_n) = \frac{1}{2}(\omega(A) - \omega(B)).$$

Now we have only to verify that no homeomorphism between $\mathcal{M}$ and $\mathcal{N}$ exists for which $\Theta(f) \leq \frac{1}{2}(\omega(A) - \omega(B))$. If such a homeomorphism existed, for every $P \in \mathcal{M}$ we would have

$$|\omega(P) - \omega(f(P))| \leq \frac{\omega(A) - \omega(B)}{2}$$

and hence $\omega(f(A)) \geq \omega(G) \geq \omega(f(B))$. Therefore we could easily find points $P \in \mathcal{M}$ for which $|\omega(P) - \omega(f(P))| > \frac{1}{2}(\omega(A) - \omega(B))$, contradicting our assumption.

Example 2.3. Consider the size pairs $(\mathcal{M}, \omega)$ and $(\mathcal{N}, \omega)$ in Figure 4, where $\mathcal{M}$ and $\mathcal{N}$ are smooth surfaces embedded into $\mathbb{R}^3$. We want to prove that the natural pseudodistance between these size pairs takes the value $1/2$.

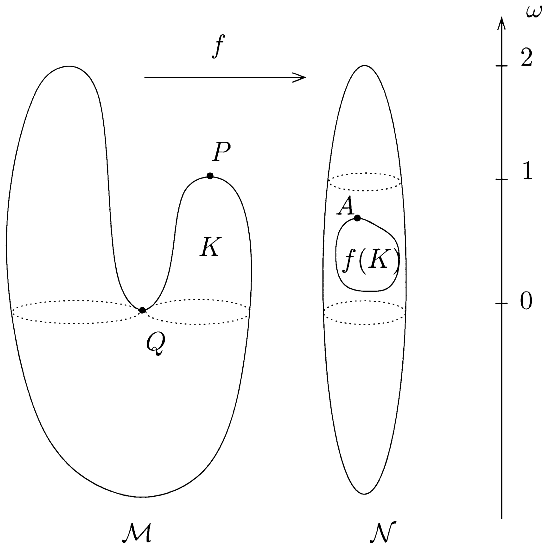

Figure 4. The natural pseudodistance between these size pairs is $d = 1/2$.

The critical points $P, Q \in \mathcal{M}$ for which $\omega(P) = 1$ and $\omega(Q) = 0$ belong to the displayed closed set $K \subset \omega^{-1}([0,1])$. First of all, we shall prove that $d \geq 1/2$, by showing that for every homeomorphism $f : \mathcal{M} \to \mathcal{N}$ the inequality

$$\Theta(f) > \frac{1}{2}(\omega(P) - \omega(Q)) = \frac{1}{2}$$

holds. Suppose $f(K)$ contains no point of $\mathcal{N}$ that is critical for ω (otherwise $\Theta(f)$ would be at least 1 and our inequality would be already satisfied). Let A be the point of $f(K)$ at which the measuring function $\omega_{|f(K)}$ takes its maximum. Since A belongs to the boundary of $f(K)$, it must be $\omega(f^{-1}(A)) = 0$ and as P is internal to K, $\omega(f(P)) < \omega(A)$. In conclusion, $\Theta(f) \geq \omega(A) > \omega(f(P))$ and hence $\Theta(f) \geq \omega(P) - \omega(f(P)) > \omega(P) - \Theta(f)$. It follows that $\Theta(f) > \omega(P)/2 = 1/2$.

In order to complete our proof that the natural pseudodistance is really 1/2, we still have to show a suitable sequence of homeomorphisms (f_n) such that

$$\lim_n \Theta(f_n) = 1/2.$$

Since the construction of such a sequence is conceptually similar to the one we gave for the previous example about curves, we skip its analytic expression.

Example 2.4. Consider the smooth surfaces $\mathcal{M}$ and $\mathcal{N}$ displayed in Figure 5 and the corresponding measuring function ω. The dotted lines are level curves for the measuring function ω.

Property 1. *The natural pseudodistance between the two size pairs vanishes.*

It is easy to see that we can isotopically deform the former surface into the latter one by a "torsion" exchanging the positions of the smallest humps. This deformation can be performed by an arbitrarily small change in the values of the height ω. Therefore we can construct a sequence of homeomorphisms (f_n) from $\mathcal{M}$ to $\mathcal{N}$ such that $\Theta(f_n) \to 0$.

Property 2. *No optimal homeomorphism exists between the two size pairs.*

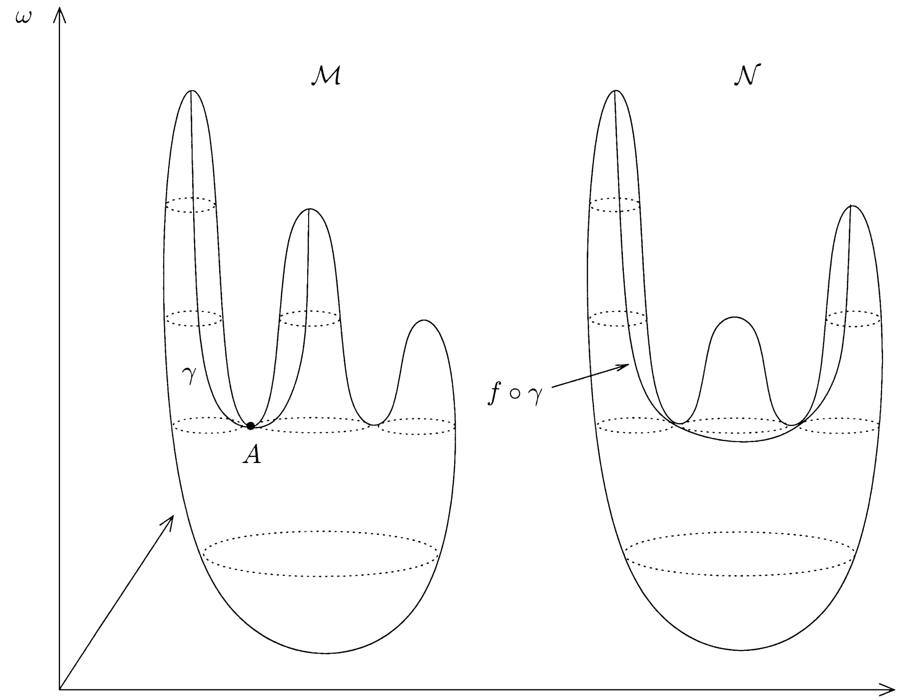

Figure 5. An example of vanishing natural pseudodistance.

