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Size Functions and Formal Series

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Abstract. In this paper we consider a mathematical tool for shape description called size function. We prove that every size function can be represented as a set of points and lines in the real plane, with multiplicities. This allows for an algebraic approach to size functions and the construction of new pseudo-distances between size functions for comparing shapes.

Keywords: Size Function, Formal series, Shape descriptor.

1 Introduction

One of the key problems in computer vision is recognizing and classifying objects using digital images. In this context shape description is usually important for recognition. It is well known that the human percept of the shape of an object remains constant despite changes in the object's appearance in images. This fact leads to the search of shape descriptors allowing to determine when shapes are perceived as the same. The possible approaches to this problem may be the most different, from statistics as in [22] to integral transforms such as in [21]. Much effort is put in the search of representations invariant for geometric transformations such as rigid, scale or projective transformations. However, in order to deal not only with rigid objects but also with natural and deformable ones, it seems convenient to combine geometric and topological aspects of shape. Indeed topological invariants such as the Euler number (see, e.g., [4]) and the winding numbers (see, e.g., [24]) have proven to be important features in many image analysis applications.

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Size functions are one of the possible approaches to the problem of describing shapes from the geometric-topological viewpoint. The emphasis on the topological aspect is common to other methods such as [1], [23], [25] and [26]. However size functions integrate topological information with the result of some kind of measurements on the image.

More precisely, measures are performed through any real valued function φ, therefore called *measuring function*, defined on the topological space $\mathscr{M}$ under study (usually a subset of some Euclidean space). The choice of the suitable measuring function depends on the specific application problem we are interested in.

The size function $\ell_{(\mathscr{M},\varphi)} : \mathbb{R}^2 \longrightarrow \mathbb{N} \cup \{+\infty\}$ describes the shape of $\mathscr{M}$ with respect to φ: $\ell_{(\mathscr{M},\varphi)}(x, y)$ is the number of equivalence classes into which the subset $\{P \in \mathscr{M} : \varphi(P) \leq x\} \subseteq \mathscr{M}$ is divided by the equivalence relation of $\langle\varphi \leq y\rangle$-homotopy, where two points $P, Q \in \mathscr{M}$ are $\langle\varphi \leq y\rangle$-homotopic if they either coincide or they can be connected by a continuous path on which the measuring function φ takes a value never greater than y.

Of course different measuring functions generate quite different size functions. By changing measuring functions the corresponding size functions furnish different descriptions of the given shape.

A fundamental property of size functions is that they inherit the invariance properties, if any, of the chosen measuring functions. Thus it is sufficient to take measuring functions with the desired invariance to obtain invariant size functions. These properties may include for instance Euclidean, affine or projective invariance. As we have already pointed out this is very useful in computer vision, where one is often interested in properties up to certain groups of transformations. Invariance of size functions is studied thoroughly in [15] and [29].

Moreover, size functions have proven to be resistant to noise and occlusions and to be easily computable ([18]). For methods to compute size functions we refer the reader to [3, 11, 12, 20]. More details can be obtained in [27] and [30].

No assumptions are made on the nature of $\mathscr{M}$, so in principle any set that can be modeled as a topological space can be represented by size functions. This means that size functions potentially have a broad range of applications, from binary to grey level or colour images but also sound waves and multidimensional medical plots. Up to now size functions have been successfully tested on a number of tasks, both with binary and grey level images: recognition of non-rigid planar shapes such as monograms ([8]), signatures ([6]), hand drawn sketches ([2]), leaves of different species of plants ([33]), hand-gestures ([28, 31]), leukocytes ([9]); viewpoint invariant recognition of rigid shapes such as manufactured objects; aspect-based recognition of 3-D rigid objects such as toy cars ([32]).

Beside applications of size functions to computer vision, this theory appears to be interesting also from the strictly mathematical viewpoint as a geometrical and topological tool to compare manifolds (see [10, 14, 19]).

The aim of this paper is to show that size functions can be regarded as formal series; expressing size functions as formal series allows for their efficient manipulation. More precisely we prove that the set of size functions is in bijection with a particular set of formal series of points and lines in the real plane. This leads to a new approach to size function theory, by translating problems about size functions into an algebraic context.

This result has not only a theoretical interest. Indeed in practice an immediate consequence of representing size functions as formal series is that it allows to reduce the computational burden and for this reason it has already been used in [2] and [8]. But what is more interesting is that it allows for the induction of new pseudo-distances on the set of all size functions in order to quantify the similarity of two shapes. In other words we can produce new and more efficient methods to compare size functions and hence shapes. The usefulness of some of these metrics derived from the formal series representation has already been assessed experimentally (see, e.g., [5]).

In Sect. 2 basic definitions and a few results about discontinuities of size functions are given. In Sect. 3 we give some lemmas and propositions needed to prove our main theorem. This result is given in the same section, showing the bijective correspondence between size functions and a class of formal series. Finally in Sect. 4 we briefly suggest how this result can be applied to the definition of new pseudo-distances between size functions.

2 Some Preliminary Results on Size Functions

2.1 Basic Definitions and Results About Size Functions

Let $\mathcal{M}$ denote a finite union of compact arcwise connected and locally arcwise connected subsets of a Euclidean space. We shall call any pair $(\mathcal{M}, \varphi)$, where $\varphi : \mathcal{M} \to \mathbb{R}$ is a continuous function, a *size pair*. Such a function φ is said to be a *measuring function*. Throughout the rest of the paper assume a size pair $(\mathcal{M}, \varphi)$ is given.

Definition 1 *For every real number y, we shall say that two points $P, Q \in \mathcal{M}$ are $\langle \varphi \le y \rangle$ -homotopic if and only if either $P = Q$ or a continuous path $\gamma : [0, 1] \to \mathcal{M}$ exists in $\mathcal{M}$ joining P and Q such that $\varphi(\gamma(t)) \le y$ for every $t \in [0, 1]$. If P and Q are $\langle \varphi \le y \rangle$-homotopic we shall write $P \cong_{\varphi \le y} Q$ and call γ a $\langle \varphi \le y \rangle$ -homotopy from P to Q.*

It is easy to see that the relation of $\langle \varphi \le y \rangle$-homotopy is an equivalence relation on $\mathcal{M}$ and all its subsets for every $y \in \mathbb{R}$.

Definition 2 *For every $x \in \mathbb{R}$ let $\mathcal{M}\langle \varphi \le x \rangle$ denote the set $\{P \in \mathcal{M} : \varphi(P) \le x\}$. Consider the function $\ell_{(\mathcal{M},\varphi)} : \mathbb{R}^2 \to \mathbb{N} \cup \{+\infty\}$ defined by setting $\ell_{(\mathcal{M},\varphi)}(x, y)$ equal to the number of equivalence classes into which $\mathcal{M}\langle \varphi \le x \rangle$*

is divided by the equivalence relation of $\langle\varphi \le y\rangle$*-homotopy. We shall call* $\ell_{(\mathcal{M},\varphi)}$ *the size function associated with the pair* $(\mathcal{M},\varphi)$.

An example of size function is given in Fig. 1. The shape $\mathcal{M}$ to be studied is depicted to the left. It is the edge of a hand-written "g". A reference frame has been fixed in the plane and the chosen measuring function is the abscissa of the point $\varphi(x, y) = x$. On the right of Fig. 1 we show the corresponding size function. More precisely, we represent only the domain of the size function. The number displayed in each region of the domain denotes the value taken by the size function in that region. For example, on the region of the size function domain with $c \le x < y < d$, this size function takes value equal to 4. This can be easily checked by looking at the figure to the left. Here the set $\mathcal{M}\langle\varphi \le x\rangle$ with $c \le x < d$ is made of 4 arcwise connected components that cannot be joined by paths whose points have abscissa less than $y < d$.

The definition of size function may recall other techniques. For instance, in [23] Kupeev and Wolfson propose a method of estimating shape similarity based on scanning a 2D closed contour along a direction l. Also, there may be some resemblance with the topological sweep studied by Edelsbrunner and Guibas in [7], thus leading to topics of computational geometry. In both cases, however, the differences are greater than the similarities.

Let us now consider the example in Fig. 2. The set $\mathcal{M}$ is depicted on the left and the chosen measuring function is the distance of each point from the barycentre B: $\varphi(P) = d(P, B)$ for every P in $\mathcal{M}$. The corresponding size function is shown on the right. Let us observe that in this case the measuring function is invariant for rotations and translations of $\mathcal{M}$. The corresponding size function

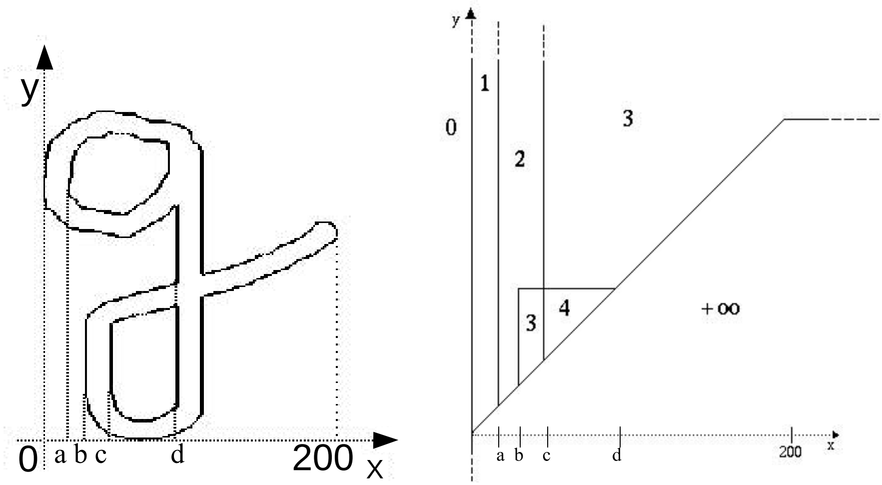

Fig. 1. A subset of the plane (the edge of a "g") and its size function calculated with respect to the measuring function abscissa of the point: $\varphi(x, y) = x$. The number displayed in each region of the domain of the size function denotes the value taken by the size function in that region

inherit this invariance: it is easy to see that by rotating or translating $\mathscr{M}$ one obtains the same size function.

