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NATURAL PSEUDODISTANCES BETWEEN CLOSED SURFACES

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Abstract. Let us consider two closed surfaces $\mathcal{M}$, $\mathcal{N}$ of class C^1 and two functions $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$ of class C^1, called measuring functions. The natural pseudodistance d between the pairs $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ is defined as the infimum of $\Theta(f) \stackrel{def}{=} \max_{P\in\mathcal{M}} |\varphi(P) - \psi(f(P))|$, as f varies in the set of all homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. In this paper we prove that the natural pseudodistance equals either $|c_1 - c_2|$ or $\frac{1}{2}|c_1 - c_2|$, or $\frac{1}{3}|c_1 - c_2|$, where c_1 and c_2 are two suitable critical values of the measuring functions. This equality shows that a previous relation between natural pseudodistance and critical values obtained in general dimension can be improved in the case of closed surfaces. Our result is based on a theorem by Jost and Schoen concerning harmonic maps between surfaces.

Introduction

The *natural pseudodistance* is a new variational approach to the comparison of manifolds endowed with real-valued functions defined on them. In [2] we proved a result about the values that such a pseudodistance δ can take in general dimension. In this work we focus on the 2-dimensional case, showing that the previous result can be improved in the case of closed surfaces. Assuming that two homeomorphic closed manifolds $\mathcal{M}$ and $\mathcal{N}$ of class C^1 are given together with two functions $\varphi : \mathcal{M} \to \mathbb{R}$, $\psi : \mathcal{N} \to \mathbb{R}$ of class C^1 (called *measuring functions*), we consider the value

$$\delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi)) \stackrel{def}{=} \inf_{f\in H(\mathcal{M},\mathcal{N})} \max_{P\in\mathcal{M}} |\varphi(P) - \psi(f(P))|,$$

where $H(\mathcal{M},\mathcal{N})$ denotes the set of all homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. The number $d = \delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi))$ is called the *natural pseudodistance between the pairs* $(\mathcal{M}, \varphi)$ *and* $(\mathcal{N}, \psi)$ (called *size pairs*).

The closeness of d to zero means that there are homeomorphisms for which the difference between the values taken by the measuring functions at corresponding points is arbitrarily small. On the other hand, if the infimum is large, we find that every homeomorphism between the considered manifolds must change the values taken by our measuring function considerably.

In [2] we proved (Theorem 6.2) that a suitable multiple of d by a positive integer k coincides with the distance between two critical values of the functions φ, ψ. It is interesting to observe that in every known example, the minimum possible value for k is 1 or 2. In this paper we shall show that, in the 2-dimensional case, the minimum value for the integer k is either 1, 2, or 3. We remark that an analogous statement has been proved in [4] for curves, too, using different techniques, but in this case we are able to prove that only the values 1 and 2 are possible.

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Besides its intrinsic interest from a purely mathematical point of view, the natural pseudodistance between closed surfaces associated with measuring functions can also be used for shape comparison purposes, together with the "twin" and strictly related concept of *size function*. For more theoretical details and examples of practical applications we refer to [1, 10, 18, 19, 20, 21].

In the following Section 1 we sketch the main ideas in this paper. In Section 2 we give the main definitions and some examples, while in Section 3 further examples are given, highlighting some characteristic phenomena. In Section 4 the concepts of train and minimal d-approximating sequence are illustrated, together with some related results. In Section 5 we prove our main result (Theorem 5.7) about the natural pseudodistance between closed surfaces endowed with measuring functions. In Section 6 open problems and further research are briefly described.

1. The point of this paper

As reported in the previous section, it was proved in [2] that the natural pseudodistance between size pairs always equals $|c_1 - c_2|/k$, where c_1, c_2 are two suitable critical values of the measuring functions and k is an appropriate integer number. The minimum possible value for k is called the *analytic folding number*.

It is interesting to observe that in every known example, the analytic folding number is 1 or 2.

Two questions naturally arise: Are there examples showing an analytic folding number strictly greater than 2? Is this question related to the dimension of our manifolds?

In this paper we take a first step forward in order to answer these questions.

It is important to observe right now that the attempt to minimize the change $\Theta(f) \stackrel{def}{=} \max_{P\in\mathcal{M}} |\varphi(P) - \psi(f(P))|$ in the measuring functions under the action of f does not, in general, lead to a homeomorphism, as we are going to show in the following section. Degeneracies can arise, and hence we cannot confine ourselves to studying a single optimal homeomorphism. Instead of a single homeomorphism, approximating sequences of homeomorphisms must be considered. In some senses, "optimal" approximating sequences (f_i) of homeomorphisms exist, converging to relations that represent the best way to take one manifold to another with respect to the change in the measuring functions. The study of these relations leads us to define the concept of train of "limit d-jumps", describing some degeneracies corresponding to the sequence (f_i). As we are going to see in the following sections, the properties of these structures imply the properties of the analytic folding number.

How can we study these properties?

In [2] local deformations were used, based on the flow diffeomorphism of the gradient of the measuring functions but, unfortunately, this approach does not seem to be sufficient to answer the questions we posed. The main idea of this paper is to use the theory of harmonic maps to confront the bidimensional case. A result by Jost and Schoen exists, allowing us to study the case of surfaces.

We shall proceed this way. We shall consider each "optimal" sequence (f_i) of homeomorphisms between the manifolds $\mathcal{M}$ and $\mathcal{N}$ we are examining, where optimal means that $\inf_i \max_{P\in\mathcal{M}} |\varphi(P) - \psi(f_i(P))|$ equals $\delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi))$. Then we shall describe the degeneracies related to (f_i) using a train of "limit d-jumps", and assume that the degeneracies of (f_i) are minimal with respect to a suitable order $\preceq$ we are going to define.

Finally, we shall apply a local harmonization procedure to each f_i far away from the critical points, using Jost and Schoen's Theorem. The key remark will be that the change we are going to apply produces a new sequence that is "smaller" than (f_i) with respect to $\preceq$. Since (f_i) will already be minimal, some further information about the length of the trains of d-jumps for (f_i) will be derived, implying the main result obtained in this paper.

Some technicalities will be necessary in order to use our ideas in practice, but the key point is simply the possibility (in some senses unexpected) of reducing the change of the measuring functions by locally decreasing the energy of the transformations we use between our manifolds. The following sections will formalize the ideas we have just described.

2. The natural pseudodistance

2.1. **The main definition.** The definition of natural pseudodistance can be introduced for n-dimensional manifolds. Let us consider the set $Size_n$ of all pairs $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a closed n-manifold of class C^k and $\varphi : \mathcal{M} \to \mathbb{R}$ is a function of class C^k. We shall call $(\mathcal{M}, \varphi)$ an (n-dimensional) *size pair* of class C^k and φ a *measuring function*.

Assume $(\mathcal{M}, \varphi), (\mathcal{N}, \psi)$ are two size pairs. $H(\mathcal{M}, \mathcal{N})$ will denote the set of all homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$.

Definition 2.1. If $H(\mathcal{M}, \mathcal{N}) \neq \emptyset$, the function $\Theta : H(\mathcal{M}, \mathcal{N}) \to \mathbb{R}$

$$\Theta(f) = \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$$

is called *natural size measure* with respect to the measuring functions φ and ψ.

In other words, Θ measures how much f changes the values taken by the measuring functions, at corresponding points.

Definition 2.2. We shall call *natural size pseudodistance* the pseudodistance $\delta : Size_n \times Size_n \to \mathbb{R} \cup \{+\infty\}$ so defined:

$$\delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi)) = \begin{cases} \inf_{f \in H(\mathcal{M},\mathcal{N})} \Theta(f) & \text{if } H(\mathcal{M}, \mathcal{N}) \neq \emptyset \\ +\infty & \text{otherwise.} \end{cases}$$

In the following, the symbol d will denote the value of the natural pseudodistance δ computed between the pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ that we are considering. As we previously explained, this pseudodistance gives a method for comparing two manifolds with respect to the measuring functions chosen.

We point out that δ is not a distance, since two size pairs can have a vanishing pseudodistance without being equal. On the other hand, the symmetry property and the triangle inequality can be trivially proved.

Remark 2.3. The presence of the word "size" in our definitions is due to the link existing between the pseudodistance δ, size functions and size homotopy groups (cf. [7, 11]). However, for the sake of simplicity, we shall often drop the word "size" in the expressions "natural size measure" and "natural size pseudodistance". The term "natural" is used in order to distinguish the pseudodistance studied here from some pseudodistances we can define between the submanifolds of the Euclidean space (cf. [6]) and from other pseudodistances between manifolds paired with measuring functions.

In spite of the considerable difficulty in computing natural size pseudodistances, the following result holds for the general dimension n (cf. [2]):

Theorem 2.4. *Assume that $\mathcal{M}$ and $\mathcal{N}$ are two homeomorphic closed manifolds of class C^1 and that $\varphi : \mathcal{M} \to \mathbb{R}$ and $\psi : \mathcal{N} \to \mathbb{R}$ are two functions of class C^1. Then, if d denotes the natural pseudodistance between the size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$, a positive integer k exists for which one of the following properties holds:*

(i) *k is odd and kd equals the distance between a critical value of φ and a critical value of ψ;*
(ii) *k is even and kd equals either the distance between two critical values of φ or the distance between two critical values of ψ.*

The smallest positive integer k for which either (i) or (ii) of Theorem 2.4 holds is called the *analytic folding number* for the pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$. It is interesting to observe that in every known example, the analytic folding number is 1 or 2.

In this paper we shall prove that in the case of two homeomorphic closed surfaces of class C^1, endowed with C^1 measuring functions, the analytic folding number always equals either 1, 2 or 3. This fact, besides showing a particular property of the 2-dimensional case, allows us to make a direct computation of natural pseudodistances for closed surfaces easier.

However, the hypothesis $n = 2$ will be not used until Section 5.

In the following Section 3, we shall show that the infimum of $\Theta(f)$ varying f in $H(\mathcal{M}, \mathcal{N})$ is not always a minimum. When such an infimum is also a minimum, we shall say that each homeomorphism $f \in H(\mathcal{M}, \mathcal{N})$ with $d = \Theta(f)$ is an *optimal homeomorphism.*

In the case where an optimal homeomorphism exists, the following result holds (Theorem 6.3 in [2]).

Theorem 2.5. *Assume that $\mathcal{M}$ and $\mathcal{N}$ are two C^1 closed homeomorphic manifolds and that $\varphi : \mathcal{M} \to \mathbb{R}$ and $\psi : \mathcal{N} \to \mathbb{R}$ are of class C^1. If an optimal homeomorphism $f \in H(\mathcal{M}, \mathcal{N})$ between the size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$ exists, then the natural pseudodistance $d = \delta((\mathcal{M}, \varphi), (\mathcal{N}, \psi))$ equals the distance between a critical value of φ and a critical value of ψ.*

N.B.: For the sake of conciseness, all through this paper we shall use the expression "closed surface" to mean a closed 2-manifold (we shall *not* require this manifold to be connected).