Suppose a homeomorphism f exists such that $\Theta(f) = 0$. Consider a path γ as in Figure 5, chosen in such a way that, in the image of the path, no point P different from A exists for which $\omega(P) = \omega(A)$. We can easily verify that the image of the path $f \circ \gamma$ must contain more than one point at which ω takes the value $\omega(A)$. This is against our assumptions, since $\Theta(f) = 0$ implies $\omega(f(P)) = \omega(P)$ for every point P in the image of γ.

3 The concept of train of "limit *d*-jumps"

In order to prove our main theorem we need some new definitions and technical results. Assume two size pairs $(\mathcal{M}, \omega)$, $(\mathcal{N}, \omega)$ are given.

The symbol $S_H(\mathcal{M},\mathcal{N})$ will denote the set of all sequences of homeomorphisms (f_n) in $H(\mathcal{M},\mathcal{N})$ such that $\Theta(f_n) \to d$. Every sequence in $S_H(\mathcal{M},\mathcal{N})$ will be called a *d-approximating sequence*.

Let us consider a sequence $(f_n) \in S_H(\mathcal{M},\mathcal{N})$. We shall say that a pair of points $(P,Q) \in \mathcal{M} \times \mathcal{N}$ is in relation with respect to (f_n) if a sequence (P_r) in $\mathcal{M}$ exists together with a strictly increasing sequence (i_r) in $\mathbb{N}$ such that

$$(P,Q) = \lim_r (P_r, f_{i_r}(P_r)).$$

In this case we shall write either $P\rho Q$ or $Q\rho P$, indifferently.

In the following part of this section we shall assume that $0 < d < +\infty$. The following compact sets are defined with respect to each d-approximating sequence (f_n):

$$N^+_{\mathcal{M}} = N^+_{\mathcal{M}}((f_n)) = \{P \in \mathcal{M} \mid \exists Q \in \mathcal{N} : P\rho Q, \omega(Q) - \omega(P) = d\}$$

$$N^-_{\mathcal{M}} = N^-_{\mathcal{M}}((f_n)) = \{P \in \mathcal{M} \mid \exists Q \in \mathcal{N} : P\rho Q, \omega(P) - \omega(Q) = d\}$$

$$N^+_{\mathcal{N}} = N^+_{\mathcal{N}}((f_n)) = \{Q \in \mathcal{N} \mid \exists P \in \mathcal{M} : P\rho Q, \omega(P) - \omega(Q) = d\}$$

$$N^-_{\mathcal{N}} = N^-_{\mathcal{N}}((f_n)) = \{Q \in \mathcal{N} \mid \exists P \in \mathcal{M} : P\rho Q, \omega(Q) - \omega(P) = d\}.$$

In plain words, the points P in $N^+_{\mathcal{M}}$ are those for which a point $Q \in \mathcal{N}$ exists, such that the pair (P,Q) can be approximated arbitrarily well by a pair $(P_n, f_{i_n}(P_n))$ whose "jump" $\omega(f_{i_n}(P_n)) - \omega(P_n)$ is arbitrarily close to d. Hence, if we think of ω as a "height" function (cf. the examples in the previous section), the points P_n have images with height approximated by $\omega(P_n) + d$. In $N^+_{\mathcal{M}}$, the symbol $\mathcal{M}$ recalls the manifold to which P belongs, while the symbol $+$ recalls that by taking P to Q we increase the value of the measuring function. The notations used for the other three sets are quite analogous.

It is clear that for every point $P \in N^+_{\mathcal{M}}$ a point $Q \in N^-_{\mathcal{N}}$ exists such that $P\rho Q$ (and vice versa) and that an analogous relation holds for the sets $N^-_{\mathcal{M}}$ and $N^+_{\mathcal{N}}$. For every sequence of homeomorphisms in $S_H(\mathcal{M},\mathcal{N})$ the sets $N_{\mathcal{M}} = N^+_{\mathcal{M}} \cup N^-_{\mathcal{M}}$ and $N_{\mathcal{N}} = N^+_{\mathcal{N}} \cup N^-_{\mathcal{N}}$ are non-empty because of the compactness of the manifolds and the continuity of the measuring functions.

Now we shall define the concept of "train" for a d-approximating sequence:

Definition 3.1. Let $(N_0, N_1, \dots, N_k)$ be an ordered $(k+1)$-tuple of points in $\mathcal{M} \cup \mathcal{N}$ with $k \geq 1$ such that, for $i = 0, \dots, k-1$ the following properties hold:

a) $\omega(N_{i+1}) = \omega(N_i) + d$;

b) $N_i \rho N_{i+1}$.

In this case the ordered set $(N_0, N_1, \dots, N_k)$ will be called a *train of limit d-jumps for the sequence* (f_n) (or, in short, a *train*) and its points will be called *nodes*. The pairs (N_i, N_{i+1}) will be known as the *wagons* of the train. The number k will be called *length of the train* and each train that is not included in any other train will be said to be *maximal*. If $(N_0, \dots, N_k)$ is a maximal train, its wagons (N_0, N_1) and (N_{k-1}, N_k) will be called *initial* and *final train wagons* (respectively), while N_0 and N_k will be the *initial* and *final train nodes*. The remaining nodes will be called *internal nodes*. The symbol $W((f_n))$ will denote the set of all the train wagons (for all the existing trains).