The next example shows the behaviour of size functions in presence of noise. In Fig. 3 we show an ellipse and its size function with respect to the measuring function distance from the barycentre. In Fig. 4 we show the same ellipse perturbed with noise and its size function with respect to the same measuring function. In the size function the noise is revealed by small triangles near the diagonal.

We shall now provide some propositions to point out some simple properties of size functions (see [13, 16]):

Proposition 1 *$\ell_{(\mathscr{M},\varphi)}(x, y)$ is non-decreasing in the variable x and non-increasing in the variable y.*

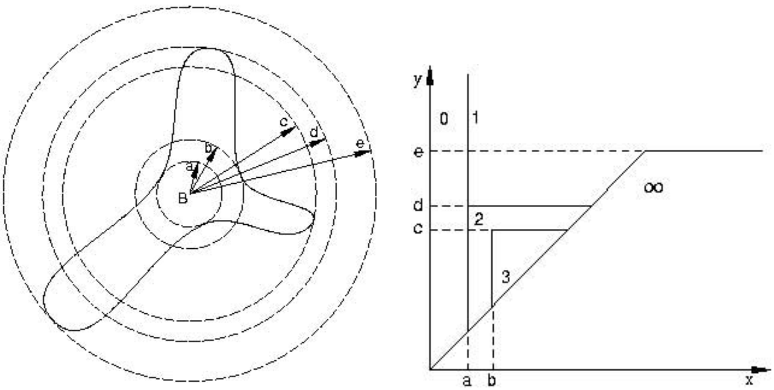

Fig. 2. Size function of the curve depicted on the left with respect to the distance from the barycentre used as measuring function

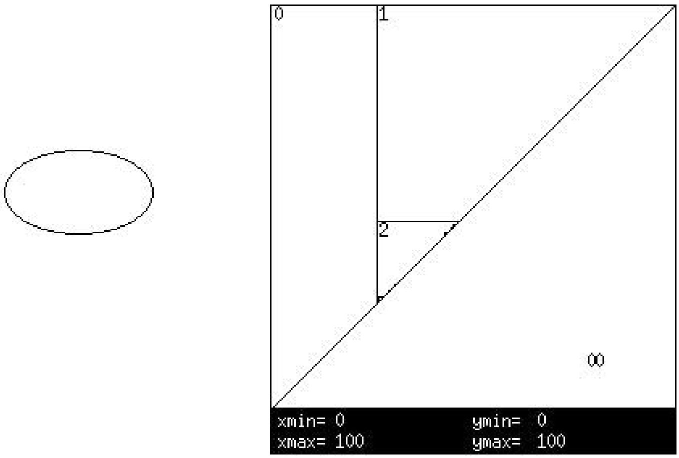

Fig. 3. Size function of an ellipse with respect to the distance from the barycentre

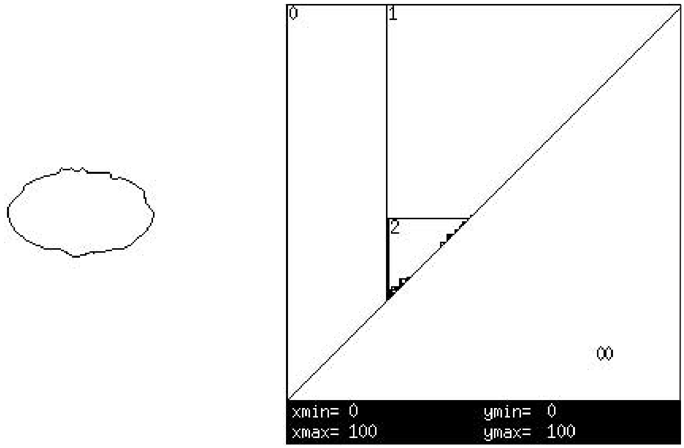

Fig. 4. Size function of a deformed ellipse with respect to the distance from the barycentre

Proposition 2 $\ell_{(\mathscr{M},\varphi)}(x, y) < +\infty$ *for* $x < y$.

In [16] this result is given for subsets of the plane but the same proof holds also for greater dimensions.

Proposition 3 $\ell_{(\mathscr{M},\varphi)}(x, y) = 0$ *for* $x < \min_{P\in\mathscr{M}} \varphi(P)$.

Proposition 4 $\ell_{(\mathscr{M},\varphi)}(x, y) = +\infty$ *for any* x, y *such that there exists a non-isolated point* $Q \in \mathscr{M}$ *with* $y < \varphi(Q) < x$.

Proposition 5 *For every* $y \geq \max_{P\in\mathscr{M}} \varphi(P)$, $\ell_{(\mathscr{M},\varphi)}(x, y)$ *is equal to the number of arcwise connected components* $\mathscr{N}$ *of* $\mathscr{M}$ *such that* $x \geq \min_{P\in\mathscr{N}} \varphi(P)$.

Proposition 3 and Proposition 4 show that, in general, outside the region $\{(x, y) \in \mathbb{R}^2 : x < y, x \geq \min_{P\in\mathscr{M}} \varphi(P)\}$ the size function $\ell_{(\mathscr{M},\varphi)}$ does not convey considerable information about the size pair under study.

Notations: when $\bar{y} \in \mathbb{R}$ is fixed, we shall use the symbol $\ell_{(\mathscr{M},\varphi)}(\cdot, \bar{y})$ to denote the function that takes each real number x to the value $\ell_{(\mathscr{M},\varphi)}(x, \bar{y})$. An analogous convention will hold for the symbol $\ell_{(\mathscr{M},\varphi)}(\bar{x}, \cdot)$.

The symbol $\mathrm{card}(X)$ will denote the cardinality of the set X.

The expression $[r : x = k]$ will denote the line r of equation $x = k$.

2.2 Some Remarks About Discontinuities of Size Functions

Since size functions are natural-valued, their discontinuities are always integer jumps. It is interesting to observe that the discontinuities of size functions behave in a regular way: for $x < y$, discontinuities in the variable x propagate downwards to the diagonal $\Delta = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and discontinuities in the variable y propagate towards the right up to the diagonal Δ. Furthermore jumps of size functions in the x and y directions are monotonic: horizontal jumps and vertical jumps cannot increase as y and x increase, respectively. In order to prove these facts (Corollary 1 and Remark 1) we need the following:

Lemma 1 *Let x_1, x_2, y_1, y_2 be real numbers such that $x_1 \leq x_2 < y_1 \leq y_2$. It holds that*

$$\ell_{(\mathscr{M},\varphi)}(x_2, y_1) - \ell_{(\mathscr{M},\varphi)}(x_1, y_1) \geq \ell_{(\mathscr{M},\varphi)}(x_2, y_2) - \ell_{(\mathscr{M},\varphi)}(x_1, y_2).$$

Proof. Since $x_1 \leq x_2$ we can consider the injective map

$$f : \mathscr{M}\langle\varphi \leq x_1\rangle \big/ {\cong_{\varphi\leq y_1}} \longrightarrow \mathscr{M}\langle\varphi \leq x_2\rangle \big/ {\cong_{\varphi\leq y_1}}$$

induced by the inclusion of $\mathscr{M}\langle\varphi \leq x_1\rangle$ into $\mathscr{M}\langle\varphi \leq x_2\rangle$. Therefore

$$\begin{aligned}
&\ell_{(\mathscr{M},\varphi)}(x_2, y_1) - \ell_{(\mathscr{M},\varphi)}(x_1, y_1)\\
&\quad = \mathrm{card}(\mathscr{M}\langle\varphi \leq x_2\rangle \big/ {\cong_{\varphi\leq y_1}}) - \mathrm{card}(\mathscr{M}\langle\varphi \leq x_1\rangle \big/ {\cong_{\varphi\leq y_1}})\\
&\quad = \mathrm{card}(\mathscr{M}\langle\varphi \leq x_2\rangle \big/ {\cong_{\varphi\leq y_1}}) - \mathrm{card}(f(\mathscr{M}\langle\varphi \leq x_1\rangle \big/ {\cong_{\varphi\leq y_1}}))\\
&\quad = \mathrm{card}(\mathscr{M}\langle\varphi \leq x_2\rangle \big/ {\cong_{\varphi\leq y_1}} - f(\mathscr{M}\langle\varphi \leq x_1\rangle \big/ {\cong_{\varphi\leq y_1}}))
\end{aligned}$$

that is, the number of equivalence classes in the quotient set

$$\{P \in \mathscr{M} : x_1 < \varphi(P) \leq x_2\} \big/ {\cong_{\varphi\leq y_1}}$$

(we are using the hypothesis $x_1 \leq x_2 < y_1$ for the finiteness of $\ell_{(\mathscr{M},\varphi)}(x_2, y_1)$ and $\ell_{(\mathscr{M},\varphi)}(x_1, y_1)$).

Analogously,

$$\ell_{(\mathscr{M},\varphi)}(x_2, y_2) - \ell_{(\mathscr{M},\varphi)}(x_1, y_2) = \mathrm{card}\,\{P \in \mathscr{M} : x_1 < \varphi(P) \leq x_2\} \big/ {\cong_{\varphi\leq y_2}} .$$

Hence the claim immediately follows from the inequality $y_1 \leq y_2$ and the definition of $\langle\varphi \leq y\rangle$-homotopy. □

Remark 1. The inequality in Lemma 1 simply means that in the half-plane $x < y$ horizontal jumps of a size function are non-increasing in the variable y (see Fig. 5). From the same lemma one of course gets the analogous statement that vertical jumps are non-increasing in the variable x by rewriting the above inequality as

$$\ell_{(\mathscr{M},\varphi)}(x_1, y_2) - \ell_{(\mathscr{M},\varphi)}(x_1, y_1) \geq \ell_{(\mathscr{M},\varphi)}(x_2, y_2) - \ell_{(\mathscr{M},\varphi)}(x_2, y_1).$$

Now we can prove the previously mentioned result about the propagation of discontinuities towards the diagonal Δ.