In order to simplify our notations, we shall assume that the manifolds $\mathcal{M}$ and $\mathcal{N}$ do not meet, and that the corresponding measuring functions are obtained by restriction of a function $\omega : \mathcal{M} \cup \mathcal{N} \to \mathbb{R}$, so that $\varphi = \omega_{|\mathcal{M}}$ and $\psi = \omega_{|\mathcal{N}}$. In this way we can use just one symbol to denote both measuring functions. These hypotheses are not restrictive, since we can always replace the size pair $(\mathcal{N}, \psi)$ with a new size pair $(\hat{\mathcal{N}}, \hat{\psi})$, having vanishing pseudodistance from the previous one and such that $\mathcal{M} \cap \hat{\mathcal{N}} = \emptyset$. Sometimes, when not confusing, we shall use the symbol ω to denote both $\omega_{|\mathcal{M}}$ and $\omega_{|\mathcal{N}}$.

Moreover, it is easy to prove that, for every 2-dimensional size pair $(\mathcal{M}, \omega)$ of class C^k, an integer m and an embedding $g : \mathcal{M} \to \mathbb{R}^m$ of class C^k exist such that $x_m(P) = \omega(g^{-1}(P))$ for each point $P \in g(\mathcal{M})$. If ω is Morse (i.e., smooth and having invertible Hessian at each critical point), we can assume that x_m is Morse on $g(\mathcal{M})$, too. In other words, there is no lack of generality in assuming that the measuring functions associated with the studied closed surfaces $\mathcal{M}, \mathcal{N}$ are obtained by restriction of the x_m-coordinate in $\mathbb{R}^m$. Sometimes, when not confusing, we shall use the symbol x_m to denote both $x_{m|\mathcal{M}}$ and $x_{m|\mathcal{N}}$ and use the expression "height of a point". For the sake of clarity, in our examples and figures we shall often assume that our measuring function is the z-coordinate in $\mathbb{R}^3$.

Example 2.6. In $\mathbb{R}^3$ consider the unit sphere $\mathcal{S}$ of equation $x^2+y^2+z^2=1$ and the ellipsoid $\mathcal{E}$ of equation $x^2+4y^2+9z^2=1$. On $\mathcal{S}$ and $\mathcal{E}$ consider respectively the measuring functions φ and ψ that take every point of $\mathcal{S}$ and $\mathcal{E}$ to the Gaussian curvature of the considered manifold at that point. We have $\delta\left((\mathcal{S},\varphi),(\mathcal{E},\psi)\right)=35$. In fact $\varphi(\mathcal{S})=\{1\}$, while $\psi(\mathcal{E})=[4/9,36]$, and therefore for every $f\in H\left(\mathcal{S},\mathcal{E}\right)$ it results that $\Theta(f)=35$.

Example 2.7. Consider the two tori $\mathcal{T},\mathcal{T}'\subset\mathbb{R}^3$ generated by the rotation around the y-axis of the circles lying in the plane yz, with centres $A=(0,0,3)$ and $B=(0,0,4)$, and radii 2 and 1, respectively (see Figure 1). As a measuring function φ (resp. φ') on $\mathcal{T}$ (resp. on $\mathcal{T}'$) we take the restriction to $\mathcal{T}$ (resp. to $\mathcal{T}'$) of the function $\zeta:\mathbb{R}^3\to\mathbb{R}$, $\zeta(x,y,z)=z$. We point out that, for both $\mathcal{T}$ and $\mathcal{T}'$, the image of the measuring function is the closed interval $[-5,5]$. We can easily prove that the natural size pseudodistance between $(\mathcal{T},\varphi)$ and $(\mathcal{T}',\varphi')$ is 2 (for a proof involving size homotopy groups see [11]). Moreover, the homeomorphism f, taking each point of $\mathcal{T}$ to the point having the same toroidal coordinates in $\mathcal{T}'$, has natural size measure $\Theta(f)=2$.

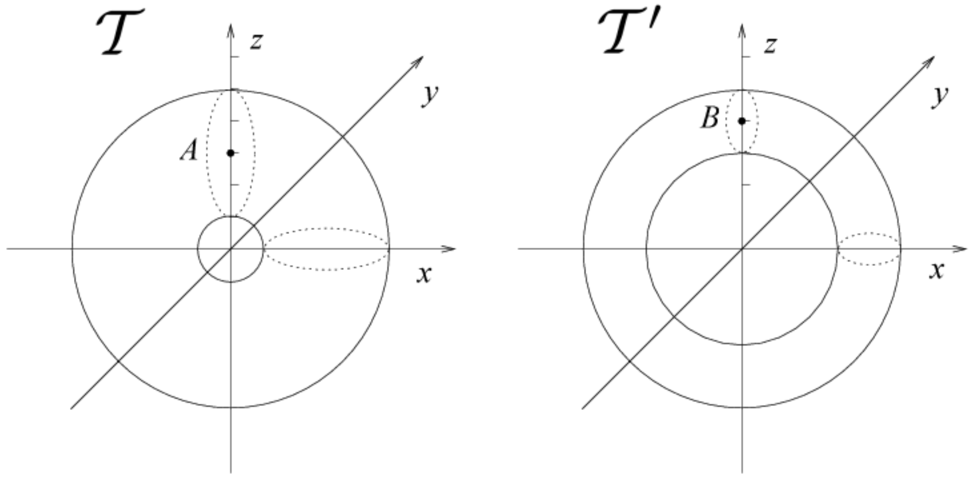

FIGURE 1. In this case an optimal (i.e. minimizing Θ) homeomorphism exists and $d=2$; d equals the distance between a critical value of φ and a critical value of φ'.

In general, d is far from being easily computable as in the previous Examples 2.6 and 2.7. In Example 2.6, for every homeomorphism $f\in H(\mathcal{S},\mathcal{E})$ we have that $\Theta(f)$ equals the Hausdorff distance $\delta_H\left(\varphi(\mathcal{S}),\psi(\mathcal{E})\right)$ between the sets $\varphi(\mathcal{S})$ and $\psi(\mathcal{E})$ in $\mathbb{R}$. Now it is clear that the natural size pseudodistance $\delta\left((\mathcal{M},\varphi),(\mathcal{N},\psi)\right)$ is always greater than or equal to $\delta_H\left(\varphi(\mathcal{M}),\psi(\mathcal{N})\right)$ and therefore $\Theta\left(f\right)$ must be the natural

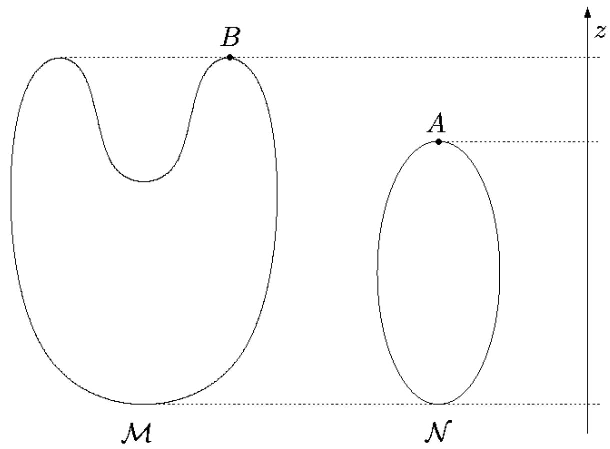

FIGURE 2. The natural pseudodistance between the size pairs $(\mathcal{M}, z)$ and $(\mathcal{N}, z)$ is $z(B) - z(A)$.

size pseudodistance we want to compute. We also point out that, in Example 2.6, the images of φ and ψ are different sets and so the natural size pseudodistance is trivially positive.

In Example 2.7 the natural size pseudodistance is strictly greater than the (vanishing) Hausdorff distance between the images of the two measuring functions.

Computing natural size pseudodistances is usually difficult. For this reason the concepts of *size function* and *size homotopy group* have been developed, making it easier to compute the value d, using some lower-bound theorems. Anyway, here we cannot illustrate these strongly correlated concepts, and hence we refer to [6, 7, 10, 11] for more details.

3. Some interesting examples about curves and surfaces

For the sake of clarity, even if this paper focuses on the bidimensional case, we shall begin our formal treatment from 1-dimensional examples.

Example 3.1. The first example we give is shown in Figure 2. Here $\mathcal{M}$ and $\mathcal{N}$ are smooth closed curves in $\mathbb{R}^3$, embedded in the xz-plane. It is clear that the natural pseudodistance d between the size pairs $(\mathcal{M}, z)$ and $(\mathcal{N}, z)$ equals $z(B) - z(A)$, that is, the distance between a critical value of $z_{|\mathcal{M}}$ and a critical value of $z_{|\mathcal{N}}$.

In this example no optimal homeomorphism exists, since it ought to take both the maximum points for $z_{|\mathcal{M}}$ to A, against injectivity.

Example 3.2. Let us consider the smooth closed curves $\mathcal{M}$ and $\mathcal{N}$ in Figure 3. The points A and B are critical points of the function z and $z(C) = \frac{1}{2}(z(A) + z(B)) = z(G)$. We want to prove that the natural pseudodistance between the size pairs $(\mathcal{M}, z)$ and $(\mathcal{N}, z)$ takes the value

$$d = \frac{1}{2}(z(A) - z(B))$$

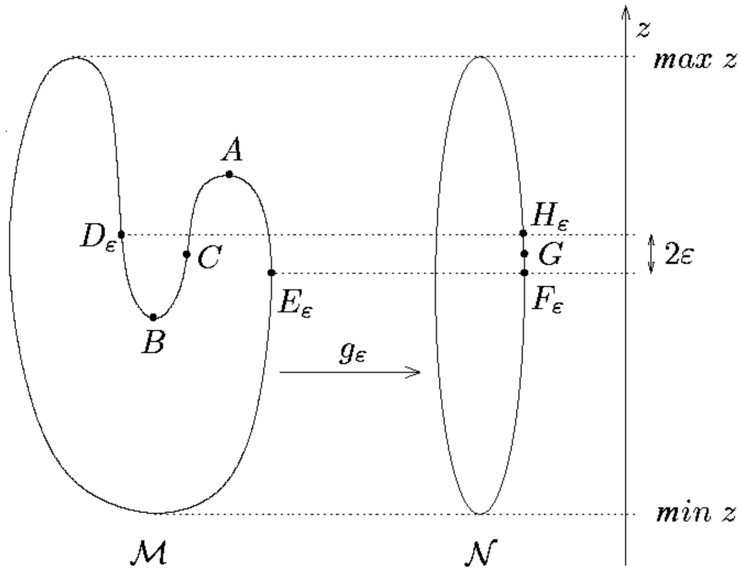

FIGURE 3. Construction of the homeomorphism g_ε for which $\Theta(g_\varepsilon) \leq d + \varepsilon$.

and that no optimal homeomorphism exists. In order to do that we shall construct a sequence of homeomorphisms (f_i) for which $\lim_i \Theta(f_i) = \frac{1}{2}(z(A) - z(B))$, and show that $\Theta(f) > \frac{1}{2}(z(A) - z(B))$ for every homeomorphism $f \in H(\mathcal{M}, \mathcal{N})$.

Let us start by proving that, for every $\varepsilon > 0$, a homeomorphism $g_\varepsilon : \mathcal{M} \to \mathcal{N}$ exists, such that $\Theta(g_\varepsilon) \leq \frac{1}{2}(z(A) - z(B)) + 2\varepsilon$. Consider the points D_ε, E_ε, H_ε and F_ε in Figure 3, verifying $z(D_\varepsilon) = z(H_\varepsilon) = z(C) + \varepsilon$ and $z(E_\varepsilon) = z(F_\varepsilon) = z(C) - \varepsilon$. We choose a homeomorphism g_ε, taking the arc $D_\varepsilon C E_\varepsilon$ to the arc $H_\varepsilon G F_\varepsilon$ in such a way that $g_\varepsilon(D_\varepsilon) = H_\varepsilon$ and $g_\varepsilon(E_\varepsilon) = F_\varepsilon$. Outside the arc $D_\varepsilon C E_\varepsilon$ in $\mathcal{M}$ we define g_ε by taking every point P to a point $g_\varepsilon(P)$, verifying $z(P) = z(g_\varepsilon(P))$.