Since each point belonging either to $N_{\mathcal{M}}$ or to $N_{\mathcal{N}}$ is a node for at least one train, the set of all trains is not empty. Notice that the point P is an initial node for at least a maximal train if and only if either $P \in N_{\mathcal{M}}^+ - N_{\mathcal{M}}^-$ or $P \in N_{\mathcal{N}}^+ - N_{\mathcal{N}}^-$, whereas it is a final node if and only if either $P \in N_{\mathcal{M}}^- - N_{\mathcal{M}}^+$ or $P \in N_{\mathcal{N}}^- - N_{\mathcal{N}}^+$.

In Figure 6 we provide a graphic representation of a maximal train (A, B, C, D). In particular, we have that $A \in N_{\mathcal{N}}^+$, $B \in N_{\mathcal{M}}^+ \cap N_{\mathcal{M}}^-$, $C \in N_{\mathcal{N}}^+ \cap N_{\mathcal{N}}^-$ and $D \in N_{\mathcal{M}}^-$. Hence A is the initial node and D is the final train node, while B and C are internal nodes. The three ordered pairs $(A, B), (B, C), (C, D)$ are the three wagons in the train; (A, B) and (C, D) are its initial and final wagon, respectively.

In Figure 7 we find the maximal train (B, G, A) associated to the d-approximating sequence we described in Example 2.2. In fact we can easily prove that $B\rho G$, $G\rho A$, $\omega(G) - \omega(B) = d$ and $\omega(A) - \omega(G) = d$. Hence $B \in N_{\mathcal{M}}^+$, $G \in N_{\mathcal{N}}^+ \cap N_{\mathcal{N}}^-$ and $A \in N_{\mathcal{M}}^-$.

Remark 3.2. The last example shows that the existence of a train of length 2, such that its initial node (in this case B) and its final node (in this case A) are critical points of the measuring function ω, guarantees that the natural pseudodistance equals half the distance between two critical values of the measuring function. In the next two sections, our main goal will be to prove that it is always possible to construct a sequence of d-approximating homeomorphisms such that all maximal trains begin and end at critical points of the measuring function. The example we have just seen justifies our task, since it points out a simple relation between d and the critical values of ω.

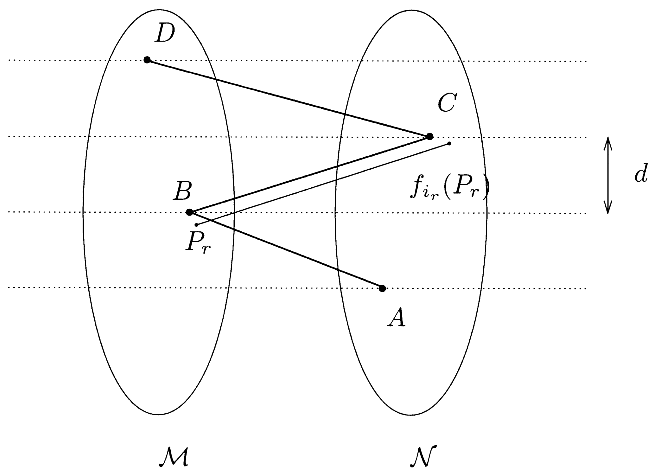

Figure 6. A train of limit d-jumps given by the quadruple (A, B, C, D).

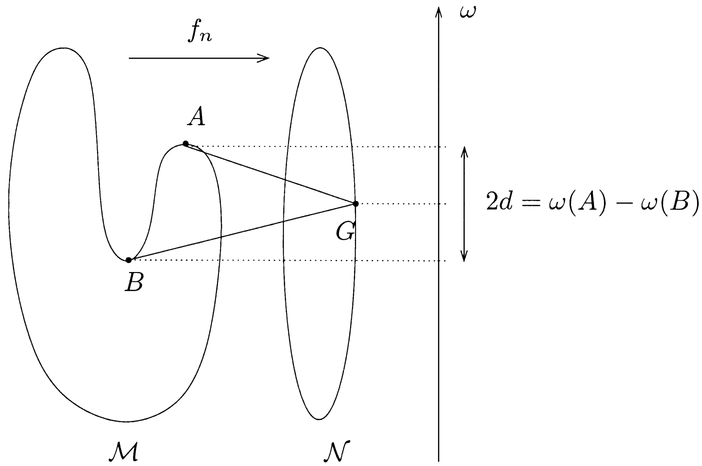

Figure 7. An example of train of limit d-jumps given by the triple (B, G, A).

4 Some technical results

All this section is devoted to some boring but necessary computations that will allow us to prove our key Lemma 5.1 in Section 5. In this section and in the following one $\mathcal{M}$ and $\mathcal{N}$ will be assumed to be smooth closed homeomorphic manifolds, while φ and ψ will be smooth Morse functions. We shall assume $0 < d < +\infty$. Let K_φ and K_ψ be the sets of all critical points of the functions φ and ψ. Since they are Morse functions, K_φ and K_ψ are finite sets, as are the sets $\varphi(K_\varphi)$ and $\psi(K_\psi)$ of all critical values for the measuring functions.

Lemma 4.1. *Suppose (f_n) is a d-approximating sequence and C is a compact subset of $\mathcal{M}$ such that $C \cap N^-_{\mathcal{M}} = \emptyset$. Then $\varepsilon = \varepsilon(C) > 0$ and $\bar{n} = \bar{n}(\varepsilon)$ exist such that*

$$\max_{P \in C}(\varphi(P) - \psi(f_n(P))) \le d - \varepsilon \quad \forall n \ge \bar{n}.$$

Proof. Suppose our statement is false. Then a strictly increasing sequence (n_i) in $\mathbb{N}$ and a sequence (P_i) in C exist, such that $\varphi(P_i) - \psi(f_{n_i}(P_i)) \ge d - 1/i$ for every index i.