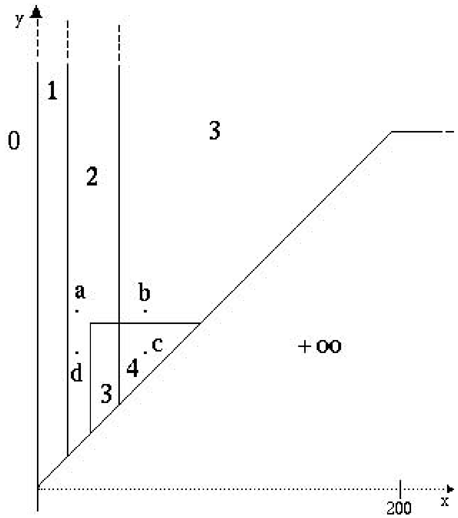

Fig. 5. Horizontal jumps are non-increasing in the variable y: $\ell_{(\mathcal{M},\varphi)}(b) - \ell_{(\mathcal{M},\varphi)}(a) = 1 \leq \ell_{(\mathcal{M},\varphi)}(c) - \ell_{(\mathcal{M},\varphi)}(d) = 2$; vertical jumps are non-increasing in the variable x: $\ell_{(\mathcal{M},\varphi)}(b) - \ell_{(\mathcal{M},\varphi)}(c) = -1 \leq \ell_{(\mathcal{M},\varphi)}(a) - \ell_{(\mathcal{M},\varphi)}(d) = 0$

Corollary 1 *The following statements hold*:

(i) *If* $\bar{x} \in \mathbb{R}$ *is a discontinuity point for* $\ell_{(\mathcal{M},\varphi)}(\cdot, \bar{y})$ *and* $\bar{x} < y < \bar{y}$ *then* $\bar{x}$ *is a discontinuity point also for* $\ell_{(\mathcal{M},\varphi)}(\cdot, y)$;

(ii) *If* $\bar{y} \in \mathbb{R}$ *is a discontinuity point for* $\ell_{(\mathcal{M},\varphi)}(\bar{x}, \cdot)$ *and* $\bar{x} < x < \bar{y}$ *then* $\bar{y}$ *is a discontinuity point also for* $\ell_{(\mathcal{M},\varphi)}(x, \cdot)$.

Proof. (i) Contrary to our claim assume that for some y with $\bar{x} < y < \bar{y}$ $\ell_{(\mathcal{M},\varphi)}(\cdot, y)$ is continuous at $\bar{x}$. Then $\lim_{x \to \bar{x}^+} \ell_{(\mathcal{M},\varphi)}(x, y) - \ell_{(\mathcal{M},\varphi)}(\bar{x}, y) = 0$. Hence Lemma 1 together with the fact that size functions are non-decreasing in x (Proposition 1) imply $\lim_{x \to \bar{x}^+} \ell_{(\mathcal{M},\varphi)}(x, \bar{y}) - \ell_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}) = 0$. Analogously, $\lim_{x \to \bar{x}^-} \ell_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}) - \ell_{(\mathcal{M},\varphi)}(x, \bar{y}) = 0$. Hence $\bar{x}$ would be a continuity point for $\ell_{(\mathcal{M},\varphi)}(\cdot, \bar{y})$, thus contradicting our hypothesis.

The proof for (ii) is similar. □

We shall see later (Corollary 6) that every discontinuity point for a size function is necessarily a discontinuity point in the x or y direction. So far we can only prove a weaker result:

Lemma 2 *Any open arcwise connected neighborhood of a discontinuity point for a size function contains at least one discontinuity point in the variable* x *or* y.

Proof. Let $p \in \mathbb{R}^2$ be a discontinuity point for $\ell_{(\mathcal{M},\varphi)}$. Then, in any open arcwise connected neighborhood $U \subseteq \mathbb{R}^2$ of p, a point q exists such that $\ell_{(\mathcal{M},\varphi)}(p) \neq \ell_{(\mathcal{M},\varphi)}(q)$. We can connect p and q by a path entirely contained in U made of segments parallel to the x and y axes. $\ell_{(\mathscr{M},\varphi)}$ cannot be constant along this path. Hence the claim. $\square$

Moreover, some constraints on the presence of discontinuities in size functions exist.

Proposition 6 *For every point* $\bar{p} = (\bar{x}, \bar{y}) \in \mathbb{R}^2$ *with* $\bar{x} < \bar{y}$ *an* $\epsilon > 0$ *exists such that the open set*

$$W_\epsilon(\bar{p}) = \{(x, y) \in \mathbb{R}^2 : |\bar{x} - x| < \epsilon, |\bar{y} - y| < \epsilon, x \neq \bar{x}, y \neq \bar{y}\}$$

does not contain any discontinuity point for $\ell_{(\mathscr{M},\varphi)}$.

Proof. Suppose, contrary to our assertion, that for every $n \in \mathbb{N}^+$ a discontinuity point $p_n = (x_n, y_n)$ in $W_{1/n}(\bar{p})$ exists. By applying the previous Lemma 2, possibly by extracting a subsequence from $(p_n)_{n\in\mathbb{N}^+}$, we can assume that each p_n is a discontinuity point in either the x or y direction. In the following, we shall assume that each p_n is a discontinuity point in the variable x. The case in which each p_n is a discontinuity point in the variable y has a similar proof.

Let us fix a natural number N that is sufficiently large so that $\bar{x} + 1/N < \bar{y} - 1/N$, i.e. the sets $W_{1/n}(\bar{p})$ with $n \geq N$ lie entirely above the diagonal Δ. Let us consider the function $\ell_{(\mathscr{M},\varphi)}(\cdot, \bar{y} - 1/N) : (\bar{x} - 1/N, \bar{x} + 1/N) \subset \mathbb{R} \longrightarrow \mathbb{N}$. From Corollary 1 we know that discontinuities in x spread downwards. Thus the function $\ell_{(\mathscr{M},\varphi)}(\cdot, \bar{y} - 1/N)$ should have an infinite number of integer jumps. Now, since size functions are non-decreasing in the variable x, this fact would imply that $\ell_{(\mathscr{M},\varphi)}(\bar{x} + 1/N, \bar{y} - 1/N) = +\infty$, thus contradicting Proposition 2. $\square$

Similarly, by recalling that for $y \geq \max_{P\in\mathscr{M}} \varphi(P)$ each path in $\mathscr{M}$ is a $\langle\varphi \leq y\rangle$-homotopy, the following result can be proven:

Proposition 7 *For every vertical line* $[\bar{r} : x = \bar{x}]$ *an* $\epsilon > 0$ *exists such that the open set*

$$V_\epsilon(\bar{r}) = \{(x, y) \in \mathbb{R}^2 : |\bar{x} - x| < \epsilon, y > 1/\epsilon, x \neq \bar{x}\}$$

does not contain any discontinuity point for $\ell_{(\mathscr{M},\varphi)}$.

The previous results show that discontinuities divide the part of the domain of a size function lying above the diagonal Δ into overlapping triangular regions (possibly of infinite area) leaning against the diagonal. For example, in the size function represented in Fig. 6 there are two triangles of infinite area above the diagonal Δ bounded by the diagonal itself and, respectively, by the vertical lines r_1 and r_2. Moreover, there are four overlapping triangles with one side on the diagonal and the opposite vertex respectively at p_1, p_2, p_4 and p_5. As we shall clarify later, the triangle with vertex at p_3 can be seen as a result of the overlapping of those with vertices at p_1 and p_2. The fact that this function is really the size function of a suitable size pair will be shown in the next section.

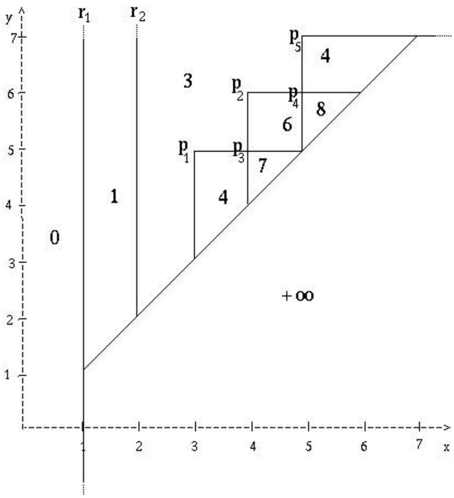

Fig. 6. How discontinuities of a size function divide its domain into overlapping triangular regions with one side on the diagonal Δ

These facts suggest a way to encode the information contained in size functions into a more compact and manageable structure.

Remark 2. Let us finally point out that the behaviour of discontinuities in the variable x differs from that in the variable y in the following sense. The definition of size function implies that, for $\bar{x} < \bar{y}$, the equality $\ell_{(\mathscr{M},\varphi)}(\bar{x}, \bar{y}) = \lim_{x\to\bar{x}^+} \ell_{(\mathscr{M},\varphi)}(x, \bar{y})$ always holds, i.e. $\ell_{(\mathscr{M},\varphi)}(\cdot, \bar{y})$ is right-continuous.

On the other hand it has been proven in [14] that at a discontinuity point in the variable y, say $\tilde{p} = (\tilde{x}, \tilde{y})$, it may happen that $\ell_{(\mathscr{M},\varphi)}(\tilde{p}) \neq \lim_{y\to\tilde{y}^+} \ell_{(\mathscr{M},\varphi)}(\tilde{x}, y)$, i.e. $\ell_{(\mathscr{M},\varphi)}(\tilde{x}, \cdot)$ may not be right-continuous.

3 Size Functions and Formal Series

3.1 Preparing the Proof of Theorem 1: Some Definitions and Results

Our aim is to capture information (about the shape under study) contained in a size function, i.e. its values and discontinuities, in algebraic language. The formal object we shall obtain will enable easier computations and reduced information storage. In order to do this we shall represent a size function by giving the list of triangular regions in which its domain is divided by discontinuities. Furthermore, we shall assign a multiplicity to each triangle so that the sum of the multiplicities of triangles containing a given point gives the value of the size function at that point.

Let us now formalize this idea.

We shall denote the set $\{(x, y) \in \mathbb{R}^2 : x < y\}$ by $\mathscr{S}_0$ and the set of vertical lines of $\mathbb{R}^2$, i.e. lines of equation $x = k$ with $k \in \mathbb{R}$, by $\mathscr{R}$.

Moreover, we shall call $\mathscr{X}$ the set $\mathscr{S}_0 \cup \mathscr{R}$. We recall the following definitions:

Definition 3 *Any function $m : \mathscr{X} \to \mathbb{Z}$ is said to be a formal series in $\mathscr{X}$. The set supp$(m) = \{X \in \mathscr{X} : m(X) \neq 0\}$ is called the support of m.*

The set of formal series in $\mathscr{X}$ is a commutative group with respect to the usual sum of functions. Thus it makes sense to denote each such formal series m by the symbol $\sum_{X \in \mathscr{X}} m(X)X$.