For every $i \in \mathbb{N} - \{0\}$ we set $f_i = g_{1/i}$. It is easy to prove that

$$\lim_i \Theta(f_i) = \frac{1}{2}(z(A) - z(B)).$$

Now we have only to verify that no homeomorphism between $\mathcal{M}$ and $\mathcal{N}$ exists for which $\Theta(f) \leq \frac{1}{2}(z(A) - z(B))$. If such a homeomorphism existed, for every $P \in \mathcal{M}$ we would have

$$|z(P) - z(f(P))| \leq \frac{z(A) - z(B)}{2}$$

and hence $z(f(A)) \geq z(G) \geq z(f(B))$. Therefore we could easily find points $P \in \mathcal{M}$, for which $|z(P) - z(f(P))| > \frac{1}{2}(z(A) - z(B))$, contradicting our assumption.

Example 3.3. Consider the size pairs $(\mathcal{M}, \omega)$ and $(\mathcal{N}, \omega)$ in Figure 4, where $\mathcal{M}$ and $\mathcal{N}$ are smooth surfaces embedded into $\mathbb{R}^3$. We want to prove that the natural pseudodistance between these size pairs takes the value $1/2$.

The critical points $P, Q \in \mathcal{M}$ for which $\omega(P) = 1$ and $\omega(Q) = 0$ belong to the displayed closed set $K \subset \omega^{-1}([0, 1])$. First of all, we shall prove that $d \geq 1/2$, by

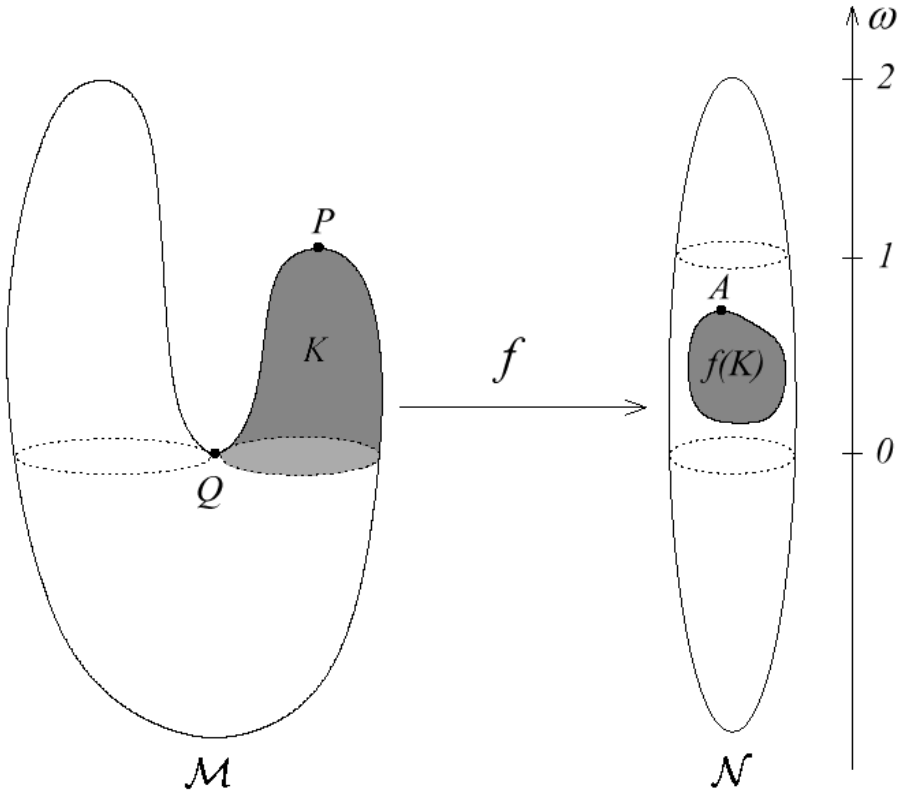

FIGURE 4. The natural pseudodistance between these size pairs is $d = 1/2$.

showing that for every homeomorphism $f : \mathcal{M} \to \mathcal{N}$ the inequality

$$\Theta(f) > \frac{1}{2}(\omega(P) - \omega(Q)) = \frac{1}{2}$$

holds. Suppose $f(K)$ contains no point of $\mathcal{N}$ that is critical for ω (otherwise $\Theta(f)$ would be at least 1 and our inequality would already be satisfied). Let A be the point of $f(K)$ at which the measuring function $\omega_{|f(K)}$ takes its maximum. Since A belongs to the boundary of $f(K)$, it must be $\omega(f^{-1}(A)) = 0$ and as P is internal to K, $\omega(f(P)) < \omega(A)$. In conclusion, $\Theta(f) \geq \omega(A) > \omega(f(P))$ and hence $\Theta(f) \geq \omega(P) - \omega(f(P)) > \omega(P) - \Theta(f)$. It follows that $\Theta(f) > \omega(P)/2 = 1/2$.

In order to complete our proof that the natural pseudodistance is really $1/2$, we still have to show a suitable sequence of homeomorphisms (f_i) such that

$$\lim_i \Theta(f_i) = 1/2.$$

Since the construction of such a sequence is conceptually similar to the one we gave for the previous example about curves, we skip its analytic expression.

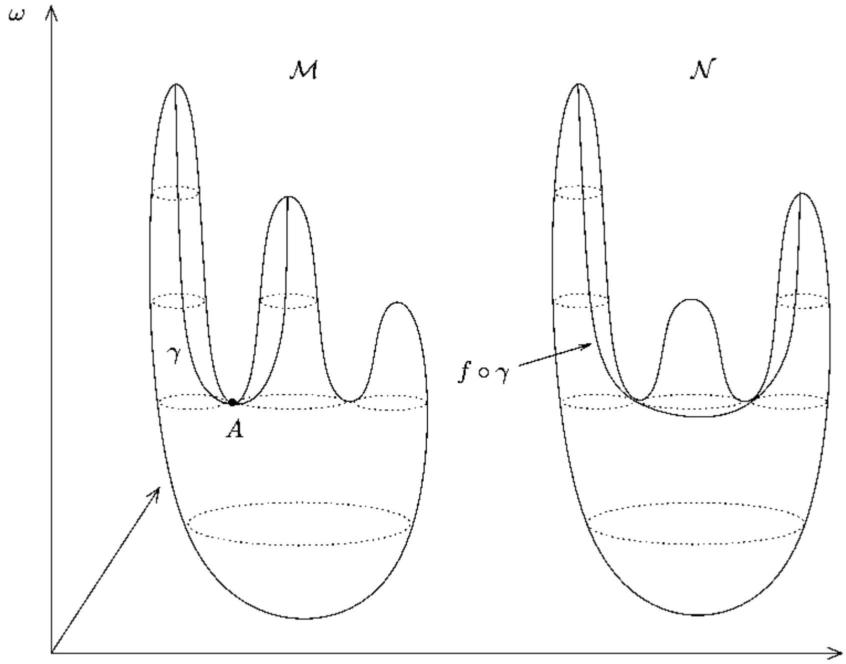

FIGURE 5. An example of vanishing natural pseudodistance.

Example 3.4. Consider the smooth surfaces $\mathcal{M}$ and $\mathcal{N}$ displayed in Figure 5 and the corresponding measuring function ω. The dotted lines are level curves for the measuring function ω.

Property 1. *The natural pseudodistance between the two size pairs vanishes.*

It is easy to see that we can isotopically deform the former surface into the latter one by a "torsion", exchanging the positions of the two smallest humps. This deformation can be performed by an arbitrarily small change in the values of the height ω. Therefore, we can construct a sequence of homeomorphisms (f_i) from $\mathcal{M}$ to $\mathcal{N}$ such that $\Theta(f_i) \to 0$.

Property 2. *No optimal homeomorphism exists between the two size pairs.*

Suppose a homeomorphism f exists such that $\Theta(f) = 0$. Consider a path γ as in Figure 5, chosen in such a way that, in the image of the path, no point P different from A exists for which $\omega(P) = \omega(A)$. We can easily verify that the image of the path $f \circ \gamma$ must contain more than one point at which ω takes the value $\omega(A)$. This is against our assumptions, since $\Theta(f) = 0$ implies $\omega(f(P)) = \omega(P)$ for every point P in the image of γ.

4. Some technical tools and definitions

4.1. **The concept of train of "limit d-jumps".** In order to prove our main theorem, we need some new definitions and technical results. Assume two size pairs $(\mathcal{M}, \omega)$, $(\mathcal{N}, \omega)$ are given.

The symbol $S_H(\mathcal{M}, \mathcal{N})$ will denote the set of all sequences of homeomorphisms (f_i) in $H(\mathcal{M}, \mathcal{N})$ such that $\Theta(f_i) \to d$. Every sequence in $S_H(\mathcal{M}, \mathcal{N})$ will be called a *d-approximating sequence* from $(\mathcal{M}, \omega)$ to $(\mathcal{N}, \omega)$.

Let us consider a sequence $(f_i) \in S_H(\mathcal{M}, \mathcal{N})$. We shall say that a pair of points $(P, Q) \in \mathcal{M} \times \mathcal{N}$ is in relation with respect to (f_i) if a sequence (P_r) in $\mathcal{M}$ exists, together with a strictly increasing sequence (i_r) in $\mathbb{N}$ such that

$$(P, Q) = \lim_r (P_r, f_{i_r}(P_r)).$$

In this case we shall write either $P\rho Q$ or $Q\rho P$, indifferently.

In the remaining part of this section we shall assume that $0 < d < +\infty$. The following compact sets are defined with respect to each d-approximating sequence (f_i):

$$\begin{aligned}
\mathbf{N}^+_{\mathcal{M}} = \mathbf{N}^+_{\mathcal{M}}\big((f_i)\big) &= \{P \in \mathcal{M} \mid \exists Q \in \mathcal{N} : P\,\rho\,Q,\ \omega(Q) - \omega(P) = d\} \\
\mathbf{N}^-_{\mathcal{M}} = \mathbf{N}^-_{\mathcal{M}}\big((f_i)\big) &= \{P \in \mathcal{M} \mid \exists Q \in \mathcal{N} : P\,\rho\,Q,\ \omega(P) - \omega(Q) = d\} \\
\mathbf{N}^+_{\mathcal{N}} = \mathbf{N}^+_{\mathcal{N}}\big((f_i)\big) &= \{Q \in \mathcal{N} \mid \exists P \in \mathcal{M} : P\,\rho\,Q,\ \omega(P) - \omega(Q) = d\} \\
\mathbf{N}^-_{\mathcal{N}} = \mathbf{N}^-_{\mathcal{N}}\big((f_i)\big) &= \{Q \in \mathcal{N} \mid \exists P \in \mathcal{M} : P\,\rho\,Q,\ \omega(Q) - \omega(P) = d\}\,.
\end{aligned}$$

In other words, the points P in $\mathbf{N}^+_{\mathcal{M}}$ are those for which a point $Q \in \mathcal{N}$ exists, such that the pair (P, Q) can be approximated arbitrarily well by a pair $(P_r, f_{i_r}(P_r))$ whose "jump" $\omega(f_{i_r}(P_r)) - \omega(P_r)$ is arbitrarily close to d. Hence, if we think of ω as a "height" function (cf. the examples in the previous section), the points P_r have images with height approximated by $\omega(P_r) + d$. In $\mathbf{N}^+_{\mathcal{M}}$, the symbol $\mathcal{M}$ recalls the manifold to which P belongs, while the symbol $+$ recalls that, by taking P to Q, we increase the value of the measuring function, i.e. the "jump" starting from the node in $\mathcal{M}$ is "upwards". The notations used for the other three sets are quite analogous. The symbol $-$ is used as a sign to denote nodes from which "downwards jumps" start (the starting node belonging to the manifold shown as subscript).