Because of the compactness of $\mathcal{M}$ and $\mathcal{N}$, possibly by extracting a subsequence, we can assume (P_i) and $(f_{n_i}(P_i))$ to be converging. Then the point $P = \lim_i P_i$ belongs to $C \cap N^-_{\mathcal{M}}$, that is a contradiction. □

Lemma 4.1 can be naturally extended to the sets $N^+_{\mathcal{M}}, N^-_{\mathcal{N}}, N^+_{\mathcal{N}}$.

By defining a suitable tangent vector field the following result can be proved.

Lemma 4.2. *Let $D \subset \mathcal{M}$ be a non-empty open set, for which $D \cap K_\varphi = \emptyset$. Then, for each $\eta > 0$, a homeomorphism $F : \mathcal{M} \to \mathcal{M}$ exists such that $F = id$ on $\mathcal{M} - D$ and verifying $\eta \geq \varphi(P) - \varphi(F(P)) > 0$, $\eta \geq \|P - F(P)\| > 0$ for every $P \in D$.*

Proof. Consider the tangent vector field $v := -\lambda \nabla \varphi$ where $\lambda : \mathcal{M} \to [0, 1]$ is a regular function such that $\lambda = 0$ on $\mathcal{M} - D$ and $\lambda > 0$ in D. The flow diffeomorphism Φ_t of v takes each point $P \in \mathcal{M}$ to a new point $\Phi_t(P) \in \mathcal{M}$ (here we are using the regularity of φ). If $\bar{t}$ is small enough, we get the wanted property by setting $F = \Phi_{\bar{t}}$. □

The following two Lemmas 4.3 and 4.4 will be fundamental for the next results. Our goal is to "improve" a d-approximating sequence of homeomorphisms (f_n) by eliminating all the maximal trains that either do not begin or do not end at critical points. The basic idea is that we can do it by means of a small perturbation of the homeomorphisms, which does not create new wagons.

As a first step in this direction, we shall get the following lemma, showing how to eliminate the initial wagons (P, Q) when $P \in \mathcal{M} - K_\varphi$. The technical details of the proof of this lemma require the definition of two families $(D_i)_{i \in \mathbb{N}}$ and $(E_i)_{i \in \mathbb{N}}$ of disjoint open subsets of $\mathcal{M}$. We shall use these families in two distinct but similar procedures.

Lemma 4.3. *For every fixed d-approximating sequence (f_n), a new d-approximating sequence (g_n) exists such that, for every $(P, Q) \in \mathcal{M} \times \mathcal{N}$, we have*

i) *$(P, Q) \in W((g_n))$ if and only if $(P, Q) \in W((f_n))$ with $P \in K_\varphi \cup N^-_{\mathcal{M}}((f_n))$;*

ii) *$(Q, P) \in W((g_n))$ if and only if $(Q, P) \in W((f_n))$.*

The previous lemma assures that the nodes of the new sequence (g_n) we get are already nodes for (f_n) and that the initial node of every maximal train beginning in $\mathcal{M}$ is a critical point for the measuring function φ.

Proof. (*Lemma 4.3*) We use the symbol C to denote the compact set $K_\varphi \cup N^-_{\mathcal{M}}((f_n))$ and, for every $i > 0$, and the symbol D_i to denote the open set

$$D_i = C^{1/(2i)} - \overline{C^{1/(2i+2)}}$$

where C^α is the open dilation in $\mathcal{M}$ of C with radius α. Set $D_0 = \mathcal{M} - \overline{C^{1/2}}$. In Figure 8 the sets D_i and D_{i+1} are displayed (the sets E_i in the picture will be defined in the following).

C is the set where the given sequence (f_n) and the sequence (g_n) we are going to define will concide. We shall change (f_n) in $\mathcal{M} - C$ in such a way that $N^+_{\mathcal{M}}((g_n)) \subseteq C$.

Set $D = \bigcup_{i \in \mathbb{N}} D_i$. Since $\bar{D}_i \cap C = \emptyset$ for every i, because of Lemma 4.1, $\varepsilon_i > 0$ and $n_i \in \mathbb{N}$ exist such that

$$(4.1) \qquad \max_{P \in \bar{D}_i} (\varphi(P) - \psi(f_n(P)) \leq d - \varepsilon_i$$

for every $n \geq n_i$. We can also assume that $\lim_{i \to \infty} \varepsilon_i = 0$.

Lemma 4.2 assures the existence of a homeomorphism $F_i : \mathcal{M} \to \mathcal{M}$ such that

$$\frac{\varepsilon_i}{2} \geq \varphi(P) - \varphi(F_i(P)) > 0, \quad \frac{\varepsilon_i}{2} \geq \|P - F_i(P)\| > 0 \quad \text{for } P \in D_i$$

and $F_i(P) = P \Leftrightarrow \varphi(F_i(P)) = \varphi(P) \Leftrightarrow P \in \mathcal{M} - D_i$.

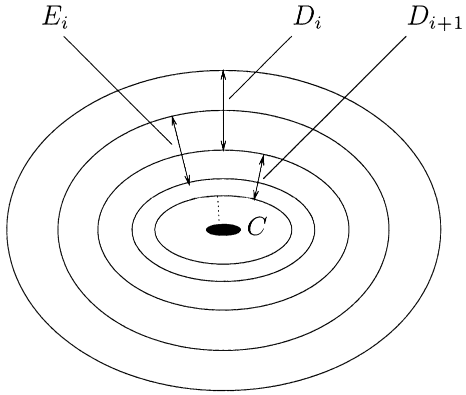

Figure 8. The sets D_i and E_i.

Since $D_i \cap D_j = \emptyset$ for $i \neq j$ and $F_i(P) = F_j(P) = P$ for $P \in \mathcal{M} - (D_i \cup D_j)$, the map $F : \mathcal{M} \to \mathcal{M}$ defined by setting $F(P) = F_i(P)$ on D_i and $F(P) = P$ elsewhere is a homeomorphism. Let us study the properties of trains for the new sequence $(\tilde{f}_n)$ where

$$\tilde{f}_n = f_n \circ F : \mathcal{M} \to \mathcal{N}.$$

Now, if $P \in \mathcal{M}$ and $Q \in \mathcal{N}$, we want to prove that $(P, Q) \in W((\tilde{f}_n))$ if and only if $(P, Q) \in W((f_n))$ with $P \notin D$ and that $(Q, P) \in W((\tilde{f}_n))$ if and only if $(Q, P) \in W((f_n))$.

