Size homotopy groups for computation of natural size distances

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Abstract

For every manifold $\mathcal{M}$ endowed with a structure described by a function from $\mathcal{M}$ to the vector space $\mathbb{R}^k$, a parametric family of groups, called size homotopy groups, is introduced and studied. Some lower bounds for natural size distances are obtained in this way.

1 Introduction

Size Theory is a new approach to the problem of comparing manifolds endowed with a structure represented by an $\mathbb{R}^k$-valued function. Apart from the intrinsic and classical mathematical interest in this subject (think of the study of metric and conformal structures on manifolds), this theory has a strong motivation in Computer Vision for problems of Shape Recognition and Image Analysis and has turned out to be useful for several applications (see [4], [12], [13], [14], [15], [16] and [17]). On the other hand, mathematical problems arising in Computer Vision require new geometrical techniques (cf. [3], [11] and also the nice informal paper [2]).

In previous papers ([5], [6], [7], [8] and [9]) Size Theory was basically founded on two related concepts: natural size distances and size functions. Natural size distances are a tool for measuring the "difference" between two homeomorphic manifolds, on each of which a continuous $\mathbb{R}^k$-valued function, called *measuring function*,

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is defined. The basic idea is very simple: we shall measure how much the measuring function changes when going homeomorphically from the first manifold to the second. This approach can then be used to measure how similar, or isometric, or conformal, etc, two manifolds are. Natural size distances are very useful from the applicative viewpoint, too, since they allow us to compare "shapes of objects" (think of the problem of classifying two "bottles" as the same object and of distinguishing them from a "glass").

Size functions are used for the same task but they are much easier to compute than natural size distances, for which they give lower bounds. More precisely, size functions are integer functions of two real variables giving metric obstructions to the classical notion of homotopy (see also Remark 2). Thus, size functions convey information both on topological and metric properties of the manifold describing the viewed shape. However, such a point of view lacks algebraic structure and therefore does not have a satisfactory connection with the classical concept of homotopy groups. In this paper we shall try to fill this gap by means of *size homotopy groups*: algebraic structures easily able to give computable lower bounds for natural size distances. Concerning the use of metric constraints on homotopy groups, although used in a different way from this article, see also [10].

A size homotopy group is a group depending on two vector parameters $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^k$, such that $\boldsymbol{\xi} \preceq \boldsymbol{\eta}$ (i.e. $\xi_i \leq \eta_i$, for every index i). The leading idea is to consider the class of loops in the topological space $\mathcal{M}_{\boldsymbol{\xi}} \stackrel{def}{=} \{P \in \mathcal{M} \mid \boldsymbol{\varphi}(P) \preceq \boldsymbol{\xi}\}$, based at a fixed point $P' \in \mathcal{M}_{\boldsymbol{\xi}}$, where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_k) : \mathcal{M} \to \mathbb{R}^k$ is the chosen measuring function on $\mathcal{M}$. Two based loops in $\mathcal{M}_{\boldsymbol{\xi}}$ are considered equivalent if they are (pointed) homotopic in the larger topological space $\mathcal{M}_{\boldsymbol{\eta}}$. The set of equivalence classes of loops has a natural group structure and is called *the first size homotopy group*. The core of this paper is the proof that we can obtain lower bounds for natural size distances using size homotopy groups (Theorems 7 and 10).

In Section 2 we shall review the concept of natural size distance, while the main results regarding the relations between natural size distances and size homotopy groups will be given in Section 3.

From now on, the symbol $\cong$ will denote both homeomorphism between topological spaces and isomorphism between groups.

2 Natural size distances

The environment in which we formalize this concept is the following. Consider the category $Size_k$ whose objects are the *size pairs* $(\mathcal{M}, \boldsymbol{\varphi})$, where $\mathcal{M}$ is a closed C^0-manifold and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_k) : \mathcal{M} \to \mathbb{R}^k$ is a continuous function called *k-dimensional measuring function*, and whose morphisms from $(\mathcal{M}, \boldsymbol{\varphi})$ to $(\mathcal{N}, \boldsymbol{\psi})$ are the homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$. The set of such morphisms will be denoted by $H(\mathcal{M}, \mathcal{N})$. As usual, by $Obj(Size_k)$ and $Mor(Size_k)$ we shall denote respectively the class of objects and the class of morphisms of $Size_k$. Now we shall give the definition of natural size distance, assuming $\mathbb{R}^k$ endowed with the usual norm of maximum: $\|(\xi_1, \xi_2, \ldots, \xi_k)\|_\infty = \max_{1 \leq i \leq k} |\xi_i|$.