Definition 4 *For every point $p = (x, y) \in \mathbb{R}^2$ and real positive numbers α, β with $(x + \alpha, y - \beta) \in \mathscr{S}_0$ let us define the number $\mu_{\alpha,\beta}(p)$ as*

$$\ell_{(\mathscr{M},\varphi)}(x + \alpha, y - \beta) - \ell_{(\mathscr{M},\varphi)}(x + \alpha, y + \beta)$$
$$-\ell_{(\mathscr{M},\varphi)}(x - \alpha, y - \beta) + \ell_{(\mathscr{M},\varphi)}(x - \alpha, y + \beta).$$

The number $\mu(p) := \min\{\mu_{\alpha,\beta}(p) : \alpha, \beta > 0, x + \alpha < y - \beta\}$ will be called multiplicity of p for $\ell_{(\mathscr{M},\varphi)}$. Moreover, we shall call cornerpoint for $\ell_{(\mathscr{M},\varphi)}$ any point $p \in \mathscr{S}_0$ such that the number $\mu(p)$ is strictly positive.

Remark 3. Obviously, for each $\alpha, \beta > 0$ with $x + \alpha < y - \beta$, $\mu_{\alpha,\beta}(p)$ is an integer number and from Lemma 1 we see that it is non-negative. Hence $\mu(p)$ is well defined and non-negative for every $p \in \mathscr{S}_0$.

Proposition 6 implies that for sufficiently small α and β each term in the sum defining $\mu_{\alpha,\beta}(p)$ is constant. Furthermore, by using Lemma 1 twice, we can easily prove that $\mu_{\alpha,\beta}(p)$ is non-decreasing in α and β. It follows that $\mu(p) = \lim_{\alpha,\beta \to 0^+} \mu_{\alpha,\beta}(p)$, thus giving an alternative definition of $\mu(p)$.

In the size function represented in Fig. 6 the only cornerpoints are p_1, p_2, p_4 and p_5 with multiplicities $\mu(p_1) = \mu(p_4) = \mu(p_5) = 1$ and $\mu(p_2) = 3$. The point p_3 is not a cornerpoint since $\mu(p_3) = 0$.

The key role of cornerpoints is demonstrated by the following proposition, which shows that each of them creates discontinuity points spreading downwards and towards the right up to the diagonal Δ.

Proposition 8 *If $\bar{p} = (\bar{x}, \bar{y})$ is a cornerpoint for $\ell_{(\mathscr{M},\varphi)}$ then the following statements hold*:

(i) *If $\bar{x} \le x < \bar{y}$ then $\bar{y}$ is a discontinuity point for $\ell_{(\mathscr{M},\varphi)}(x, \cdot)$;*
(ii) *If $\bar{x} < y < \bar{y}$ then $\bar{x}$ is a discontinuity point for $\ell_{(\mathscr{M},\varphi)}(\cdot, y)$.*

Proof. Since $\bar{p} = (\bar{x}, \bar{y})$ is a cornerpoint for $\ell_{(\mathscr{M},\varphi)}$,

$$\ell_{(\mathscr{M},\varphi)}(\bar{x} + \alpha, \bar{y} - \beta) - \ell_{(\mathscr{M},\varphi)}(\bar{x} + \alpha, \bar{y} + \beta)$$
$$-\ell_{(\mathscr{M},\varphi)}(\bar{x} - \alpha, \bar{y} - \beta) + \ell_{(\mathscr{M},\varphi)}(\bar{x} - \alpha, \bar{y} + \beta) > 0$$

for every positive α and β for which $\bar{x} + \alpha < \bar{y} - \beta$.

By recalling Proposition 1 we obtain that $\ell_{(\mathscr{M},\varphi)}(\bar{x}-\alpha,\bar{y}-\beta) \geq \ell_{(\mathscr{M},\varphi)}(\bar{x}-\alpha,\bar{y}+\beta)$ and $\ell_{(\mathscr{M},\varphi)}(\bar{x}+\alpha,\bar{y}+\beta) \geq \ell_{(\mathscr{M},\varphi)}(\bar{x}-\alpha,\bar{y}+\beta)$. It follows that

$$\ell_{(\mathscr{M},\varphi)}(\bar{x}+\alpha,\bar{y}-\beta) - \ell_{(\mathscr{M},\varphi)}(\bar{x}+\alpha,\bar{y}+\beta) > 0 \tag{1}$$

and

$$\ell_{(\mathscr{M},\varphi)}(\bar{x}+\alpha,\bar{y}-\beta) - \ell_{(\mathscr{M},\varphi)}(\bar{x}-\alpha,\bar{y}-\beta) > 0 \tag{2}$$

for every positive α and β, such that $\bar{x}+\alpha < \bar{x}-\beta$.

Let us now prove assertion (i). Since α is arbitrary and inequality (1) holds, we obtain $\lim_{\epsilon\to 0^+} \ell_{(\mathscr{M},\varphi)}(\bar{x}+\epsilon,\bar{y}-\beta) - \lim_{\epsilon\to 0^+} \ell_{(\mathscr{M},\varphi)}(\bar{x}+\epsilon,\bar{y}+\beta) > 0$. Let us now recall that $\ell_{(\mathscr{M},\varphi)}(\bar{x},\bar{y}-\beta) = \lim_{\epsilon\to 0^+} \ell_{(\mathscr{M},\varphi)}(\bar{x}+\epsilon,\bar{y}-\beta)$ and $\ell_{(\mathscr{M},\varphi)}(\bar{x},\bar{y}+\beta) = \lim_{\epsilon\to 0^+} \ell_{(\mathscr{M},\varphi)}(\bar{x}+\epsilon,\bar{y}+\beta)$ (Remark 2). Therefore $\ell_{(\mathscr{M},\varphi)}(\bar{x},\bar{y}-\beta) - \ell_{(\mathscr{M},\varphi)}(\bar{x},\bar{y}+\beta) > 0$. Since β can be chosen to be arbitrarily small it follows that $\bar{y}$ is a discontinuity point for $\ell_{(\mathscr{M},\varphi)}(\bar{x},\cdot)$. Now it is sufficient to apply Corollary 1 to conclude.

Let us now consider assertion (ii). By inequality (2) and by taking α to be arbitrarily small we can show that $\bar{x}$ is a discontinuity point for $\ell_{(\mathscr{M},\varphi)}(\cdot,\bar{y}-\beta)$. By making β smaller and smaller and by recalling Corollary 1 we complete the proof. □

The previous proposition implies that every cornerpoint is a discontinuity point in the y direction.

The converse of Proposition 8 fails to be true, as can be checked by looking at the point p_3 in Fig. 6. Indeed, in the size function represented in Fig. 6 the only cornerpoints are p_1, p_2, p_4 and p_5.

Corollary 2 *$\mathscr{S}_0$ does not contain accumulation points for the set of cornerpoints. In particular, cornerpoints are isolated points.*

Proof. If this claim were not true, by Proposition 8 we would get a contradiction of Proposition 6. □

Furthermore, by Propositions 3, 5 and their definition, cornerpoints lie in the region $\{(x,y)\in\mathbb{R}^2 : x<y, x\geq \min_{p\in\mathscr{M}}\varphi(p), y\leq \max_{p\in\mathscr{M}}\varphi(p)\}$. Hence, in case a size function has an infinite number of cornerpoints, they must accumulate onto the diagonal Δ. An example of size function with cornerpoints accumulating onto the diagonal is given in Fig. 7.

For every $\rho>0$ let $\mathscr{S}_\rho = \{(x,y)\in\mathbb{R}^2 : x<y-\rho\}$. In view of the above discussion we have:

Corollary 3 *Size functions have at most a finite number of cornerpoints in $\mathscr{S}_\rho$ for any $\rho>0$.*

The main purpose of this paper is to use cornerpoints to identify triangular regions of finite area which, as we have seen in Sect. 2, are a major feature of size functions.

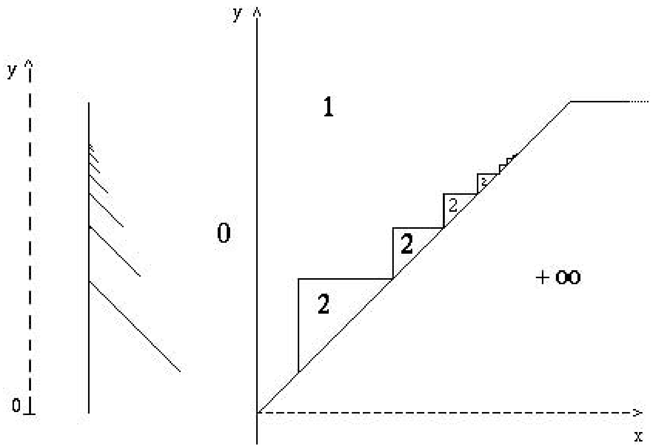

Fig. 7. A topological space (on the left) whose size function, with respect to the measuring function $\varphi(x, y) = y$, has cornerpoints accumulating onto the diagonal Δ

Following this idea, let us now represent triangles of infinite area that can be cut out by discontinuities of a size function.

Definition 5 *For every vertical line $[r : x = k] \in \mathscr{R}$ we shall call multiplicity of r for $\ell_{(\mathscr{M},\varphi)}$ the number*

$$\mu(r) := \min_{\alpha>0, k+\alpha<y} \ell_{(\mathscr{M},\varphi)}(k+\alpha, y) - \ell_{(\mathscr{M},\varphi)}(k-\alpha, y).$$

When $\mu(r)$ is strictly positive, the line r will be said to be a cornerline for the size function.

Observe that the notion of cornerline is a natural extension of the notion of cornerpoint for "points at infinity", where $\ell_{(\mathscr{M},\varphi)}$ can be assumed to be vanishing. In the size function of Fig. 6 the only cornerlines are given by r_1 and r_2 with the multiplicities $\mu(r_1) = 1$ and $\mu(r_2) = 2$.

Remark 4. Obviously, from Proposition 1 we know that $\mu(r)$ is well defined and non-negative for every $r \in \mathscr{R}$.