As an informal note, we point out that in D_i the composition with F increases the absolute value of every negative jump by a quantity less than $\varepsilon_i/2$ and that, all over D, every positive jump decreases. By using the inequality (4.1), we can easily prove that $(\tilde{f}_n)$ is still a d-approximating sequence.

Let us study the properties of nodes and wagons of the new sequence $(\tilde{f}_n) \in S_H(\mathcal{M}, \mathcal{N})$. Note that $P\rho Q$ for the sequence $(\tilde{f}_n)$ if and only if $F(P)\rho Q$ for the sequence (f_n). So we have the following:

a) if $(P, Q) \in W((f_n))$ and $P \notin D$, then $\psi(Q) - \varphi(P) = d$, $P\rho Q$ for (f_n) and $P = F(P)$. It follows that $F^{-1}(P)\rho Q$ for $(\tilde{f}_n)$ with $F^{-1}(P) = P$. Therefore $(P, Q) \in W((\tilde{f}_n))$.

b) if $(P, Q) \in W((\tilde{f}_n))$ then $P\rho Q$ for $(\tilde{f}_n)$ and $\psi(Q) - \varphi(P) = d$. Hence

$$\psi(Q) - \varphi(F(P)) = d + \varphi(P) - \varphi(F(P)) \geq d.$$

Since $F(P)\rho Q$ for (f_n), it must be that $\psi(Q) - \varphi(F(P)) \leq d$ (as a simple consequence of (f_n) being a d-approximating sequence) and therefore $\varphi(F(P)) = \varphi(P)$. It follows that $P \notin D$, $F(P) = P$ and $(P, Q) \in W((f_n))$.

So we have proved that $(P, Q) \in W((\tilde{f}_n))$ if and only if $(P, Q) \in W((f_n))$ and $P \notin D$. We still have to examine the case the considered wagon begins at a point Q of $\mathcal{N}$. We have that

a$'$) if $(Q, P) \in W((f_n))$ then $P \in N^-_{\mathcal{M}}((f_n))$ and hence $P \notin D$, so that $P = F^{-1}(P)$. Moreover, since $F^{-1}(P)\rho Q$ for $(\tilde{f}_n)$ and $\varphi(P) - \psi(Q) = d$, we have $(Q, P) \in W((\tilde{f}_n))$.

b$'$) if $(Q, P) \in W((\tilde{f}_n))$ we have to prove that $(Q, P) \in W((f_n))$. This case requires more attention. We must verify that changing (f_n) into $(\tilde{f}_n)$ does not create new nodes for $N^-_{\mathcal{M}}$.

Since $P\rho Q$ for $(\tilde{f}_n)$ then $F(P)\rho Q$ for (f_n). If we prove that $F(P) = P$ then, as $\varphi(P) - \psi(Q) = d$, we get $(Q, P) \in W((f_n))$.

Suppose that $F(P) \neq P$ (and hence $P \in D$) and choose a sequence (P_n) of points in $\mathcal{M}$ such that $P_n \to P$. Then $i, \bar{n} \in \mathbb{N}$ will exist for which $P_n \in D_i$ and $F(P_n) \in D_i$ for every $n \geq \bar{n}$. From (4.1) it follows that $\varphi(F(P_n)) - \psi(f_n(F(P_n))) \leq d - \varepsilon_i$ for $n \geq n_i, \bar{n}$, and therefore

$$\begin{aligned} \varphi(P_n) - \psi(\tilde{f}_n(P_n)) &= (\varphi(P_n) - \varphi(F(P_n))) + (\varphi(F(P_n)) - \psi(f_n \circ F(P_n))) \\ &\leq \frac{\varepsilon_i}{2} + d - \varepsilon_i < d - \frac{\varepsilon_i}{2}. \end{aligned}$$

So $P \notin N^-_{\mathcal{M}}((\tilde{f}_n))$, against the hypothesis (Q, P) is a wagon for $(\tilde{f}_n)$. Therefore it must be that $F(P) = P$, and hence $(Q, P) \in W((f_n))$.

The sequence $(\tilde{f}_n)$ is not yet the one we wanted, since we have proved that $(P, Q) \in W((\tilde{f}_n))$ if and only if $(P, Q) \in W((f_n))$ and $P \in \mathcal{M} - D$, while we wanted to have $(P, Q) \in W((\tilde{f}_n))$ if and only if $(P, Q) \in W((f_n))$ and $P \in C$. By means of a procedure analogous to the one we used for constructing $(\tilde{f}_n)$, we shall now show that we can get a new sequence (g_n) for which the wanted property is fulfilled.

For every $i \in \mathbb{N}$ let E_i be the open set

$$E_i = C^{1/(2i+1)} - \overline{C^{1/(2i+3)}}$$

and define $E = \bigcup_{i \in \mathbb{N}} E_i$. Because of our definition, we can easily verify that $\bigcup_{i=0}^{\infty} \partial D_i \subset E$ and $\mathcal{M} - (D \cup E) = C$.

By applying the procedure we used for (f_n) and the set D to the sequence $(\tilde{f}_n)$ and to the set E, we get a new homeomorphism $F' : \mathcal{M} \to \mathcal{M}$ and then a new sequence

$(g_n) = (\tilde{f}_n \circ F')$ with the following property: $(P,Q) \in W((g_n))$ if and only if $(P,Q) \in W((\tilde{f}_n))$ with $P \in \mathscr{M} - E$ and $(Q,P) \in W((g_n))$ if and only if $(Q,P) \in W((f_n))$.