It is clear that, for every point $P \in \mathbf{N}^+_{\mathcal{M}}$, a point $Q \in \mathbf{N}^-_{\mathcal{N}}$ exists such that $P\,\rho\,Q$ (and vice versa), and that an analogous relation holds for the sets $\mathbf{N}^-_{\mathcal{M}}$ and $\mathbf{N}^+_{\mathcal{N}}$. For every sequence of homeomorphisms in $S_H(\mathcal{M}, \mathcal{N})$ the sets $\mathbf{N}_{\mathcal{M}} = \mathbf{N}^+_{\mathcal{M}} \cup \mathbf{N}^-_{\mathcal{M}}$ and $\mathbf{N}_{\mathcal{N}} = \mathbf{N}^+_{\mathcal{N}} \cup \mathbf{N}^-_{\mathcal{N}}$ are non-empty because of the compactness of the manifolds.

Now we shall define the concept of "train" for a d-approximating sequence:

Definition 4.1. Let $(N_0, N_1, \ldots, N_k)$ be an ordered $(k+1)$-tuple of points in $\mathcal{M} \cup \mathcal{N}$ with $k \geq 1$ such that, for $j = 0, \ldots, k-1$ the following properties hold:

(a) $\omega(N_{j+1}) = \omega(N_j) + d$;
(b) $N_j\,\rho\,N_{j+1}$.

In this case the ordered set $(N_0, N_1, \ldots, N_k)$ will be called a *train of limit d-jumps for the sequence* (f_i) (or, in short, a *train*) and its points will be called *nodes*. The pairs (N_j, N_{j+1}) will be known as the *wagons* of the train. The number k will be called *length of the train* and each train that is not included (in the obvious sense) in any other train will be said to be *maximal.* If $(N_0, \ldots, N_k)$ is a maximal train, its wagons (N_0, N_1) and (N_{k-1}, N_k) will be called *initial* and *final train*

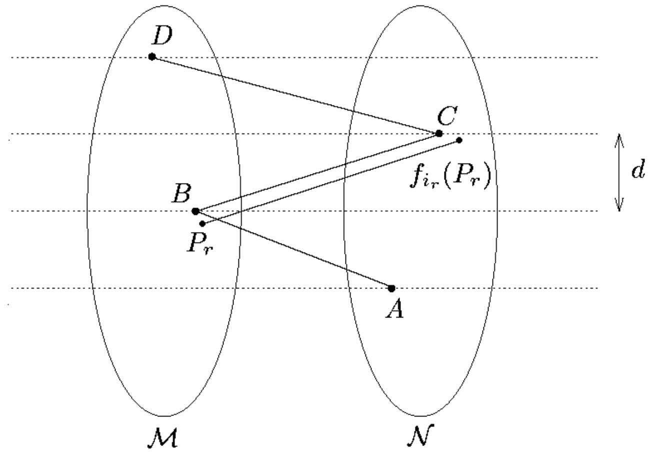

FIGURE 6. A train of limit d-jumps given by the quadruple (A, B, C, D).

wagons (respectively), while N_0 and N_k will be the *initial* and *final train nodes*. The remaining nodes will be called *internal nodes*. The symbol $W\big((f_i)\big)$ will denote the set of all the train wagons (for all the existing trains).

Since each point belonging either to $\mathbf{N}_{\mathcal{M}}$ or to $\mathbf{N}_{\mathcal{N}}$ is a node for at least one train, the set of all trains is not empty. Notice that the point P is an initial node for at least a maximal train if and only if either $P \in \mathbf{N}^+_{\mathcal{M}} - \mathbf{N}^-_{\mathcal{M}}$ or $P \in \mathbf{N}^+_{\mathcal{N}} - \mathbf{N}^-_{\mathcal{N}}$, whereas it is a final node if and only if either $P \in \mathbf{N}^-_{\mathcal{M}} - \mathbf{N}^+_{\mathcal{M}}$ or $P \in \mathbf{N}^-_{\mathcal{N}} - \mathbf{N}^+_{\mathcal{N}}$.

In Figure 6 we provide a graphic representation of a maximal train (A, B, C, D). In this particular case, we have that $A \in \mathbf{N}^+_{\mathcal{N}}$, $B \in \mathbf{N}^+_{\mathcal{M}} \cap \mathbf{N}^-_{\mathcal{M}}$, $C \in \mathbf{N}^+_{\mathcal{N}} \cap \mathbf{N}^-_{\mathcal{N}}$ and $D \in \mathbf{N}^-_{\mathcal{M}}$. Hence, A is the initial node and D is the final train node, while B and C are internal nodes. The three ordered pairs $(A, B), (B, C), (C, D)$ are the three wagons in the train; (A, B) and (C, D) are its initial and final wagons, respectively.

In Figure 7 we can find the maximal train (B, G, A) associated with the d-approximating sequence we described in Example 3.2. In fact, we can easily prove that $B\rho G$, $G\rho A$, $z(G) - z(B) = d$ and $z(A) - z(G) = d$. Hence $B \in \mathbf{N}^+_{\mathcal{M}}$, $G \in \mathbf{N}^+_{\mathcal{N}} \cap \mathbf{N}^-_{\mathcal{N}}$ and $A \in \mathbf{N}^-_{\mathcal{M}}$.

Remark 4.2. The example described in Figure 7 shows that the existence of a train of length 2, such that its initial node (in this case B) and its final node (in this case A) are critical points of the measuring function z, guarantees that the natural pseudodistance d equals half the distance between two critical values of the measuring function.

Our main goal will be to show that in the case of closed surfaces it is always possible to construct a sequence of d-approximating homeomorphisms for which we can demonstrate the existence of a train of length 1, 2 or 3, beginning and ending at critical heights for the measuring functions. We shall do that in the next subsection, 4.2, and in Section 5. The example we have just seen justifies our task, since it points out a simple relation between d and the critical values of z.

Now, in order to attain our goal, we need to introduce the concept of *minimal d-approximating sequence*.

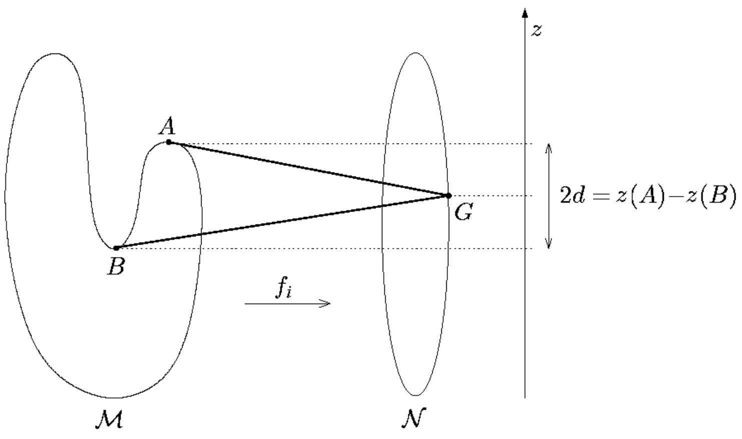

FIGURE 7. An example of a train of limit d-jumps given by the triple (B, G, A).

4.2. **Minimal d-approximating sequences.** The concept of train that we have just mentioned allows us to prove Theorem 2.4 cited in Section 2, and will be central in the following sections, devoted to the proof of the main result in this paper (Theorem 5.7). In this subsection we shall assume that $\mathcal{M}$ and $\mathcal{N}$ are smooth homeomorphic closed manifolds and φ and ψ are Morse measuring functions on $\mathcal{M}$ and $\mathcal{N}$, respectively. We shall weaken these hypotheses at the end of this paper.

As we explained in the introduction, the main goal of this paper is to show that the analytic folding number is either 1, 2 or 3 in the case of closed surfaces.

The idea is to extend the reasoning applied in Remark 4.2, about the example described in Figure 7. In order to do that, from a constructive point of view we need to take a d-approximating sequence and improve it by shortening its trains as much as possible, until we get a train of length 1, 2 or 3, beginning and ending at critical heights for the measuring functions.

This procedure will be carried out in two steps. The former will consist in a reduction of the trains applicable in any dimension, which has been developed and applied in [2] (Lemma 4.6 in this paper) in order that only trains beginning and ending at critical points for the measuring functions remain.

The latter will be a reduction process, expressly developed for the case of surfaces, allowing us to get a further shortening of trains.

Our goal requires a formal definition of "improving" a d-approximating sequence.

Hence we need to define the following preordering $\preceq$ on the set $S_H(\mathcal{M}, \mathcal{N})$ of the d-approximating sequences.

Definition 4.3. If (f_i) and (g_i) are two d-approximating sequences, we set

$$(g_i) \preceq (f_i) \quad (\text{or, equivalently,} \quad (f_i) \succeq (g_i))$$

if $\varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((g_i)\big)\big) \subseteq \varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((f_i)\big)\big)$ and $\varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((g_i)\big)\big) \subseteq \varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((f_i)\big)\big)$.

Definition 4.4. Let (f_i) and (g_i) be two d-approximating sequences. We say that $(g_i) \prec (f_i)$ (or, equivalently, $(f_i) \succ (g_i)$) if $(g_i) \preceq (f_i)$ and either $\varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((g_i)\big)\big) \neq$

$\varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((f_i)\big)\big)$ or $\varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((g_i)\big)\big) \neq \varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((f_i)\big)\big)$ (i.e., at least one of the two inclusions in Definition 4.3 is proper).

We shall say that $(f_i) \in S_H(\mathcal{M},\mathcal{N})$ is a *minimal sequence* if no sequence $(g_i) \in S_H(\mathcal{M},\mathcal{N})$ exists such that $(g_i) \prec (f_i)$.

Remark 4.5. The relations $\preceq$ and $\prec$ could be defined by referring to the nodes in $\mathcal{N}$ in place of the nodes in $\mathcal{M}$. In fact, our definitions immediately imply that the inclusion $\varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((g_i)\big)\big) \subseteq \varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((f_i)\big)\big)$ is equivalent to the inclusion $\psi\big(\mathbf{N}^-_{\mathcal{N}}\big((g_i)\big)\big) \subseteq \psi\big(\mathbf{N}^-_{\mathcal{N}}\big((f_i)\big)\big)$ and the inclusion $\varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((g_i)\big)\big) \subseteq \varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((f_i)\big)\big)$ is equivalent to the inclusion $\psi\big(\mathbf{N}^+_{\mathcal{N}}\big((g_i)\big)\big) \subseteq \psi\big(\mathbf{N}^+_{\mathcal{N}}\big((f_i)\big)\big)$. An analogous statement holds for proper inclusions.