Moreover, Proposition 1 and Lemma 1 show that the value $\ell_{(\mathscr{M},\varphi)}(k+\alpha, y) - \ell_{(\mathscr{M},\varphi)}(k-\alpha, y)$ is non-decreasing in α and non-increasing in y, respectively.

From all this it follows that $\mu(r) = \lim_{\epsilon\to 0^+} \ell_{(\mathscr{M},\varphi)}(k+\epsilon, 1/\epsilon) - \ell_{(\mathscr{M},\varphi)}(k-\epsilon, 1/\epsilon)$, thus giving an alternative definition of $\mu(r)$.

On the other hand, by recalling Proposition 5 and since $\mathscr{M}$ contains only a finite number of arcwise connected components, it follows that for a small enough ϵ the number $\ell_{(\mathscr{M},\varphi)}(k+\epsilon, 1/\epsilon) - \ell_{(\mathscr{M},\varphi)}(k-\epsilon, 1/\epsilon)$ counts the arcwise connected components of $\mathscr{M}$ on which φ takes k as minimum value. All this proves the following proposition, yielding yet another definition of $\mu(r)$.

Proposition 9 *Let $c(k)$ be the number of arcwise connected components of $\mathscr{M}$ on which φ takes k as a global minimum. Consider the vertical line $[r : x = k]$. Then $\mu(r) = c(k)$. Therefore, r is a cornerline for $\ell_{(\mathscr{M},\varphi)}$ if and only if $k = \min_{P\in\mathscr{N}} \varphi(P)$ for at least one arcwise connected component $\mathscr{N}$ of $\mathscr{M}$.*

The previous proposition together with the definition of size function easily imply the following result:

Corollary 4 *If $[r : x = k]$ is a cornerline for $\ell_{(\mathscr{M},\varphi)}$ and $y > k$ then k is a discontinuity point for $\ell_{(\mathscr{M},\varphi)}(\cdot, y)$.*

In other words, any point on a cornerline is a discontinuity point in the variable x for $\ell_{(\mathscr{M},\varphi)}$.

Proposition 9 also implies that the number of cornerlines for a size function $\ell_{(\mathscr{M},\varphi)}$ is never greater than the number of arcwise connected components of $\mathscr{M}$. From our assumption on $\mathscr{M}$ we conclude:

Corollary 5 *Size functions have a finite number of cornerlines.*

Cornerpoints and cornerlines are naturally related to bounded and unbounded triangles, as the following definition reveals:

Definition 6 *If $\bar{p} = (\bar{x}, \bar{y})$ is a cornerpoint for $\ell_{(\mathscr{M},\varphi)}$ we shall call the set $\{(x, y) \in \mathbb{R}^2 : \bar{x} \le x < y < \bar{y}\}$ the main triangle of $\bar{p}$. Analogously, if $[r : x = k]$ is a cornerline for $\ell_{(\mathscr{M},\varphi)}$ we shall call the set $\{(x, y) \in \mathbb{R}^2 : k \le x < y\}$ the main (unbounded) triangle of r.*

The multiplicity of a cornerpoint or a cornerline will also be called the multiplicity $\mu(\tau)$ of the related main triangle τ.

Moreover, for each point $p \in \mathscr{S}_0$, we shall denote by $T(p)$ the set of all main triangles containing p.

Note that each main triangle contains only a proper subset of its boundary, namely the left vertical segment.

Now we give a lemma which will be useful for further results in this paper. In plain words it states that for each discontinuity point in the variable x we find either a cornerpoint above it or a cornerline through it; similarly, on the left of each discontinuity point in the variable y there is always a cornerpoint.

Lemma 3 (i) *If $\bar{x}$ is a discontinuity point for $\ell_{(\mathscr{M},\varphi)}(\cdot, \bar{y})$ with $\bar{x} < \bar{y}$ then either there is a cornerpoint for $\ell_{(\mathscr{M},\varphi)}$ on the closed half-line $\{(\bar{x}, y) \in \mathbb{R}^2 : \bar{y} \le y\}$ or line $x = \bar{x}$ is a cornerline, or both cases occur.*

(ii) *If $\bar{y}$ is a discontinuity point for $\ell_{(\mathscr{M},\varphi)}(\bar{x}, \cdot)$ with $\bar{x} < \bar{y}$ then there is a cornerpoint for $\ell_{(\mathscr{M},\varphi)}$ on the closed half-line $\{(x, \bar{y}) \in \mathbb{R}^2 : x \le \bar{x}\}$.*

Proof. (i) If the line $x = \bar{x}$ is not a cornerline then $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\bar{x}+\epsilon, 1/\epsilon) - \ell_{(\mathcal{M},\varphi)}(\bar{x}-\epsilon, 1/\epsilon) = 0$ (see Remark 4) and hence $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\bar{x}+\epsilon, y) - \ell_{(\mathcal{M},\varphi)}(\bar{x}-\epsilon, y) = 0$ for every y large enough. Since $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\bar{x}+\epsilon, \bar{y}) - \ell_{(\mathcal{M},\varphi)}(\bar{x}-\epsilon, \bar{y}) > 0$, a value $\hat{y} \geq \bar{y}$ must exist such that $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\bar{x}+\epsilon, \hat{y}+\eta) - \ell_{(\mathcal{M},\varphi)}(\bar{x}-\epsilon, \hat{y}+\eta) = 0$ and $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\bar{x}+\epsilon, \hat{y}-\eta) - \ell_{(\mathcal{M},\varphi)}(\bar{x}-\epsilon, \hat{y}-\eta) > 0$ for every $\eta > 0$. Since $\mu_{\alpha,\beta}(p)$ is constant for small enough α and β and for fixed p (see Remark 3), it follows that $\lim_{\alpha,\beta \to 0^+} \mu_{\alpha,\beta}(\bar{x}, \hat{y}) > 0$, so that $(\bar{x}, \hat{y})$ is a cornerpoint. This proves (i).

(ii) Proposition 3 implies that $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(x, \bar{y}-\epsilon) - \ell_{(\mathcal{M},\varphi)}(x, \bar{y}+\epsilon) = 0$ for every $x < \min_{P \in \mathcal{M}} \varphi(P)$, while $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}-\epsilon) - \ell_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}+\epsilon) > 0$. Hence a value $\hat{x} \leq \bar{x}$ must exist such that $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\hat{x}-\eta, \bar{y}-\epsilon) - \ell_{(\mathcal{M},\varphi)}(\hat{x}-\eta, \bar{y}+\epsilon) = 0$ and $\lim_{\epsilon \to 0^+} \ell_{(\mathcal{M},\varphi)}(\hat{x}+\eta, \bar{y}-\epsilon) - \ell_{(\mathcal{M},\varphi)}(\hat{x}+\eta, \bar{y}+\epsilon) > 0$ for every $\eta > 0$. As above, it follows that $\lim_{\alpha,\beta \to 0^+} \mu_{\alpha,\beta}(\bar{x}, \hat{y}) > 0$, and so $(\hat{x}, \bar{y})$ is a cornerpoint. This proves (ii). $\square$

The value taken by a size function at a point is related to multiplicities of main triangles as the following results show:

Lemma 4 *Assume that two points $\bar{p}$ and $\tilde{p}$ in $\mathbb{R}^2$ are given, satisfying the following conditions: $\bar{p} = (\bar{x}, \bar{y})$, $\tilde{p} = (\tilde{x}, \bar{y})$ with $\tilde{x} < \bar{x} < \bar{y}$ and $\bar{p}$ is a continuity point in the variable y for the size function $\ell_{(\mathcal{M},\varphi)}$. Moreover, assume that the closed segment connecting $\tilde{p}$ to $\bar{p}$ meets one and only one discontinuity point $\bar{q}$ in the variable x for $\ell_{(\mathcal{M},\varphi)}$, and that $\bar{q} \neq \tilde{p}$.*

Then $T(\tilde{p})$ is properly contained in $T(\bar{p})$ and the number $\ell_{(\mathcal{M},\varphi)}(\bar{p}) - \ell_{(\mathcal{M},\varphi)}(\tilde{p})$ equals the sum of the multiplicities of main triangles in $T(\bar{p}) - T(\tilde{p})$.

Proof. Let $[r : x = k]$ be the vertical line through $\bar{q}$. Finiteness and monotonicity of size functions easily imply that in the closed half-line $\{(x, y) \in \mathbb{R}^2 : x = k, y \geq \bar{y}\}$ there must exist only finitely many discontinuity points in the variable y for $\ell_{(\mathcal{M},\varphi)}$, say $q^i = (k, y^i)$ with i varying in a finite set of indexes $\mathcal{I}$. Let $\mathcal{I} = \{1, 2, \ldots, h\}$ in the case that it is non-empty. We point out that $\bar{q} \neq q^i$ for every $i \in \mathcal{I}$. In fact, as $\bar{p}$ is a continuity point in the variable y for $\ell_{(\mathcal{M},\varphi)}$, by Corollary 1 (ii) the same must hold for all points in the closed segment $\tilde{p}\bar{p}$.

Since $\bar{q}$ is the only discontinuity point in the variable x on the closed segment $\tilde{p}\bar{p}$, by Corollary 1(i) there cannot exist discontinuity points in the variable x in the strip $[\tilde{x}, \bar{x}] \times [\bar{y}, +\infty)$ except on the line r. From Proposition 8 and Corollary 4 it follows that the strip $[\tilde{x}, \bar{x}] \times [\bar{y}, +\infty)$ can contain neither cornerpoints nor cornerlines except on the line r. Hence Lemma 3(ii) and the fact that every cornerpoint is a discontinuity point in the y direction show that if y is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(x, \cdot)$, and $(x, y) \in [\tilde{x}, \bar{x}] \times [\bar{y}, +\infty)$ then $y = y^i$ for a suitable index $i \in \mathcal{I}$. All this implies that $\ell_{(\mathcal{M},\varphi)}$ is constant in the rectangles $[\tilde{x}, k) \times (y^i, y^{i+1})$ and $[k, \bar{x}] \times (y^i, y^{i+1})$ for $i = 1, \ldots, h-1$, as well as in the rectangles $[\tilde{x}, k) \times (\bar{y}, y^1)$ and $[k, \bar{x}] \times (\bar{y}, y^1)$, and in the two strips $[\tilde{x}, k) \times (y^h, +\infty)$ and $[k, \bar{x}] \times (y^h, +\infty)$ (see Fig. 8).