In conclusion, $(P,Q) \in W((g_n))$ if and only if $(P,Q) \in W((f_n))$ with $P \in \mathscr{M} - (D \cup E) = C$, while $(Q,P) \in W((g_n))$ if and only if $(Q,P) \in W((f_n))$. Hence our lemma is proved. □

By exchanging the role of $\mathscr{M}$ and $\mathscr{N}$ and considering the sequence (f_n^{-1}) in the previous lemma, we get

Lemma 4.4. *For every fixed d-approximating sequence (f_n) a new d-approximating sequence (g_n) exists such that, for every $(P,Q) \in \mathscr{M} \times \mathscr{N}$, we have*

i) $(P,Q) \in W((g_n))$ *if and only if* $(P,Q) \in W((f_n))$*;*

ii) $(Q,P) \in W((g_n))$ *if and only if* $(Q,P) \in W((f_n))$ *with* $Q \in K_\psi \cup N_{\mathscr{N}}^-((f_n))$.

This result, in analogy with the previous one, allows us to obtain a new sequence such that all the nodes in $N_{\mathscr{N}}^+$ are either critical points or nodes in $N_{\mathscr{N}}^-$.

In conclusion, the last two Lemmas 4.3 and 4.4 allow us to assume that all initial nodes of maximal trains are critical points. By applying the two previous lemmas to the size pairs $(\mathscr{M}, -\varphi)$ and $(\mathscr{N}, -\psi)$ we get the following two results, allowing us to assume that all final nodes of maximal trains are critical points.

Lemma 4.5. *For every fixed d-approximating sequence (f_n) a new d-approximating sequence (g_n) exists such that, for every $(P,Q) \in \mathscr{M} \times \mathscr{N}$, we have*

i) $(Q,P) \in W((g_n))$ *if and only if* $(Q,P) \in W((f_n))$ *with* $P \in K_\varphi \cup N_{\mathscr{M}}^+((f_n))$*;*

ii) $(P,Q) \in W((g_n))$ *if and only if* $(P,Q) \in W((f_n))$.

Lemma 4.6. *For every fixed d-approximating sequence (f_n) a new d-approximating sequence (g_n) exists such that, for every $(P,Q) \in \mathscr{M} \times \mathscr{N}$, we have*

i) $(Q,P) \in W((g_n))$ *if and only if* $(Q,P) \in W((f_n))$*;*

ii) $(P,Q) \in W((g_n))$ *if and only if* $(P,Q) \in W((f_n))$ *with* $Q \in K_\psi \cup N_{\mathscr{N}}^+((f_n))$.

5 The main result (weaker form)

By using the previous lemmas we can prove the existence of a d-approximating sequence of homeomorphisms whose maximal trains begin and end at critical points. As we pointed out in Remark 3.2, the existence of such a sequence allows us to establish a link between the natural pseudodistance and the distance between critical values of the measuring functions.

Lemma 5.1. *Assume $0 < d < +\infty$. For every sequence of homeomorphisms (f_n) in $S_H(\mathscr{M}, \mathscr{N})$ a new sequence (g_n) exists in $S_H(\mathscr{M}, \mathscr{N})$ such that all maximal trains begin and end at critical points of the measuring functions and $W((g_n)) \subseteq W((f_n))$.*

Proof. Set $\hat{H} = \{(h_n) \in S_H(\mathscr{M}, \mathscr{N}) \mid W((h_n)) \subseteq W((f_n))\}$. For every $(h_n) \in \hat{H}$ consider the set $T((h_n))$ of all maximal trains which either do not begin or do not end with critical points of the measuring functions. Define $k((h_n))$ as the length of the longest train in $T((h_n))$ (we set $k((h_n)) = 0$ if $T((h_n)) = \emptyset$). It should be noted that $\hat{H} \neq \emptyset$ since it contains at least (f_n).

Our lemma is proved if a $(g_n) \in \hat{H}$ exists such that $k((g_n)) = 0$. So choose one sequence $(g_n) \in \hat{H}$ such that $k((g_n)) = \min\{k((h_n)) \mid (h_n) \in \hat{H}\}$. Suppose $k((g_n)) > 0$. By applying Lemmas 4.3, 4.4, 4.5, 4.6 to the sequence (g_n) one after the other, we get a sequence $(\tilde{g}_n)$. Since all trains in $T((g_n))$ became strictly shorter by changing (g_n) into $(\tilde{g}_n)$, it must be that $k((\tilde{g}_n)) < k((g_n))$, against our assumption. □

We underline that the measuring functions take the set of all nodes of maximal trains for the new sequence (g_n) we got in previous lemma to a finite set of real numbers. In fact the length of the maximal trains is finite and the sets K_φ and K_ψ (to which the initial and final nodes of the maximal trains belong) are finite sets.

Now we can prove the main result in this section. As we previously said, this result will allow us to obtain an interesting relation between the natural pseudodistance and the critical values of the measuring functions.

Theorem 5.2. *Assume that $\mathscr{M}$ and $\mathscr{N}$ are two smooth and closed homeomorphic manifolds and that $\varphi : \mathscr{M} \to \mathbb{R}$ and $\psi : \mathscr{N} \to \mathbb{R}$ are two smooth Morse functions. Then, if d denotes the natural size pseudodistance between the size pairs $(\mathscr{M}, \varphi)$ and $(\mathscr{N}, \psi)$, a positive integer k exists for which one of the following properties holds:*

i) *k is odd and kd equals the distance between a critical value of φ and a critical value of ψ;*

ii) *k is even and kd equals either the distance between two critical values of φ or the distance between two critical values of ψ.*

Proof. If $d = 0$ then properties i) and ii) are trivially verified for $k = 1$ and $k = 2$. Therefore we can assume $d > 0$. By applying Lemma 5.1, choose a sequence in $S_H(\mathscr{M}, \mathscr{N})$ such that all its maximal trains begin and end at critical points, and take a maximal train $(N_0, \ldots, N_k)$ of this sequence. We have $\omega(N_k) - \omega(N_0) = kd$ with $N_0, N_k \in K_\omega$, where ω is the usual extension to $\mathscr{M} \cup \mathscr{N}$ of the measuring functions. This equality ends our proof. □

Notice that our theorem do not set hypotheses about the dimensions of the considered manifolds. Moreover it provides information about the admissible values for the natural pseudodistance, although δ is defined by means of the set of all possible homeomorphisms between $\mathscr{M}$ and $\mathscr{N}$, which may be quite difficult to study.