We observe that, in our definition, $(g_i) \preceq (f_i)$ *does not* mean that either $(g_i) \prec (f_i)$ or $(g_i) = (f_i)$.

The minimal sequences for $\prec$ are, in some ways, the best sequences of homeomorphisms whose measure approximates the natural size pseudodistance, since they minimize the sets $\varphi\big(\mathbf{N}^+_{\mathcal{M}}\big)$ and $\varphi\big(\mathbf{N}^-_{\mathcal{M}}\big)$ (and hence the sets $\psi\big(\mathbf{N}^+_{\mathcal{N}}\big)$ and $\psi\big(\mathbf{N}^-_{\mathcal{N}}\big)$, too, i.e. the sets of node heights for the four types of node we have considered). Afterwards, we shall see that it is always possible to construct a d-approximating sequence of homeomorphisms such that the sets $\varphi\big(\mathbf{N}_{\mathcal{M}}\big)$ and $\psi\big(\mathbf{N}_{\mathcal{N}}\big)$ are finite, and that this can be done by using minimal sequences, too.

The existence of minimal sequences with respect to the preordering $\prec$ will be important in the following Section 5.

The following lemma is the main tool used in [2] to prove Theorem 2.4 cited in this paper (for a proof of this lemma see [2], pg. 710).

Lemma 4.6. *Assume that $0 < d < +\infty$ and the measuring functions φ, ψ are Morse. For every sequence of homeomorphisms (f_i) in $S_H(\mathcal{M},\mathcal{N})$ a new sequence (g_i) exists in $S_H(\mathcal{M},\mathcal{N})$ such that all maximal trains begin and end at critical points of the measuring functions and $W\big((g_i)\big) \subseteq W\big((f_i)\big)$.*

Remark 4.7. We observe that in Lemma 4.6 the relation $(g_i) \preceq (f_i)$ is easily implied by the inclusion $W\big((g_i)\big) \subseteq W\big((f_i)\big)$.

The following proposition shows some properties of the minimal sequences we are going to use, under the hypotheses that our measuring functions are Morse. In the next pages the symbols K_φ and K_ψ will represent the sets of critical points of the measuring functions φ and ψ, respectively. The sets of critical values of φ and ψ will be denoted by $\varphi(K_\varphi)$ and $\psi(K_\psi)$.

Proposition 4.8. *Assume that $0 < d < +\infty$ and the measuring functions φ, ψ are Morse, and set $\mathcal{A} = \{z \in \mathbb{R} | \exists c_1, c_2 \in \varphi(K_\varphi) \cup \psi(K_\psi), r, s \in \mathbb{N} : z - c_1 = rd, c_2 - z = sd\}$. Then the following statements hold:*

(a) *If a train for a d-approximating sequence begins and ends at critical points of the measuring functions, the heights of its nodes belong to the finite set $\mathcal{A}$.*
(b) *For every d-approximating sequence (f_i), a minimal sequence $(h_i) \preceq (f_i)$ exists whose maximal trains begin and end at critical points of the measuring functions.*
(c) *If a d-approximating sequence (g_i) is minimal, the height of every node of its trains belongs to the finite set $\mathcal{A}$.*

Proof. (a) It trivially follows from the definition of train. The finiteness of $\mathcal{A}$ follows from the finiteness of the sets K_φ and K_ψ, and hence of the sets $\varphi(K_\varphi)$ and $\psi(K_\psi)$ (here we are using the hypothesis that the measuring functions are Morse).

(b) Lemma 4.6 ensures that we can take a sequence $(g_i) \preceq (f_i)$ whose maximal trains begin and end at critical points of the measuring functions. The previous statement (a) and the definition of the relation $\prec$ imply that no infinite descending chain $\big((g_i) \succ (g_i^1) \succ (g_i^2) \succ \dots\big)$ beginning at (g_i) can exist. Let us consider the last term (g_i') in a maximal descending chain beginning at (g_i). Obviously, (g_i') is a minimal d-approximating sequence. Unfortunately, statement (b) is still not proved, since some maximal train of (g_i') could either begin or end at regular points of the measuring functions, contrary to what happens for (g_i). However, by applying Lemma 4.6 to (g_i') we get a new d-approximating sequence (h_i) that is still minimal and has the required property about maximal trains.

(c) Because of the previous statement (b), a minimal sequence $(h_i) \preceq (g_i)$ exists whose maximal trains begin and end at critical points of the measuring functions. Since (g_i) is already minimal, it follows that $\varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((h_i)\big)\big) = \varphi\big(\mathbf{N}^+_{\mathcal{M}}\big((g_i)\big)\big)$ and $\varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((h_i)\big)\big) = \varphi\big(\mathbf{N}^-_{\mathcal{M}}\big((g_i)\big)\big)$ (and hence $\psi\big(\mathbf{N}^-_{\mathcal{N}}\big((h_i)\big)\big) = \psi\big(\mathbf{N}^-_{\mathcal{N}}\big((g_i)\big)\big)$ and $\psi\big(\mathbf{N}^+_{\mathcal{N}}\big((h_i)\big)\big) = \psi\big(\mathbf{N}^+_{\mathcal{N}}\big((g_i)\big)\big)$). Statement (a) ensures that $\varphi\big(\mathbf{N}_{\mathcal{M}}(h_i)\big) \cup \psi\big(\mathbf{N}_{\mathcal{N}}(h_i)\big)$ is included in the finite set $\mathcal{A}$, and therefore the same happens to the set $\varphi\big(\mathbf{N}_{\mathcal{M}}\big((g_i)\big)\big) \cup \psi\big(\mathbf{N}_{\mathcal{N}}\big((g_i)\big)\big)$. □

5. Our main result

In Section 2 we have recalled (Theorem 2.4) that the natural pseudodistance between two size pairs is related to the critical values of their measuring functions.

However, the examples we have displayed suggest that our results can be improved. In fact our examples show an analytic folding number k that is never greater than 2. In the first part of this section we shall prove (Theorem 5.4) that the analytic folding number is never greater than 3, under the assumption that $\mathcal{M}$ and $\mathcal{N}$ are two homeomorphic smooth closed surfaces and the measuring functions φ, ψ are Morse. These hypotheses will make our proofs easier from a technical point of view. In subsection 5.1 we shall weaken our assumptions and come back to the case of class C^1 (Theorem 5.7).

Now we introduce two lemmas. The former is trivial and clarifies the local nature of the concept of node.

Lemma 5.1. *Assume $0 < d < +\infty$. Let U be an open subset of $\mathcal{M}$ and (f_i) and (g_i) be two d-approximating sequences such that, for every $i \in \mathbb{N}$, f_i coincides with g_i in U. Then $\mathbf{N}^+_{\mathcal{M}}\big((f_i)\big) \cap U = \mathbf{N}^+_{\mathcal{M}}\big((g_i)\big) \cap U$ and $\mathbf{N}^-_{\mathcal{M}}\big((f_i)\big) \cap U = \mathbf{N}^-_{\mathcal{M}}\big((g_i)\big) \cap U$.*

Proof. It immediately follows from the definitions of the sets $\mathbf{N}^+_{\mathcal{M}}$ and $\mathbf{N}^-_{\mathcal{M}}$. □

A similar result obviously holds for an open subset V of $\mathcal{N}$, and can easily be obtained by interchanging the roles of the sequences (f_i), (g_i) and (f_i^{-1}), (g_i^{-1}) in Lemma 5.1.

The useful property described by the following key lemma justifies the introduction of the concept of minimal sequence in the case of closed surfaces.

Lemma 5.2. *Assume that $\mathcal{M}$, $\mathcal{N}$ are smooth homeomorphic closed surfaces and φ, ψ are Morse measuring functions on $\mathcal{M}$ and $\mathcal{N}$, respectively. Suppose that $0 < d < +\infty$, and (f_i) is a minimal d-approximating sequence from $(\mathcal{M}, \varphi)$ to*

$(\mathcal{N},\psi)$. If $N \in \mathbf{N}_{\mathcal{M}}\big((f_i)\big)$ and $\varphi(N)$ is not a critical value for the function φ, then at least one of the values $\varphi(N)-d$, $\varphi(N)+d$ is a critical value for the function ψ.

In other words, under the hypotheses of the lemma (possibly by exchanging the roles of the two surfaces), if we consider the heights of three consecutive nodes in a train of a minimal sequence, at least one of them is a critical value. The proof of this property involves Jost and Schoen's Theorem about harmonic maps between surfaces and is the key passage in proving the main result of this paper (Theorem 5.7).

Proof. We shall prove that if $\varphi(N)$ is a regular value for φ and both $\varphi(N)-d$ and $\varphi(N)+d$ are regular values for ψ then we can get a new d-approximating sequence $(\tilde{f}_i)$ such that $(\tilde{f}_i) \prec (f_i)$, contradicting the assumption that (f_i) is minimal. So, in the following we shall assume that $\varphi(N) \notin \varphi(K_\varphi)$ and $\varphi(N)+d, \varphi(N)-d \notin \psi(K_\psi)$.

Let us define the open sets $\mathcal{D}_\varepsilon = \{P \in \mathcal{M} : |\varphi(P)-\varphi(N)| < \varepsilon\}$ and $V_\varepsilon = \{Q \in \mathcal{N} : \min_{\bar{Q}\in K_\psi} |\psi(Q)-\psi(\bar{Q})| < \varepsilon\}$ (in other words $\mathcal{D}_\varepsilon$ is the set of all points of $\mathcal{M}$ whose height differs less than ε from the height of N, while V_ε is the set of all points of $\mathcal{N}$ whose height differs less than ε from the height of a critical point of ψ). Moreover, let us choose $\varepsilon > 0$ so small that

(1) $\overline{\mathcal{D}_\varepsilon}$ does not contain critical points for φ;
(2) $\partial\mathcal{D}_\varepsilon$ does not contain nodes belonging to $\mathbf{N}_{\mathcal{M}}\big((f_i)\big)$;
(3) for i large enough, if $Q \in f_i(\overline{\mathcal{D}_\varepsilon})$ and $|\varphi(N)-\psi(Q)| \geq d-2\varepsilon$ then $Q \notin \overline{V_\varepsilon}$.

The existence of an $\varepsilon > 0$ verifying (1) and (2) is ensured by the assumption that $\varphi(N) \notin \varphi(K_\varphi)$ and the fact that the set of heights of the nodes is finite (see Proposition 4.8(c)), always recalling that the measuring functions are Morse. As regards (3), if for every positive ε we could find an arbitrarily large i and a $\bar{Q} \in \overline{V_\varepsilon} \cap f_i(\overline{\mathcal{D}_\varepsilon})$ verifying $|\varphi(N)-\psi(\bar{Q})| \geq d-2\varepsilon$, then a wagon $(\hat{N},\hat{Q})$ should exist for (f_i), with $\psi(\hat{Q})$ equal to a critical value of ψ, $\varphi(\hat{N}) = \varphi(N)$ and $|\varphi(\hat{N})-\psi(\hat{Q})| = d$, since $\lim_i \Theta(f_i) = d$. Therefore either $\varphi(N)+d$ or $\varphi(N)-d$ should be a critical value for ψ, against our hypothesis.