Thus, if $\mathscr{I} \neq \emptyset$ for each q^i we can take four points $(\tilde{x}, y_1^i)$, $(\tilde{x}, y_2^i)$, $(\bar{x}, y_1^i)$, $(\bar{x}, y_2^i)$, such that $\bar{x} < y_1^i < y^i < y_2^i$, $\ell_{(\mathscr{M},\varphi)}(\bar{x}, y_1^i) - \ell_{(\mathscr{M},\varphi)}(\tilde{x}, y_1^i) - \ell_{(\mathscr{M},\varphi)}(\bar{x}, y_2^i) + \ell_{(\mathscr{M},\varphi)}(\tilde{x}, y_2^i) = \mu(q^i)$ (possibly $\mu(q^i) = 0$) and, for every $i = 1, \ldots, h-1$, $y_2^i = y_1^{i+1}$. Moreover since $\bar{p}$ and $\tilde{p}$ are continuity points in the variable y for $\ell_{(\mathscr{M},\varphi)}$, in case $\mathscr{I} \neq \emptyset$ we can also assume $y_1^1 = \bar{y}$ (see Fig. 9).

Therefore, for $\mathscr{I} \neq \emptyset$,

$$\ell_{(\mathscr{M},\varphi)}(\bar{p}) - \ell_{(\mathscr{M},\varphi)}(\tilde{p}) = \mu(q^1) + \ell_{(\mathscr{M},\varphi)}(\bar{x}, y_2^1) - \ell_{(\mathscr{M},\varphi)}(\tilde{x}, y_2^1)$$

$$= \cdots = \sum_{i=1,\ldots,h} \mu(q^i) + \ell_{(\mathscr{M},\varphi)}(\bar{x}, y_2^h) - \ell_{(\mathscr{M},\varphi)}(\tilde{x}, y_2^h).$$

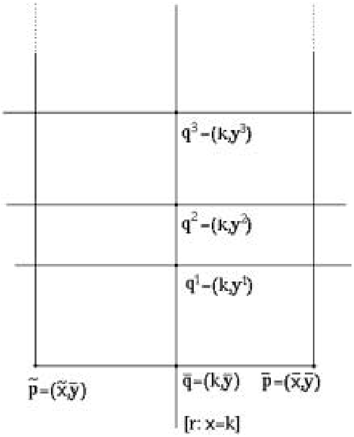

Fig. 8. Proof of Lemma 4: in the displayed rectangles and strips the size function $\ell_{(\mathscr{M},\varphi)}$ is constant

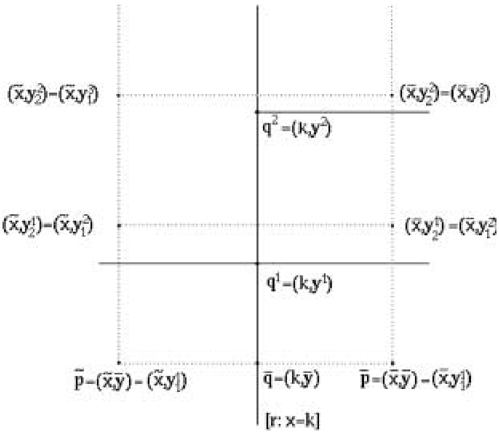

Fig. 9. How to choose the points $(\tilde{x}, y_1^i)$, $(\tilde{x}, y_2^i)$, $(\bar{x}, y_1^i)$, $(\bar{x}, y_2^i)$ as described in the proof of Lemma 4

Now let us observe that $\ell_{(\mathcal{M},\varphi)}(\bar{x}, y_2^h) - \ell_{(\mathcal{M},\varphi)}(\tilde{x}, y_2^h) = \mu(r)$ (possibly $\mu(r) = 0$). Thus

$$\ell_{(\mathcal{M},\varphi)}(\bar{p}) - \ell_{(\mathcal{M},\varphi)}(\tilde{p}) = \mu(r) + \sum_{i=1,\dots,h} \mu(q^i). \tag{3}$$

Now Proposition 8(i) implies that all cornerpoints lying on the closed half-line $\{(x, y) \in \mathbb{R}^2 : x = k, y \geq \bar{y}\}$ belong to $\{q^i : i \in \mathscr{I}\}$. Hence the right-hand side in (3) reduces to the sum of the multiplicities of cornerpoints on the half-line $\{(x, y) \in \mathbb{R}^2 : x = k, y \geq \bar{y}\}$ plus the multiplicity of r in the case that r is a cornerline. This number is easily seen to equal the sum of the multiplicities of main triangles in the set $T(\bar{p}) - T(\tilde{p})$.

If $\mathscr{I} = \emptyset$ we simply obtain $\ell_{(\mathcal{M},\varphi)}(\bar{p}) - \ell_{(\mathcal{M},\varphi)}(\tilde{p}) = \mu(r) > 0$, and r is a cornerline. Also in this case $\ell_{(\mathcal{M},\varphi)}(\bar{p}) - \ell_{(\mathcal{M},\varphi)}(\tilde{p})$ equals $\sum_{\tau \in (T(\bar{p}) - T(\tilde{p}))} \mu(\tau)$.

In both cases $\mathscr{I} \neq \emptyset$ and $\mathscr{I} = \emptyset$ there is at least one main triangle containing $\bar{p}$ and not containing $\tilde{p}$. Moreover, $T(\tilde{p}) \subseteq T(\bar{p})$ because $\tilde{x} < \bar{x}$. Hence, in any case $T(\tilde{p})$ is properly contained in $T(\bar{p})$. This completes the proof. □

Now we can give the key tool in the proof of our main theorem:

Proposition 10 *Assume $\bar{p} = (\bar{x}, \bar{y}) \in \mathscr{S}_0$ and no point $(\tilde{x}, \bar{y})$ with $\tilde{x} \leq \bar{x}$ is a cornerpoint for $\ell_{(\mathcal{M},\varphi)}$. Then*

$$\ell_{(\mathcal{M},\varphi)}(\bar{p}) = \sum_{\tau \in T(\bar{p})} \mu(\tau)$$

if $T(\bar{p}) \neq \emptyset$, $\ell_{(\mathcal{M},\varphi)}(\bar{p}) = 0$ otherwise.

Proof. Let $s(\bar{p}) = \sum_{\tau \in T(\bar{p})} \mu(\tau)$ (i.e. the sum of multiplicities of all the main triangles in $T(\bar{p})$) if $T(\bar{p}) \neq \emptyset$, $s(\bar{p}) = 0$ otherwise.

We must prove that $\ell_{(\mathcal{M},\varphi)}(\bar{p}) = s(\bar{p})$. Since $\bar{x} < \bar{y}$, we point out that there is a $\bar{\rho} > 0$ such that $\bar{p} \in \mathscr{S}_{\bar{\rho}}$. Moreover, Lemma 3(ii) and our assumptions imply that $\ell_{(\mathcal{M},\varphi)}$ is continuous in the variable y at $\bar{p}$.

We shall proceed by induction on the cardinality $t(\bar{p})$ of $T(\bar{p})$. We first observe that $t(\bar{p})$ is finite, since $\bar{p} \in \mathscr{S}_{\bar{\rho}}$ and because of Corollaries 3 and 5.

Let us assume that $t(\bar{p}) = 0$. By Proposition 3 a point $q \in \mathscr{S}_{\bar{\rho}}$ exists with the same ordinate as $\bar{p}$, such that $\ell_{(\mathcal{M},\varphi)}(q) = 0$. Because of Lemma 3(i) and our assumptions, condition $t(\bar{p}) = 0$ implies that the closed segment between q and $\bar{p}$ never meets any discontinuity point in x for the size function, and hence $\ell_{(\mathcal{M},\varphi)}(\bar{p}) = 0$. Moreover, $T(\bar{p}) = \emptyset$, so that $s(\bar{p}) = 0$. Therefore $\ell_{(\mathcal{M},\varphi)}(\bar{p}) = s(\bar{p})$.

Now, assume the claim true when $t(\bar{p}) < n$ ($n > 0$). If $t(\bar{p}) = n$, then there is a point $\tilde{p} = (\tilde{x}, \bar{y})$, where $\tilde{x} < \bar{x}$, such that the closed segment connecting $\tilde{p}$ to $\bar{p}$ meets only one discontinuity point $\bar{q} \neq \tilde{p}$ in the variable x for the size function. According to Lemma 4, this means that $t(\tilde{p}) < t(\bar{p})$ and that $\ell_{(\mathcal{M},\varphi)}(\bar{p}) - \ell_{(\mathcal{M},\varphi)}(\tilde{p}) = s(\bar{p}) - s(\tilde{p})$ (we recall that $\ell_{(\mathcal{M},\varphi)}$ is continuous in the variable y at $\bar{p}$).

On the other hand the induction hypothesis implies that $\ell_{(\mathcal{M},\varphi)}(\tilde{p}) = s(\tilde{p})$. This concludes the proof. □

In plain words Proposition 10 says that the value taken by a size function at a point for which no cornerpoint has the same ordinate and smaller abscissa is equal to the sum of multiplicities of all the main triangles containing such a point.

Corollary 6 *Each discontinuity point $\bar{p} = (\bar{x}, \bar{y}) \in \mathcal{S}_0$ for $\ell_{(\mathcal{M},\varphi)}$ is in fact such that either $\bar{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\cdot, \bar{y})$ or $\bar{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\bar{x}, \cdot)$, or both hold.*

Proof. If $\bar{p}$ is neither a discontinuity point in the y direction nor a discontinuity point in the x direction for the size function, then Proposition 8 shows that there is no main triangle τ for which $\bar{p}$ belongs to the boundary of τ. Hence an open arcwise connected neighborhood of $\bar{p}$ completely contained in $\cap_{\tau \in T(\bar{p})} \tau$ exists, and this intersection is finite because of Corollaries 3 and 5. By Proposition 10, $\ell_{(\mathcal{M},\varphi)}$ is constant on such a neighborhood, contradicting the assumption that $\bar{p}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}$. □

This result improves Lemma 2 (for the case $\bar{x} < \bar{y}$).

3.2 The Main Theorem

Now we are ready to state the main theorem of this paper.