The previous result leads to the following

Definition 5.3. We call *analytic folding number* the smallest positive integer k for which either i) or ii) in Theorem 5.2 holds.

We have a particularly simple case when an optimal homeomorphism exists.

Theorem 5.4. *Assume that* $\mathcal{M}$ *and* $\mathcal{N}$ *are two smooth and closed homeomorphic manifolds and that* $\varphi : \mathcal{M} \to \mathbb{R}$ *and* $\psi : \mathcal{N} \to \mathbb{R}$ *are two smooth Morse functions. If an optimal homeomorphism* $f \in H(\mathcal{M}, \mathcal{N})$ *between the size pairs* $(\mathcal{M}, \varphi)$ *and* $(\mathcal{N}, \psi)$ *exists, then the natural size pseudodistance* d *equals the distance between a critical value of* φ *and a critical value of* ψ.

Proof. If $d = 0$ then our thesis is trivially verified. Therefore we can assume $d > 0$. Choose the trivial d-approximating sequence (f_n) with $f_n = f$ for every index n. For this sequence the trains can only have length 1. Hence, by applying Lemma 5.1 we get our result. □

We point out that the preceding result does not require the knowledge of an optimal homeomorphism but only of its existence. The previous Examples 1.3 and 1.4 display two simple cases to which Theorem 5.4 applies.

Remark 5.5. Examples 2.2 and 2.3 show that we cannot avoid the hypothesis of the existence of an optimal homeomorphism in Theorem 5.4. In these cases we cannot apply Theorem 5.4, since there is no optimal homeomorphism and, in fact, the natural pseudodistance $d = \delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi))$ is *not* equal to the distance between a critical value of φ and a critical value of ψ.

Anyway, Theorem 5.2 applies, and we observe that, in our examples, d is *half* the distance between two critical values of φ.

Remark 5.6. Our main result is given by Theorem 5.2 (together with the extension we shall give in Section 6). Unfortunately, as we have seen, some boring technical passages are needed in order to prove that. The technical details must not hide the simplicity of our basic idea, that is the following: when we have a maximal train that does not begin (or end) at a critical point, it is always possible to eliminate the initial (final) node and to make the train shorter by slightly modifying the homeomorphisms in the considered d-approximating sequence. If we assume that the measuring functions are the "height" with respect to a suitable embedding of our manifolds into $\mathbb{R}^k$, then our task is to raise (lower) slightly the neighborhood of the first (last) node of the train. This procedure is quite delicate, both because we have to manage a potentially infinite set of maximal trains and because we want the shortening of a train not to cause the lengthening of another one. This compels us to carefully evaluate the displacement we are performing by providing all the lemmas given in Section 4 and also Lemma 5.1. In particular the proof of Lemma 4.3 appears to be a little bit tricky, since we have to manage an infinite number of local changes.

Lemma 5.1 requires the construction of a new sequence in order to obtain the desired property. It may be interesting, anyway, to point out that a weaker property actually holds for the original sequence (f_n). Formally, assume $(f_n) \in S_H(\mathcal{M}, \mathcal{N})$. Then we can prove that there is a train for (f_n) whose ends are critical points for the measuring functions. However, this train is *not* guaranteed to be maximal.

The ideas underlying the proof of this statement are quite similar to the ones used for proving our key Lemma 5.1, and we shall skip the technical details. We could have proved Theorem 5.2 by using such a statement in place of Lemma 5.1. We did not do that, since Lemma 5.1 appears to be much more useful for getting further results.

6 Weakening the hypotheses about the regularity of measuring functions and manifolds

Until now we have considered smooth Morse measuring functions and smooth manifolds. In this section we shall prove that this regularity can be largely weakened.

First of all we give the following useful result:

Lemma 6.1. *Let* $(\mathcal{M}, \varphi)$ *and* $(\mathcal{N}, \psi)$ *be two size pairs. Consider two sequences* (φ_n) *and* (ψ_n) *of measuring functions on* $\mathcal{M}$ *and* $\mathcal{N}$*, converging to* φ *and* ψ *with respect to the* C^0*-norm. Then*

$$\delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi)) = \lim_n \delta((\mathcal{M}, \varphi_n), (\mathcal{N}, \psi_n)).$$

Proof. It follows immediately from our definitions. □

We are now going to prove the following generalization of Theorem 5.2.

Theorem 6.2. *Assume that* $\mathcal{M}$ *and* $\mathcal{N}$ *are two closed homeomorphic manifolds of class* C^1 *and that* $\varphi : \mathcal{M} \to \mathbb{R}$ *and* $\psi : \mathcal{N} \to \mathbb{R}$ *are two functions of class* C^1*. Then, if* d *denotes the natural pseudodistance between the size pairs* $(\mathcal{M}, \varphi)$ *and* $(\mathcal{N}, \psi)$*, a positive integer* k *exists for which one of the following properties holds:*

i) *k is odd and kd equals the distance between a critical value of φ and a critical value of ψ;*

ii) *k is even and kd equals either the distance between two critical values of φ or the distance between two critical values of ψ.*