Note: As a matter of fact, the expression "for i large enough" in (3) can be replaced with the words "for $i \geq \bar{\imath}$", where $\bar{\imath}$ is a natural number *independent* of ϵ. We just require ϵ to be chosen strictly less than $\eta/3$, with η being the minimum distance between the finite set of all critical values of the (Morse) function ψ and the set $\{\varphi(N)+d, \varphi(N)-d\}$. In order to proceed this way we only have to take $\bar{\imath}$ so large that for every $i \geq \bar{\imath}$ the inequality $\Theta(f_i) \leq d+\eta/3$ holds, implying $|\varphi(N)-\psi(Q)| \leq d+\eta/3+\epsilon < d+2\eta/3$ for every $Q \in f_i(\overline{\mathcal{D}_\varepsilon})$. If we also assume that $|\varphi(N)-\psi(Q)| \geq d-2\varepsilon$, then $d-2\eta/3 < |\varphi(N)-\psi(Q)| < d+2\eta/3$ and hence the distance between $\psi(Q)$ and the set $\{\varphi(N)+d, \varphi(N)-d\}$ must be strictly less than $2\eta/3$. Therefore $\psi(Q)$ is more distant than $\eta/3 > \epsilon$ from the set of all critical values of ψ and hence $Q \notin \overline{V_\varepsilon}$. Anyway, this change of statement is not necessary for our proof, and we maintain the simpler version of (3).

Now, we are going to prove that a sequence $(\tilde{f}_i) \in S_H(\mathcal{M},\mathcal{N})$ exists such that $\tilde{f}_i = f_i$ in the closed set $\mathcal{M}-\mathcal{D}_\varepsilon$ and $\mathbf{N}_{\mathcal{M}}\big((\tilde{f}_i)\big) \cap \overline{\mathcal{D}_\varepsilon} = \emptyset$ (in other words, we can eliminate all wagons from $\mathcal{M}$ to $\mathcal{N}$, beginning in $\overline{\mathcal{D}_\varepsilon}$).

So, we start by setting $\tilde{f}_i(P) = f_i(P)$ for $P \in \mathcal{M} - \mathcal{D}_\varepsilon$.

In order to define $\tilde{f}_i$ in $\mathcal{D}_\varepsilon$ we have to consider each connected component $\mathcal{C}$ of $\mathcal{D}_\varepsilon$. Because of hypothesis (1), $\mathcal{C}$ is (homeomorphic to) a cylinder. On $\mathcal{C} \cong S^1 \times (\varphi(N) - \epsilon, \varphi(N) + \epsilon)$ let us define the product metric $d\theta^2 + d\varphi^2$, so that the function φ is linear in $\mathcal{C}$ (i.e. $\nabla^2\varphi \equiv 0$). Here, $d\theta^2$ and $d\varphi^2$ are an arbitrarily chosen Riemannian metric on S^1 and the Riemannian metric on the interval $(\varphi(N) - \epsilon, \varphi(N) + \epsilon)$ induced by the Euclidean distance, respectively (cf. [8]).

Then consider a Riemannian metric $\mu_\mathcal{N}$ on $\mathcal{N}$ in such a way that the measuring function ψ is harmonic at each point of $\mathcal{N} - \overline{V_\varepsilon}$. In other words, we require that ψ is harmonic in $\mathcal{N}$, with the possible exception of the closure of the set of those points whose height has a distance smaller than ε from some critical height of ψ. We can get this by using the construction in the previous paragraph. The set $\mathcal{N} - \overline{V_\varepsilon}$ is a union of cylinders, and the level sets of ψ slice each cylinder into circles. Using the construction in the last paragraph, it follows that we can choose a metric so that ψ is harmonic on $\mathcal{N} - \overline{V_\varepsilon}$. We refer to [8] and [9] for alternative proofs of the existence of such a Riemannian metric.

In order to apply Jost and Schoen's Theorem we need to work with diffeomorphisms. This implies that we have to approximate our homeomorphisms f_i by diffeomorphisms, without changing the trains of our d-approximating sequence.

Claim A. *A sequence of diffeomorphisms (g_i) exists such that $W((g_i)) = W((f_i))$.*

Proof of Claim A. Since $\mathcal{M}$ and $\mathcal{N}$ are smooth surfaces, for each index i we can find a diffeomorphism $g_i : \mathcal{M} \to \mathcal{N}$ such that $d_\mathcal{N}(f_i(P), g_i(P)) \leq 1/i$ for every $P \in \mathcal{M}$, where $d_\mathcal{N}$ is the distance on $\mathcal{N}$ induced by the Riemannian metric $\mu_\mathcal{N}$ (cf., e.g., Corollary 1.18 in [22], and [17]). Hence $P\rho Q$ with respect to (f_i) if and only if $P\rho Q$ with respect to (g_i) (recall subsection 4.1). This implies that $W((g_i)) = W((f_i))$.

Because of Claim A, we can assume that each f_i is a diffeomorphism, without loss of generality.

The following Theorem holds (cf. [15]):

Theorem 5.3 (Jost and Schoen)**.** *Let $\Omega \subset M_1$ be a domain with non-empty boundary $\partial\Omega$ consisting of C^1 Jordan curves. Let $h : \Omega \to M_2$ be a diffeomorphism of $\overline{\Omega}$ onto $\overline{h(\Omega)}$. Suppose the curves $h(\partial\Omega)$ are of class $C^{2+\alpha}$ and are locally convex with respect to $h(\Omega)$, i.e. $h(\partial\Omega)$ has non-negative geodesic curvature with respect to the normal pointing into $h(\Omega)$. There exists a harmonic diffeomorphism $\bar{h} : \Omega \to h(\Omega)$ which is homotopic to h and satisfies $\bar{h} = h$ on $\partial\Omega$. Moreover, $\bar{h}$ is of least energy among all diffeomorphisms homotopic to h and assuming the same boundary values.*

Jost and Schoen's Theorem guarantees the existence of a diffeomorphism $\bar{h} : \overline{f_i(\mathcal{C})} \to \overline{\mathcal{C}}$ that is harmonic in the set $f_i(\mathcal{C})$ and coincides with f_i^{-1} at the boundary of $f_i(\mathcal{C})$.

Now, we are ready to define $\tilde{f}_i(P)$ in the case $P \in \mathcal{D}_\varepsilon$, by setting $\tilde{f}_i(P) = \bar{h}^{-1}(P)$ for every $P \in \mathcal{C}$, and $\mathcal{C}$ varying in the set of all the connected components of $\mathcal{D}_\varepsilon$. Practically, we are going to change f_i into $\bar{h}^{-1}$ inside each cylinder $\mathcal{C}$. Notice that every $\tilde{f}_i$ is a homeomorphism from $\mathcal{M}$ to $\mathcal{N}$, verifying the equalities $\tilde{f}_i(\overline{\mathcal{D}_\varepsilon}) = f_i(\overline{\mathcal{D}_\varepsilon})$ and $\tilde{f}_i(\partial\mathcal{D}_\varepsilon) = f_i(\partial\mathcal{D}_\varepsilon)$.

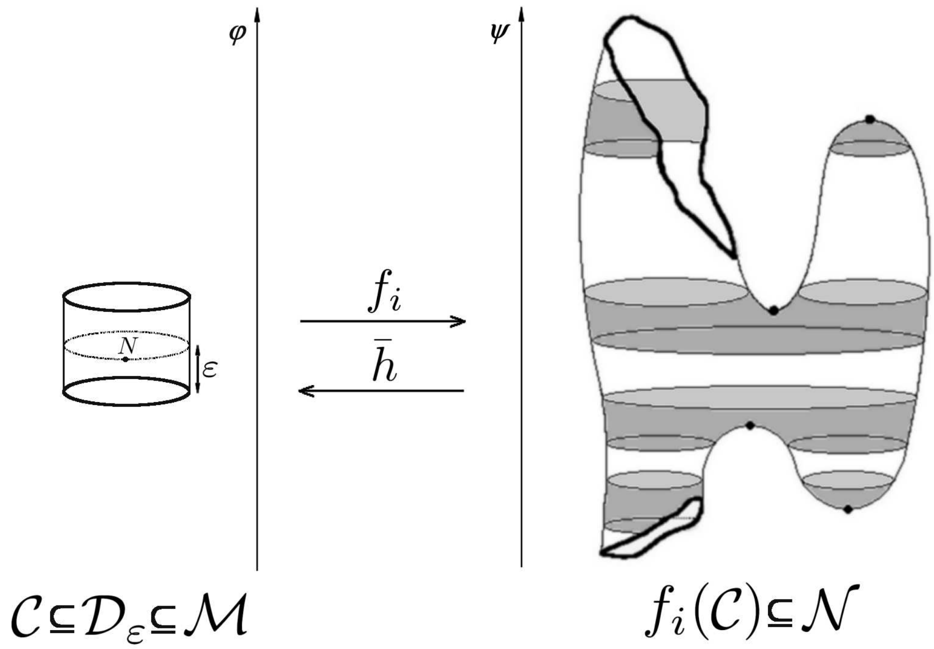

FIGURE 8. The cylinder $\mathcal{C}$ inside the set $\mathcal{D}_\varepsilon$ and its image $f_i(\mathcal{C})$. The subset $V_\varepsilon \cap f_i(\mathcal{C})$ is highlighted (in grey).

The key property of the new sequence $(\tilde{f}_i)$ is that its "jumps" starting from $\mathcal{D}_\varepsilon$ are controlled because of our hypotheses and the use of harmonic maps, and the "*largest*" jumps of $(\tilde{f}_i)$ are not larger than the corresponding jumps of (f_i). Formally, the following claims **B** and **C** hold.

Claim B. $\max_{f_i(\overline{\mathcal{D}_\varepsilon})} |\varphi \circ \tilde{f}_i^{-1} - \psi| = \max_{f_i(\partial\mathcal{D}_\varepsilon)\cup(\overline{V_\varepsilon}\cap f_i(\mathcal{D}_\varepsilon))} |\varphi \circ \tilde{f}_i^{-1} - \psi|$.

Proof of Claim B. The key remark is that the (continuous) function $\varphi \circ \tilde{f}_i^{-1} - \psi : \mathcal{N} \to \mathbb{R}$ is harmonic on $f_i(\mathcal{D}_\varepsilon) - \overline{V_\varepsilon}$. In fact, on the one hand, since φ is linear on $\mathcal{D}_\varepsilon$ and $\tilde{f}_i^{-1}$ is harmonic on $f_i(\mathcal{D}_\varepsilon)$ it immediately follows that the function $\varphi \circ \tilde{f}_i^{-1}$ is harmonic on $f_i(\mathcal{D}_\varepsilon)$ (cf., e.g., Corollary 8.7.4 in [14]). On the other hand ψ is harmonic on $f_i(\mathcal{D}_\varepsilon) - \overline{V_\varepsilon}$, because of the Riemannian metric we have chosen on $\mathcal{N}$. Hence, because of the Maximum Principle, the restriction to $\overline{f_i(\mathcal{D}_\varepsilon) - \overline{V_\varepsilon}}$ of the function $\varphi \circ \tilde{f}_i^{-1} - \psi$ must take its maximum also at a point of the set $\partial(f_i(\mathcal{D}_\varepsilon) - \overline{V_\varepsilon})$. The same holds for the minimum value. This implies that the restriction of $\varphi \circ \tilde{f}_i^{-1} - \psi$ to the compact set $f_i(\overline{\mathcal{D}_\varepsilon})$ takes each extremum either in the set $f_i(\partial\mathcal{D}_\varepsilon)$ or in the set $(\overline{V_\varepsilon} \cap f_i(\mathcal{D}_\varepsilon))$. The thesis of our claim immediately follows.