First of all, for each $\rho \geq 0$ we define an equivalence relation $\cong_\rho$ on the set of all size functions by setting $\ell_{(\mathcal{M}_1,\varphi_1)} \cong_\rho \ell_{(\mathcal{M}_2,\varphi_2)}$ if and only if $\ell_{(\mathcal{M}_1,\varphi_1)}$ and $\ell_{(\mathcal{M}_2,\varphi_2)}$ coincide almost everywhere in $\mathcal{S}_\rho$. In other words, the subset of $\mathcal{S}_\rho$ in which the two functions differ has vanishing measure.

Hereafter, for every real number $\rho \geq 0$, we shall denote by $\mathcal{L}_\rho$ the quotient of the set of all size functions by the equivalence relation $\cong_\rho$.

Ω_ρ, with $\rho > 0$, will denote the set of all natural-valued formal series σ in $\mathcal{S}_\rho \cup \mathcal{R}$, having a finite support and verifying the following property: there is a line $[r : x = k]$ in $\mathrm{supp}(\sigma)$ such that the half-plane $\{(x, y) \in \mathbb{R}^2 : x \geq k\}$ contains all the other points and lines in $\mathrm{supp}(\sigma)$. In other words Ω_ρ is the set of all finite collections of points and vertical lines of $\mathbb{R}^2$ (with positive multiplicities), with "a vertical line as the element farthest to the left".

Ω_0 will denote the set of all natural-valued formal series σ in $\mathcal{S}_0 \cup \mathcal{R}$ (possibly with $\mathrm{card}(\mathrm{supp}(\sigma)) = +\infty$), having the following properties:

(i) for every positive real number ρ the restriction of $\sigma \in \Omega_0$ to $\mathcal{S}_\rho \cup \mathcal{R}$ is a series belonging to Ω_ρ;
(ii) $\sup\{y : (x, y) \in \mathcal{S}_0 \cap \mathrm{supp}(\sigma)\} < +\infty$ (i.e., the supremum y_{sup} of the ordinates of the points in σ is finite).

It is very natural now to define a map $\tilde{\alpha}_\rho : \mathcal{L}_\rho \longrightarrow \Omega_\rho$ for every real number $\rho \geq 0$.

Let us first examine the case $\rho > 0$. For every size function ℓ define $\alpha_\rho(\ell)$ as the formal series $\sum_{X \in C(\ell)} \mu(X)X$, where $C(\ell)$ denotes the set of all cornerpoints in $\mathscr{S}_\rho$ and all cornerlines for ℓ. From Corollaries 3 and 5 we easily obtain that $\mathrm{supp}(\alpha_\rho(\ell))$ is finite. From Remarks 3 and 4 we see that $\alpha_\rho(\ell)$ is a natural-valued formal series in $\mathscr{S}_\rho \cup \mathscr{R}$. Proposition 9 implies that $\alpha_\rho(\ell)$ has the vertical line $x = \min_{P \in \mathscr{M}} \varphi(P)$ as the farthest to the left element. Therefore $\alpha_\rho(\ell)$ is actually in Ω_ρ. Moreover, if ℓ_1 and ℓ_2 belong to the same equivalence class in $\mathscr{L}_\rho$ then they must have the same cornerpoints and cornerlines in $\mathscr{S}_\rho$ with the same multiplicities, so that $\alpha_\rho(\ell_1) = \alpha_\rho(\ell_2)$. If this were not the case, Proposition 10 would allow us to find a continuity point p for both ℓ_1 and ℓ_2, at which point the two size functions differ. Hence they would also differ in an open neighborhood of p in $\mathscr{S}_\rho$, against the definition of the equivalence relation $\cong_\rho$. All this proves that the function α_ρ induces a well-defined map $\tilde{\alpha}_\rho : \mathscr{L}_\rho \longrightarrow \Omega_\rho$.

For $\rho = 0$ we set $\alpha_0(\ell_{(\mathscr{M},\varphi)})$ equal to the formal series extending all the formal series $\alpha_\rho(\ell_{(\mathscr{M},\varphi)})$ for $\rho > 0$ so that (i) holds. Since $\mathscr{M}$ is compact, also property (ii) holds for $\alpha_0(\ell_{(\mathscr{M},\varphi)})$ and it actually belongs to Ω_0. As before the map α_0 induces a map $\tilde{\alpha}_0 : \mathscr{L}_0 \longrightarrow \Omega_0$.

Theorem 1 *For every real number $\rho \geq 0$ the map $\tilde{\alpha}_\rho : \mathscr{L}_\rho \longrightarrow \Omega_\rho$ is a bijection.*

Proof. Let us first consider the case $\rho > 0$. Proposition 10 implies that if $\alpha_\rho(\ell_1) = \alpha_\rho(\ell_2)$ then $\ell_1 \cong_\rho \ell_2$. Hence we easily obtain that $\tilde{\alpha}_\rho$ is injective.

We shall now prove that $\tilde{\alpha}_\rho$ is also surjective by showing that for every formal series $\sigma = \sum_{X \in \mathscr{S}_\rho \cup \mathscr{R}} m(X)X$ in Ω_ρ, a size pair $(\mathscr{M}, \varphi)$ exists such that $\tilde{\alpha}_\rho([\ell_{(\mathscr{M},\varphi)}]) = \sigma$.

We shall actually show how such a size pair can be constructed. It may be helpful if the reader refers to the example shown in Fig. 10.

We shall choose $\mathscr{M}$ as a subset of the real plane and the measuring function φ as the function that takes each point of $\mathscr{M}$ to its ordinate: $\varphi(x, y) = y$.

We construct $\mathscr{M}$ as follows: for every vertical line $[r : x = k]$ with $m(r) \neq 0$ let us take $m(r)$ closed segments parallel to the y-axis in $\mathbb{R}^2$ whose lower endpoints all have the same ordinate equal to k. We ask that these segments be pairwise disjoined and that the upper endpoint of each segment have its ordinate equal to the greatest value b for which either a point with ordinate b or a line $[r : x = b]$ exists in $\mathrm{supp}(\sigma)$.

Now, consider one of the segments that take the maximum length and call it u (obviously, the ordinates of its endpoints will be $a = \min\{\bar{x} : [r : x = \bar{x}] \in \mathscr{R}, m(r) \neq 0\}$ and b).

For every $p = (x_p, y_p) \in \mathscr{S}_\rho$ such that $m(p) \neq 0$ we glue to u, at a height equal to y_p, a number of closed segments equal to the value of $m(p)$, so that their lower extremity has an ordinate equal to x_p and they never intersect other segments except (possibly) at their upper extremity. Since $\sigma \in \Omega_\rho$, we have to join a finite number of segments to u, and hence the set $\mathscr{M}$ we obtain is

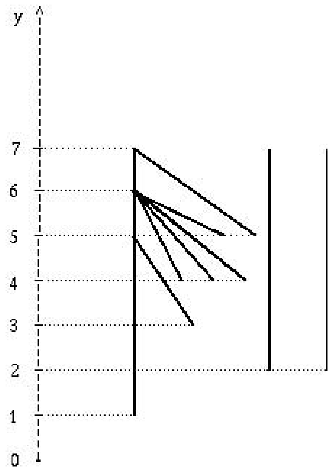

Fig. 10. How to construct a topological space $\mathscr{M}$ such that $\alpha(\ell_{(\mathscr{M},\varphi(x,y)=y)}) = r_1 + 2r_2 + p_1 + 3p_2 + p_4 + p_5$ with $[r_1 : x = 1]$ and $[r_2 : x = 2]$ cornerlines, and $p_1 = (3, 5)$, $p_2 = (4, 6)$, $p_4 = (5, 6)$, $p_5 = (5, 7)$ cornerpoints. The corresponding size function is given in Fig. 6

compact. Moreover, it is easily seen that each component of $\mathscr{M}$ is arcwise and locally arcwise connected.

The size pair $(\mathscr{M}, \varphi)$ thus constructed gives rise to a size function which is taken by α_ρ into the desired formal series σ and this shows the surjectivity of α_ρ and $\tilde{\alpha}_\rho$.

Thus for every $\rho > 0$ we have shown that $\tilde{\alpha}_\rho$ is a bijection.

Let us now consider the case $\rho = 0$. It is easy to see that $\tilde{\alpha}_0$ is injective. The surjectivity of $\tilde{\alpha}_0$ is proven by a construction similar to the one used for the case $\rho > 0$. The only difference is that the upper endpoints of our vertical segments have ordinate $\sup\{y : (x, y) \in \mathscr{S}_0 \cap \operatorname{supp}(\sigma)\}(< +\infty)$ instead of b (which may be not defined). A compact set can be obtained also in the case that we join an infinite number of segments to u. It is enough to take such segments with lengths converging to 0 (as shown, for example, in Fig. 7). In this way we also prove that each component of $\mathscr{M}$ is arcwise and locally arcwise connected. Hence also $\tilde{\alpha}_0 : \mathscr{L}_0 \to \Omega_0$ is a bijection and the claim is proven. □

Remark 5. Proposition 10 easily implies that $\ell_{(\mathscr{M}_1,\varphi_1)} \cong_\rho \ell_{(\mathscr{M}_2,\varphi_2)}$ if and only if such functions coincide in $\mathscr{S}_\rho$, outside a countable (finite, for $\rho > 0$) union of closed horizontal segments s_i, each connecting a point p_i of the half-plane $x < y$ to the diagonal Δ.

4 Construction of Pseudo-metrics

Among the advantages of representing size functions as formal series, one is certainly that we are able to define new pseudo-distances between size functions. A thorough study of such distances would be far from the aim of this paper so now we merely give hints of how some new pseudo-distances can be constructed (more details can be found in [17], for experimental results see also [5]). The key idea here is, given two size functions ℓ_1 and ℓ_2, to somehow match the cornerpoints and cornerlines of ℓ_1 respectively with the cornerpoints and cornerlines of ℓ_2 and to quantify their differences depending on such a correspondence.

One way to accomplish this is the following. Let us fix a number $\rho > 0$ and consider two size functions ℓ_1 and ℓ_2. The map α_ρ of Theorem 1 takes these two size functions into two natural-valued formal series σ_1 and σ_2 with finite support in $\mathscr{S}_\rho \cup \mathscr{R}$. It follows that any distance or pseudo-distance between σ_1 and σ_2 induces a pseudo-distance between ℓ_1 and ℓ_2.