Proof. First of all we weaken the hypothesis on the measuring functions. Assume two size pairs $(\overline{\mathcal{M}}, \varpi)$ and $(\overline{\mathcal{N}}, \varpi)$ are given ($\overline{\mathcal{M}} \cap \overline{\mathcal{N}} = \emptyset$), where $\varpi : \overline{\mathcal{M}} \cup \overline{\mathcal{N}} \to \mathbb{R}$ is a function of class C^1 and $\overline{\mathcal{M}}, \overline{\mathcal{N}}$ are assumed to be smooth. It is well-known that the set of smooth Morse functions is dense in the set of all functions of class C^1 with respect to the C^1-norm (see, e.g., [15]). Hence, for every $n \in \mathbb{N} - \{0\}$, a smooth Morse function $\omega_n : \overline{\mathcal{M}} \cup \overline{\mathcal{N}} \to \mathbb{R}$ exists such that

$$\|\omega_n - \varpi\|_{C^1} \leq \frac{1}{n}.$$

Moreover, since the size pairs $(\overline{\mathcal{M}}, \omega_n)$ and $(\overline{\mathcal{N}}, \omega_n)$ satisfy the hypotheses of Theorem 5.2, two critical points N_n and N_n' for ω_n and a positive integer k_n exist, such that

$$k_n \delta((\bar{\mathscr{M}}, \omega_n), (\bar{\mathscr{N}}, \omega_n)) = |\omega_n(N_n) - \omega_n(N_n')|.$$

Since $\bar{\mathscr{M}} \cup \bar{\mathscr{N}}$ is compact and the differential $d\omega_n$ converges to $d\bar{\omega}$, two critical points N and N' for $\bar{\omega}$ exist such that $N_n \to N$, $N_n' \to N'$ (possibly by extracting subsequences). Furthermore, because of the boundness of the set $\{k_n\}$, we can assume that a positive integer k exists such that $k_n = k$ for every index n.

Since $\omega_n(N_n) \to \bar{\omega}(N)$ and $\omega_n(N_n') \to \bar{\omega}(N')$, then Lemma 6.1 implies

$$k\delta((\bar{\mathscr{M}}, \bar{\omega}), (\bar{\mathscr{N}}, \bar{\omega})) = \lim_n k_n \delta((\bar{\mathscr{M}}, \omega_n), (\bar{\mathscr{N}}, \omega_n)) = |\bar{\omega}(N) - \bar{\omega}(N')|.$$

We have thus proved that Theorem 5.2 can be extended to measuring functions of class C^1. In the following we shall prove that even the manifolds can be assumed to be only of class C^1. Therefore, let us make the assumption that $(\mathscr{M}, \varphi)$ and $(\mathscr{N}, \psi)$ are two size pairs of class C^1 (thus $\mathscr{M}$ and $\mathscr{N}$ are manifolds of class C^1 and φ and ψ are measuring functions of class C^1). Then two smooth manifolds $\bar{\mathscr{M}}, \bar{\mathscr{N}}$ ($\bar{\mathscr{M}} \cap \bar{\mathscr{N}} = \emptyset$) and two diffeomorphisms $\alpha_1 : \bar{\mathscr{M}} \to \mathscr{M}, \alpha_2 : \bar{\mathscr{N}} \to \mathscr{N}$ of class C^1 exist (see, e.g., [13]). It is easily verified that $\delta((\mathscr{M}, \varphi), (\bar{\mathscr{M}}, \varphi \circ \alpha_1)) = 0$ and $\delta((\mathscr{N}, \psi), (\bar{\mathscr{N}}, \psi \circ \alpha_2)) = 0$.

Now apply the extension of Theorem 5.2 for measuring functions of class C^1 to the size pairs $(\bar{\mathscr{M}}, \varphi \circ \alpha_1)$ and $(\bar{\mathscr{N}}, \psi \circ \alpha_2)$. Call $\bar{\omega}$ the usual extension of $\varphi \circ \alpha_1$ and $\psi \circ \alpha_2$ to $\bar{\mathscr{M}} \cup \bar{\mathscr{N}}$, and α the extension of α_1 and α_2 to $\bar{\mathscr{M}} \cup \bar{\mathscr{N}}$. Consider two critical points N and N' for $\bar{\omega}$ such that either i) or ii) holds with respect to the critical values $\bar{\omega}(N), \bar{\omega}(N')$. Obviously, the points $\alpha(N)$ and $\alpha(N')$ belong to $\mathscr{M} \cup \mathscr{N}$ and are critical points for ω (i.e. the extension of φ and ψ). Since $\delta((\bar{\mathscr{M}}, \varphi \circ \alpha_1), (\bar{\mathscr{N}}, \psi \circ \alpha_2)) = \delta((\mathscr{M}, \varphi), (\mathscr{N}, \psi))$, we easily get the result. □

In a similar way, also Theorem 5.4 can be extended to the following.

Theorem 6.3. *Assume that $\mathscr{M}$ and $\mathscr{N}$ are two closed homeomorphic manifolds of class C^1 and that $\varphi : \mathscr{M} \to \mathbb{R}$ and $\psi : \mathscr{N} \to \mathbb{R}$ are of class C^1. If an optimal homeomorphism $f \in H(\mathscr{M}, \mathscr{N})$ between the size pairs $(\mathscr{M}, \varphi)$ and $(\mathscr{N}, \psi)$ exists, then the natural pseudodistance d equals the distance between a critical value of φ and a critical value of ψ.*

7 Conclusions and final remarks

In this paper we have proved that the natural pseudodistance and the critical values of the measuring functions involved are strongly related, even when we cannot obtain an optimal homeomorphism f with respect to the operator Θ. In fact, Theorem 6.2 shows that the natural pseudodistance is always a submultiple of the distance between two suitable critical values of the measuring functions.

In two following papers, using the tools given in this work, we shall prove that stronger constraints can be obtained for the analytical folding numbers, when the considered manifolds are curves or surfaces.

Acknowledgment. We wish to thank M. Ferri, F. Cagliari, M. Mulazzani, R. Kusner and M. Katsoulakis for their helpful discussions, and R. Weiss, R. Manmatha and C. Ravela for their fruitful advice and kind hospitality given at the University of Massachussetts in Amherst, which allowed us to start drawing up this paper. Anyway, the authors are the only people responsible for any mistakes.

We are grateful to H. Melville for his invaluable support.

Finally, thanks to Marina A. Rangelova.

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Received January 22, 2002

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