Claim C. *For every large enough index i, if $\max_{\overline{\mathcal{D}_\varepsilon}} |\varphi - \psi \circ \tilde{f}_i| \geq d - \varepsilon$ then $\max_{\overline{\mathcal{D}_\varepsilon}} |\varphi - \psi \circ \tilde{f}_i| \leq \max_{\partial\mathcal{D}_\varepsilon} |\varphi - \psi \circ f_i|$. Therefore, $\Theta(\tilde{f}_i) \leq \Theta(f_i)$, and hence the new sequence $(\tilde{f}_i)$ is a d-approximating sequence from $(\mathcal{M}, \varphi)$ to $(\mathcal{N}, \psi)$.*

Proof of Claim C. Because of Claim B, a point $\bar{Q} \in f_i(\partial\mathcal{D}_\varepsilon) \cup (\overline{V_\varepsilon} \cap f_i(\mathcal{D}_\varepsilon))$ exists such that $|\varphi \circ \tilde{f}_i^{-1}(\bar{Q}) - \psi(\bar{Q})| = \max_{f_i(\overline{\mathcal{D}_\varepsilon})} |\varphi \circ \tilde{f}_i^{-1} - \psi| = \max_{\overline{\mathcal{D}_\varepsilon}} |\varphi - \psi \circ \tilde{f}_i| \geq d - \varepsilon$, under our hypothesis. Since $\tilde{f}_i^{-1}(\bar{Q}) \in \overline{\mathcal{D}_\varepsilon}$, we have that $|\varphi \circ \tilde{f}_i^{-1}(\bar{Q}) - \varphi(N)| \leq \epsilon$. It follows that $|\varphi(N) - \psi(\bar{Q})| \geq d - 2\varepsilon$ and hence $\bar{Q} \notin \overline{V_\varepsilon}$ (for i large enough), because of the assumption (3) about $\mathcal{D}_\varepsilon$. Therefore $\bar{Q} \in f_i(\partial\mathcal{D}_\varepsilon)$. Since the diffeomorphism $\tilde{f}_i^{-1}$ coincides with f_i^{-1} in the set $f_i(\partial\mathcal{D}_\varepsilon)$, $\max_{\overline{\mathcal{D}_\varepsilon}} |\varphi - \psi \circ \tilde{f}_i| \leq \max_{\partial\mathcal{D}_\varepsilon} |\varphi - \psi \circ f_i|$. So our claim is proved.

Lemma 5.1 (local nature of the concept of node) and the coincidence of $\tilde{f}_i$ and f_i outside $\overline{\mathcal{D}_\varepsilon}$ immediately imply the next claim.

Claim D. *The set of all wagons arriving from $\mathcal{M} - \overline{\mathcal{D}_\varepsilon}$ at $\mathcal{N}$ is equal for (f_i) and $(\tilde{f}_i)$. In particular, the following equalities hold:*

$$\begin{aligned} \mathbf{N}^+_{\mathcal{M}}\big((\tilde{f}_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon}) &= \mathbf{N}^+_{\mathcal{M}}\big((f_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon}) \\ \mathbf{N}^-_{\mathcal{M}}\big((\tilde{f}_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon}) &= \mathbf{N}^-_{\mathcal{M}}\big((f_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon}) \end{aligned} \tag{5.0.1}$$

Finally, we can prove that under our hypotheses the new sequence $(\tilde{f}_i)$ is "better" than (f_i) in the sense expressed by the following statement, saying that no wagon from $\mathcal{M}$ to $\mathcal{N}$ beginning in $\overline{\mathcal{D}_\varepsilon}$ exists.

Claim E. *The set $\mathbf{N}_{\mathcal{M}}\big((\tilde{f}_i)\big) \cap \overline{\mathcal{D}_\varepsilon}$ is empty.*

Proof of Claim E. If an $N' \in \mathbf{N}_{\mathcal{M}}\big((\tilde{f}_i)\big) \cap \overline{\mathcal{D}_\varepsilon}$ existed, then a sequence (P_r) of points of $\mathcal{M}$ converging to N' and a strictly increasing sequence (i_r) in $\mathbb{N}$ would exist such that the sequence $\big(\tilde{f}_{i_r}(P_r)\big)$ converges and $|\varphi(P_r) - \psi(\tilde{f}_{i_r}(P_r))| \to d$. Since f_i and $\tilde{f}_i$ coincide outside $\mathcal{D}_\varepsilon$ and $\partial\mathcal{D}_\varepsilon$ does not contain nodes for (f_i) (hypothesis (2) about $\mathcal{D}_\varepsilon$), we can assume that all points P_r belong to $\overline{\mathcal{D}_\varepsilon}$, without loss of generality. Then, because of Claim C, a converging sequence (B_r) would also exist such that $|\varphi(B_r) - \psi(f_{i_r}(B_r))| \to d$, where each B_r is a point belonging to $\partial\mathcal{D}_\varepsilon$. This fact would imply the existence of a node for (f_i) belonging to $\partial\mathcal{D}_\varepsilon$, once more against hypothesis (2).

In summary, we have seen that $\overline{\mathcal{D}_\varepsilon}$ does not meet $\mathbf{N}_{\mathcal{M}}\big((\tilde{f}_i)\big)$, but contains at least one node of $\mathbf{N}_{\mathcal{M}}\big((f_i)\big)$, while $\mathbf{N}^+_{\mathcal{M}}\big((\tilde{f}_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon}) = \mathbf{N}^+_{\mathcal{M}}\big((f_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon})$ and $\mathbf{N}^-_{\mathcal{M}}\big((\tilde{f}_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon}) = \mathbf{N}^-_{\mathcal{M}}\big((f_i)\big) \cap (\mathcal{M} - \overline{\mathcal{D}_\varepsilon})$ (Claim D). It follows that $(\tilde{f}_i) \prec (f_i)$. This contradicts the hypothesis that (f_i) is a minimal sequence. □

Let us apply Lemma 5.2 to prove that the analytic folding number is never greater than 3 for the closed surfaces.

Theorem 5.4. *Assume that $\mathcal{M}$ and $\mathcal{N}$ are two homeomorphic smooth closed surfaces and that $\varphi : \mathcal{M} \to \mathbb{R}$ and $\psi : \mathcal{N} \to \mathbb{R}$ are two Morse functions. Then, if d denotes the natural pseudodistance between the size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$, at least one of the following properties holds:*

(i) *d equals the distance between a critical value of φ and a critical value of ψ;*
(ii) *d equals half the distance between two critical values of φ.*
(iii) *d equals half the distance between two critical values of ψ.*

(iv) *d equals one third of the distance between a critical value of φ and a critical value of ψ.*

Proof. If $d = 0$, then φ and ψ have the same global minimum μ. Hence, $d = |\mu - \mu|$ and our thesis is trivial.

So let us assume $d > 0$. Let (h_i) be a minimal sequence whose maximal trains begin and end at critical points of the measuring functions (Proposition 4.8(b)) and suppose $N_1 \in \mathcal{M}$ is the initial node of a maximal train (if no maximal train begins in $\mathcal{M}$, it is sufficient to exchange the roles of our surfaces; in this case properties (ii) and (iii) exchange in the following). Therefore $\varphi(N_1)$ is a critical value for φ. Let $N_2 \in \mathcal{N}$ be the next node in the train. If $\psi(N_2)$ is a critical value for ψ, then condition (i) holds. Otherwise, call $N_3 \in \mathcal{M}$ the next node in the train (N_3 exists because $\psi(N_2)$ is not a critical value for ψ, and hence N_2 is not the final node of the train). If $\varphi(N_3)$ is a critical value for φ, then condition (ii) holds. Otherwise, call $N_4 \in \mathcal{N}$ the next node in the train (N_4 exists because $\varphi(N_3)$ is not a critical value for φ, and hence N_3 is not the final node of the train). Lemma 5.2 applied for $N = N_3$ ensures that $\psi(N_4) = \varphi(N_1) + 3d$ is a critical value for the measuring function ψ. Therefore $d = \frac{1}{3}(\psi(N_4) - \varphi(N_1))$ and property (iv) holds. □

Remark 5.5. It may be interesting to note that Example 3.4 can be used to show that the hypothesis of $h(\partial\Omega)$ being locally convex with respect to $h(\Omega)$ is really necessary in Theorem 5.3. In fact, consider the open surfaces $\mathcal{M}^*, \mathcal{N}^*$ displayed in Figure 9, obtained from the two surfaces $\mathcal{M}$, $\mathcal{N}$ in Figure 5 by taking away suitable closed neighborhoods of the critical points. On $\mathcal{M}^*$ and $\mathcal{N}^*$ consider two Riemannian metrics $\mu_{\mathcal{M}^*}$ and $\mu_{\mathcal{N}^*}$ such that φ and ψ are linear functions with respect to $\mu_{\mathcal{M}^*}$ and $\mu_{\mathcal{N}^*}$, respectively (cf. [8]). The metrics $\mu_{\mathcal{M}^*}$ and $\mu_{\mathcal{N}^*}$ are the ones induced by the embeddings of $\mathcal{M}^*$ and $\mathcal{N}^*$ in $\mathbb{R}^3$ displayed in Figure 10. Notice that $\partial\mathcal{M}^*$ is *not* locally convex with respect to $\mathcal{M}^*$. We observe that a diffeomorphism $h : \overline{\mathcal{N}^*} \to \overline{\mathcal{M}^*}$ exists, preserving the height of the boundary points. If a diffeomorphism $\bar{h} : \overline{\mathcal{N}^*} \to \overline{\mathcal{M}^*}$ existed, harmonic in $\mathcal{N}^*$ and coinciding with h at $\partial\mathcal{N}^*$, it should preserve the height of every point in $\mathcal{N}^*$, since the function $\varphi - \psi \circ \bar{h} : \overline{\mathcal{N}^*} \to \mathbb{R}$ would be harmonic in $\mathcal{N}^*$ and take its maximum and minimum at points of $\partial\mathcal{N}^*$, where $\varphi - \psi \circ \bar{h}$ vanishes. Moreover, we could easily extend $\bar{h}$ to a diffeomorphism $h' : \mathcal{N} \to \mathcal{M}$ that preserves the height of every point outside $\mathcal{N}^*$. Therefore, an optimal diffeomorphism between the size pairs $(\mathcal{N}, \omega)$, $(\mathcal{M}, \omega)$ would exist, against what we said in Example 3.4.

Remark 5.6. Lemma 5.2 may be considered analogous to Lemma 3.2 proved in [4] for curves, but the techniques used in the proof are deeply different, since here we have to manage harmonic maps in place of linear maps in dimension 1. As a consequence, because of the hypotheses required in Jost and Schoen's Theorem, problems about the position of images of critical points arise after the harmonicization process, since, contrary to what happens in the case of curves, we don't know this position. As a result, in both the 1-dimensional and the 2-dimensional case we can prove that if we consider the heights of m consecutive nodes in a train of a minimal sequence, at least one of them is a critical value, but we have to set $m = 2$ for curves and $m = 3$ for surfaces, depending on the different techniques and dimensional constraints involved in our proofs. This difference explains why the thesis of Theorem 3.4 in [4] (for curves) is stronger than the thesis of Theorem 5.7 in this paper, concerning surfaces.