Suppose that $\sigma_1 = \sum_{X \in I_1 \cup J_1} m(X)X$ and $\sigma_2 = \sum_{Y \in I_2 \cup J_2} n(Y)Y$, where I_1 and I_2 denote two finite subsets of $\mathscr{S}_\rho$ while J_1 and J_2 denote two finite subsets of $\mathscr{R}$. Recall that m and n take values into $\mathbb{N}$. Thus it makes sense to consider as many copies of each point of I_1 (resp. I_2) as its multiplicity is, so as to form a new set of distinct points $\tilde{I}_1$ (resp. $\tilde{I}_2$). Repeat the same for the lines in J_1 and J_2 to obtain the sets $\tilde{J}_1$ and $\tilde{J}_2$. Let F be the set of all injective functions f from a subset D_f of $\tilde{I}_1$ into $\tilde{I}_2$ and let G be the set of all bijective functions g from $\tilde{J}_1$ to $\tilde{J}_2$. We allow for the possibility that D_f is empty. Therefore, F also contains the function which takes the empty set to itself.

For each pair $(f, g) \in F \times G$ we can now compute the value $v(f, g)$ as follows: we start with $v(f, g) = 0$ and then, for each $p \in D_f$, we increase $v(f, g)$ by the Euclidean distance of p from $f(p)$. Analogously, for each $r \in \tilde{J}_1$ we add the Euclidean distance between r and $g(r)$ to $v(f, g)$. Finally, for each point $p \in \tilde{I}_1 - D_f$, we increase $v(f, g)$ by the Euclidean distance of p from the diagonal $\Delta = \{(x, y) \in \mathbb{R}^2 : x = y\}$ and for each point $q \in \tilde{I}_2 - f(D_f)$ we increase $v(f, g)$ by the Euclidean distance of q from the diagonal Δ.

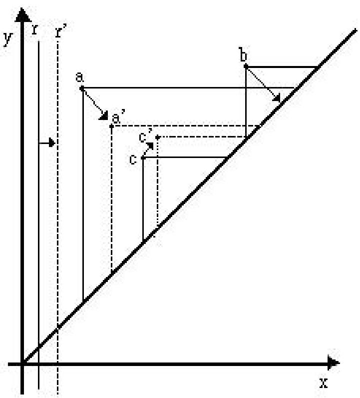

Fig. 11. How to compute the distance between two formal series

Thus we can define the following distance between σ_1 and σ_2:

$$\mathrm{dist}_\rho(\sigma_1, \sigma_2) := \begin{cases} +\infty & \text{if } G = \emptyset \\ \min_{(f,g)\in F\times G} v(f, g) & \text{otherwise.} \end{cases}$$

In plain words, what we actually do to calculate such a distance is to measure the reciprocal distances of pairs of points and pairs of lines of the two formal series under study, allowing us to "destroy" some points by sending them onto the diagonal Δ. Then we choose the matching which minimizes the sum of these distances. Thus we obtain a distance between formal series which takes into account multiplicities and is small when formal series are similar.

In Fig. 11 we show how it works on two formal series $r + a + b + c$ and $r' + a' + c'$ obtained as images of two size functions by the map α_ρ of Theorem 1, for a suitable ρ. The set of discontinuity points of the two size functions are represented by continuous and dotted lines respectively. The arrows show the action of the considered maps f and g. Here $D_f = \{a, c\}$, $\tilde{I}_1 - D_f = \{b\}$ and $\tilde{J}_1 = \{r\}$. The total amount of the displayed "movements" equals the distance between the two formal series (and corresponding size functions).

References

1. Callahan, J., Weiss, R.: A model for describing surface shape. In: Proc. of the IEEE Conf. on Computer Vision and Pattern Recognition, pp. 240–245, San Francisco 1985
2. Collina, C., Ferri, M., Frosini, P., Porcellini, E.: SketchUp: Towards qualitative shape data management. In: Chin, R., Pong, T. (eds.) Proc. ACCV'98, Lecture Notes in Computer Science 1351, Vol. I, pp. 338–345, Berlin Heidelberg: Springer 1998
3. D'Amico, M.: A new optimal algorithm for computing size function of shapes. In: CVPRIP Algorithms III, Proceedings International Conference on Computer Vision, Pattern Recognition and Image Processing, Atlantic City, pp. 107–110, 2000
4. Díaz-De-León, J. L., Sossa-Azuela, J. H.: On the computation of the Euler number of a binary object. Pattern Recognition **29**, 471–476 (1996)
5. Donatini, P., Frosini, P., Landi, C.: Deformation energy for size functions. In: Hancock, E. R., Pelillo, M. (eds.) Energy Minimization Methods in Computer Vision and Pattern Recognition, Lecture Notes in Computer Science, Vol. 1654, pp. 44–53, 1999
6. Donatini, P., Frosini, P., Lovato, A.: Size functions for signature recognition. In: Melter, R. A., Wu, A. Y., Latecki, L. J. (eds.) Vision Geometry VII, Proc. SPIE, Vol. 3454, pp. 178–183, 1998
7. Edelsbrunner, H., Guibas, L. J.: Topologically sweeping an arrangement. J. Comput. Syst. Sci. **38**, 165–194 (1989)
8. Ferri, M., Frosini, P., Lovato, A., Zambelli, C.: Point selection: A new comparison scheme for size functions (With an application to monogram recognition). In: Chin, R., Pong, T. (eds.) Proc. ACCV'98, Lecture Notes in Computer Science 1351, Vol. I, pp. 329–337, Berlin Heidelberg: Springer 1998
9. Ferri, M., Lombardini, S., Pallotti, C.: Leukocyte classification by size functions. In: Proceedings of the second IEEE Workshop on Applications of Computer Vision, pp. 223–229, IEEE Computer Society Press, Los Alamitos, CA, 1994
10. Frosini, P.: A distance for similarity classes of submanifolds of a Euclidean space. Bull. Austral. Math. Soc. **42**(3), 407–416 (1990)

11. Frosini, P.: Discrete computation of size functions. J. Combin. Inform. System Sci. **17**(3–4), 232–250 (1992)
12. Frosini, P.: Connections between size functions and critical points. Math. Meth. Appl. Sci. **19**, 555–569 (1996)
13. Frosini, P.: Measuring shapes by size functions. In: Proc. of SPIE, Intelligent Robots and Computer Vision X: Algorithms and Techniques, Boston, MA Vol. 1607, pp. 122–133, 1991
14. Frosini, P.: Natural size distances and size functions for studying shapes of manifolds. Preprint
15. Frosini, P., Ferri, M.: Range size functions. In: Proc. of SPIE, Vision Geometry III, Boston MA, Vol. 2356, pp. 243–251, 1994
16. Frosini, P., Landi, C.: Size functions and morphological transformations. Acta Appl. Math. **49**, 85–104 (1997)
17. Frosini, P., Landi, C.: New pseudo-distances for the size function space. In: Robert A. Melter, Angela Y. Wu, Longin J. Latecki (eds.) Proc. SPIE, Vision Geometry VI, S. Diego, CA, Vol. 3168, pp. 52–60, 1997
18. Frosini, P., Landi, C.: Size theory as a topological tool for Computer Vision. Pattern Recognit. Image Anal. **9**, 596–603 (1999)
19. Frosini, P., Mulazzani, M.: Size homotopy groups for computation of natural size distances. Bull. Belg. Math. Soc. **6**, 455–464 (1999)
20. Frosini, P., Pittore, M.: New methods for reducing size graphs. Int. J. Comput. Math. **70**, 505–517 (1999)
21. Hsh, Y-N., Arsenault, H. H., April, G.: Rotation-invariant digital pattern recognition using circular harmonic expansion. Appl. Opt. **21**, 4012–4015 (1982)
22. Kendall, D. G.: Shape manifolds, procrustrean metrics and complex projective spaces. Bull. London Math. Soc. **16**, 81–121 (1984)
23. Kupeev, K. Y., Wolfson, H. J.: A new method for estimating shape similarity. Pattern Recognit. Lett. **17**, 873–887 (1996)
24. Lee, C-N., Poston, T., Rosenfeld, A.: Winding and Euler numbers for 2D and 3D digital images. CVGIP: Graphical Models and Image Processing **53**, 522–537 (1991)
25. Siddiqi, K., Tannenbaum, A., Zucker, S. W.: A Hamiltonian approach to the eikonal equation. In: Hancock, E. R., Pelillo, M. (eds.) Energy Minimization Methods in Computer Vision and Pattern Recognition, Lecture Notes in Computer Science, Vol. 1654, pp. 1–13, 1999
26. Rom, H., Medioni, G.: Hierarchical decomposition and axial shape description. IEEE Trans. Pattern Anal. Mach. Intell. **15**, 973–981 (1993)
27. Uras, C., Verri, A.: Studying shape through size functions. In: Toet, Y. O. A., Foster, D., Heijmans, H., Meer, P. (eds.) Shape in Picture, NATO ASI Series F, Vol. 126, pp. 81–90, Berlin Heidelberg: Springer 1994
28. Uras, C., Verri, A.: On the recognition of the alphabet of the sign language through size functions. In: Proc. XII IAPR International Conference on Pattern Recognition, Jerusalem (Israel), Vol. II, pp. 334–338, IEEE Computer Society Press, Los Alamitos, CA, 1994
29. Uras, C., Verri, A.: Invariant size functions. In: Mundy, J. L., Zisserman, A., Forsyth, D. (eds.) Applications of invariance in computer vision. Lecture Notes in Computer Science, Vol. 825, pp. 215–234 Berlin Heidelberg: Springer 1994
30. Verri, A., Uras, C.: Metric-topological approach to shape representation and recognition. Image Vision Comput. **14**, 189–207 (1996)
31. Verri, A., Uras, C.: Computing size functions from edge maps. Int. J. Comput. Vision, **23**, 1–15 (1997)
32. Verri, A., Uras, C.: Aspect-based object recognition with size functions. In: Proc. 13th Int. Conf. Patt. Recog., Vienna, Vol 1, pp. 682–686, IEEE Comput. Soc. Press 1996
33. Verri, A., Uras, C., Frosini, P., Ferri, M.: On the use of size functions for shape analysis. Biol. Cybern. **70**, 99–107 (1993)