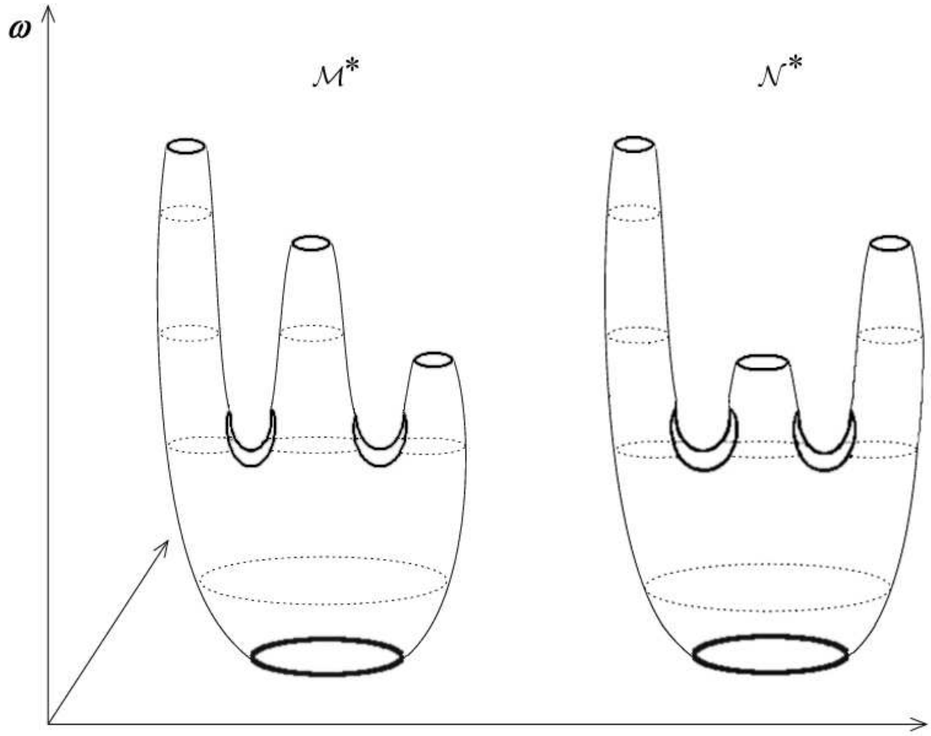

FIGURE 9. A diffeomorphism $h : \mathcal{N}^* \to \mathcal{M}^*$ exists, preserving the height of the boundary points (in bold).

5.1. **Weakening the hypotheses about the regularity of surfaces and measuring functions.** Until now we have considered smooth closed surfaces and Morse measuring functions. By repeating the same proofs that we have used in [2] to weaken our hypotheses about regularity (see Section 6 in that paper), we can get our main result via an approximation procedure:

Theorem 5.7. *Assume that $\mathcal{M}$ and $\mathcal{N}$ are two homeomorphic closed surfaces of class C^1 and that $\varphi : \mathcal{M} \to \mathbb{R}$ and $\psi : \mathcal{N} \to \mathbb{R}$ are two functions of class C^1. Then, if d denotes the natural pseudodistance between the size pairs $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$, at least one of the following properties holds:*

(i) *d equals the distance between a critical value of φ and a critical value of ψ;*
(ii) *d equals half the distance between two critical values of φ.*
(iii) *d equals half the distance between two critical values of ψ.*
(iv) *d equals one third of the distance between a critical value of φ and a critical value of ψ.*

6. Conclusions and further research

In this paper we have proved that for closed surfaces the relation between natural pseudodistance and the critical values of the measuring functions is stronger than the one we proved in [2] for general dimension. In fact, Theorem 5.7 shows that the

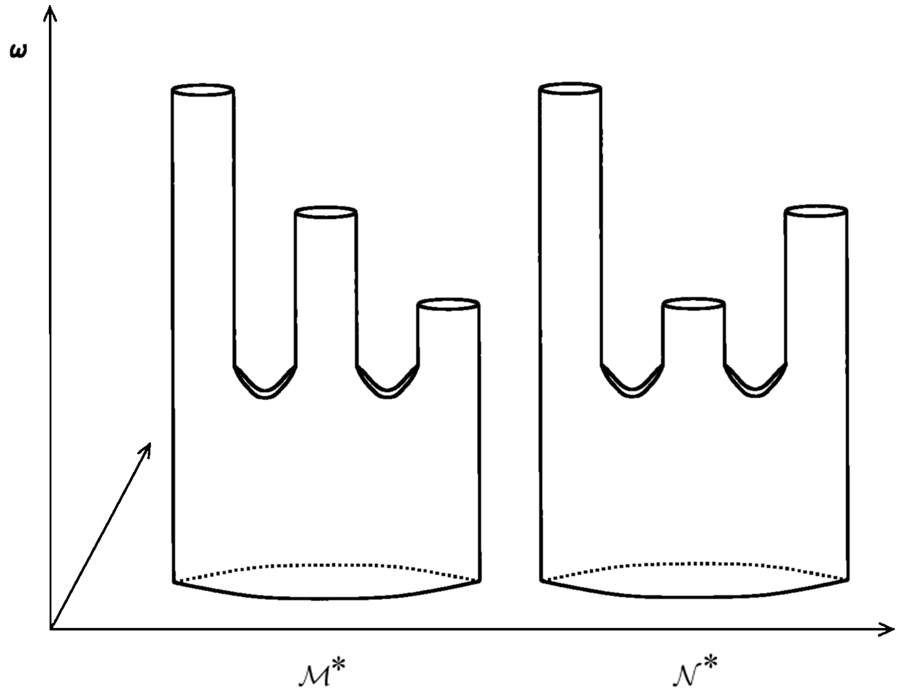

FIGURE 10. The metrics $\mu_{\mathcal{M}^*}$ and $\mu_{\mathcal{N}^*}$ are the ones induced by the displayed embeddings of $\mathcal{M}^*$ and $\mathcal{N}^*$ into $\mathbb{R}^3$.

natural pseudodistance between two homeomorphic C^1 closed surfaces associated with C^1 measuring functions is always either the distance or half the distance or one third of the distance between two suitable critical values of the measuring functions.

Unfortunately, our techniques cannot be used for larger dimensions, since the statement of Jost and Schoen's Theorem fails in dimension strictly greater than 2 (cf. [12], Section 5.8, and [5], Section 12). Moreover, the application of this theorem requires the approximability of homeomorphisms by means of diffeomorphisms. This procedure is not available for dimensions strictly larger than 3 (cf., e.g., [16]). As a consequence we don't know if results analogous to Theorem 5.7 hold for dimensions strictly larger than 2. In other words, we wonder if two n-manifolds associated with regular measuring functions φ, ψ exist such that their pseudodistance neither equals D nor $D/2$ nor $D/3$, for D varying in the set of all distances between the critical values of φ and ψ.

In our further research we intend to study this problem and the availability of new techniques for studying the general n-dimensional case.

However, it is interesting to note that we do not know examples where the analytic folding number equals 3, also in the bidimensional case. On the other hand we are not able to make our result better and to prove that the analytic folding number never equals 3, also in the case of surfaces (see Remark 5.6).

The difficulty in finding examples where the analytic folding number equals 3 deserves some further remarks. One technique that can be used for computing natural size pseudodistances is based on *size functions* (cf. [3]). The computation of size functions is usually easy and gives us a lower bound s for natural size pseudodistances. Obviously, when we are able to show a sequence (f_i) of homeomorphisms for which $\lim_i \Theta(f_i) = s$ we can claim that the natural size pseudodistance equals s. The key point is that the best lower bound s we can obtain is either the distance or half the distance between two suitable critical values of the measuring functions (cf. Theorem 2 in [3]). Therefore, in case an example where the analytic folding number equals 3 really does exist, we are not able to find and recognize it using the previously described technique. Apparently, new techniques should be developed.

As regards the use of harmonic maps in our study, this corresponds to the property that the deformation due to tension fields decreases both the energy and the maximum change of the measuring functions, provided that we are far from their critical points. The use of different kinds of deformations (e.g. curvature evolution of level lines of the measuring functions) might be investigated. The main problem seems to be the possible birth of degeneracies.

Furthermore, it might be interesting to examine the possibility of moving from the study of trains of limit d-jumps to the study of relations obtained as limits of d-approximating sequences of homeomorphisms, with respect to the Hausdorff (or another more suitable) topology.

In conclusion, various interesting questions remain open and deserve further study and research.

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References

1. F. Dibos, P. Frosini and D. Pasquignon, *The use of Size Functions for Comparison of Shapes through Differential Invariants*, Journal of Mathematical Imaging and Vision, vol. 21 (2004), n. 2, 107–118.
2. P. Donatini and P. Frosini, *Natural pseudodistances between closed manifolds*, Forum Mathematicum **16** (2004), no. 5, 695–715.
3. P. Donatini and P. Frosini, *Lower bounds for natural pseudodistances via size functions*, Archives of Inequalities and Applications, vol. 2 (2004), n. 1, 1–12.
4. P. Donatini and P. Frosini, *Natural pseudodistances between closed curves* (submitted).
5. J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. 20 (1988), no. 5, 385–524.
6. P. Frosini, *A distance for similarity classes of submanifolds of a euclidean space*, Bull. Austral. Math. Soc. **42** (1990), no. 3, 407–416.
7. ———, *Connections between size functions and critical points*, Math. Meth. Appl. Sci. **19** (1996), 555–569.
8. ———, *A note on the linearity of real-valued functions with respect to suitable metrics*, Geometriae Dedicata, **108** (2004), no. 1, 105–110.
9. P. Frosini and C. Landi, *Intrinsic harmonicity of Morse functions*, Mathematika **50** (2003), 167–170.
10. C. Landi and P. Frosini, *Size functions and morphological transformations*, Acta Appl. Math. **49** (1997), no. 1, 85–104.

11. M. Mulazzani and P. Frosini, *Size homotopy groups for computation of natural size distances*, Bull. Belg. Math. Soc. **6** (1999), 455–464.
12. J. Jost, *Harmonic mappings between Riemannian manifolds*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Vol. 4, Canberra: Centre for Mathematical Analysis, Australian National University, 1983.
13. J. Jost, *Harmonic maps between surfaces (with a special chapter on conformal mappings)*, Lecture Notes in Mathematics, 1062, Springer-Verlag, Berlin, 1984.
14. J. Jost, *Riemannian geometry and geometric analysis*, Springer, Berlin, 1995.
15. J. Jost and R. Schoen, *On the existence of harmonic diffeomorphisms between surfaces*, Invent. Math. **66** (1982), 353–359.
16. J. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2) 64 (1956), 399–405.
17. J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. Math. (2) 72 (1960), 521–554.
18. C. Uras and A. Verri, *Invariant size functions*, Applications of invariance in computer vision. Lecture Notes in Comput. Sci. **825** (1994), 215–234.
19. ———, *Metric-topological approach to shape representation and recognition*, Image Vision Comput. **14** (1996), 189–207.
20. ———, *Computing size functions from edge maps*, Internat. J. Comput. Vision **23** (1997), no. 2, 169–183.
21. A. Verri, C. Uras, P. Frosini and M. Ferri, *On the use of size functions for shape analysis*, Biol. Cybern. **70** (1993), 99–107.
22. J.H.C. Whitehead, *Manifolds with transverse fields in Euclidean space*, Ann. of Math. **73** (1961), 154–212.

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