Definition 1. Let $\Theta_k : Mor(Size_k) \to \mathbb{R}$ be the function defined by $\Theta_k(f) = \max_{P\in\mathcal{M}} \|\boldsymbol{\varphi}(P) - \boldsymbol{\psi}(f(P))\|_\infty$, for any $f \in H(\mathcal{M},\mathcal{N})$. We call Θ_k the *natural size measure* on $Mor(Size_k)$.

In plain words, Θ_k measures how much f "changes" the values taken by the measuring function.

Proposition 2. *Let $\Sigma_k : Obj(Size_k) \times Obj(Size_k) \to \mathbb{R} \cup \{+\infty\}$ be the function defined by setting $\Sigma_k((\mathcal{M},\boldsymbol{\varphi}),(\mathcal{N},\boldsymbol{\psi})) = \inf_{f\in H(\mathcal{M},\mathcal{N})} \Theta_k(f)$ if $H(\mathcal{M},\mathcal{N}) \neq \emptyset$ and $+\infty$ otherwise. Then Σ_k is a pseudometric on $Obj(Size_k)$.*

Proof. Trivial. ■

Definition 3. The metric σ_k induced by the pseudometric Σ_k will be called the *natural size distance* on $Obj(Size_k)/\approx$, where $\approx$ denotes the equivalence relation defined by setting $(\mathcal{M},\boldsymbol{\varphi}) \approx (\mathcal{N},\boldsymbol{\psi})$ if and only if $\Sigma_k((\mathcal{M},\boldsymbol{\varphi}),(\mathcal{N},\boldsymbol{\psi})) = 0$. The equivalence class of $(\mathcal{M},\boldsymbol{\varphi})$ will be denoted by the symbol $[(\mathcal{M},\boldsymbol{\varphi})]$.

More details about the passage from a pseudometric to a metric can be found in [1]. The term "natural" is used since the previous way to define a pseudometric between manifolds is a particular case of a more general method illustrated in [9]. That paper also states the main properties of natural size distances.

Remark 1. For the sake of conciseness, we shall often use the symbols Θ and σ respectively instead of Θ_k and σ_k, whenever no possibility of confusion may arise.

Before going on, we shall give some simple examples of natural size distances.

Example 1. Let $(\mathcal{M},\boldsymbol{\varphi}),(\mathcal{N},\boldsymbol{\psi}) \in Obj(Size_k)$ such that $\boldsymbol{\varphi} = \boldsymbol{\psi} \circ f$ for a certain homeomorphism $f : \mathcal{M} \to \mathcal{N}$. Since $\Theta(f) = 0$, we get $\sigma([(\mathcal{M},\boldsymbol{\varphi})],[(\mathcal{M},\boldsymbol{\psi})]) = 0$.

Example 2. Consider the unit sphere $\mathcal{S} \subseteq \mathbb{R}^3$ of equation $x^2+y^2+z^2 = 1$ and the ellipsoid $\mathcal{E} \subseteq \mathbb{R}^3$ of equation $x^2+4y^2+9z^2 = 1$. Let $\varphi : \mathcal{S} \to \mathbb{R}$ and $\psi : \mathcal{E} \to \mathbb{R}$ be the Gaussian curvatures of the surfaces. We obtain $\sigma([(\mathcal{S},\varphi)],[(\mathcal{E},\psi)]) = 35$. In fact, $\varphi(\mathcal{S}) = \{1\}$ and $\psi(\mathcal{E}) = [4/9, 36]$, and therefore $\Theta(f) = 35$, for every $f \in H(\mathcal{S},\mathcal{E})$.

Example 3. Take a torus $\mathcal{T}$ and the two Riemannian metrics defined on $\mathcal{T}$ in toroidal coordinates (α,β) respectively by $ds^2 = d\alpha^2 + d\beta^2$ and $ds^2 = d\alpha^2 + (2+\cos\alpha)^2 d\beta^2$. Now, we consider the size pairs $(\mathcal{T},\varphi)$ and $(\mathcal{T},\psi)$, where $\varphi,\psi : \mathcal{T} \to \mathbb{R}$ are the Gaussian curvatures of $\mathcal{T}$ associated to the two metrics.
We then have $\sigma([(\mathcal{T},\varphi)],[(\mathcal{T},\psi)]) = 1$. In fact, $\varphi(\mathcal{T}) = \{0\}$ and $\psi(\mathcal{T}) = [-1, 1/3]$ and therefore $\Theta(f) = 1$, for every $f \in H(\mathcal{T},\mathcal{T})$.

Example 4. Let $\mathcal{C}_1,\mathcal{C}_2 \subseteq \mathbb{R}^3$ be the closed curves of parametric equations $\gamma_1(t) = (\cos t, \sin t, 0)$ and $\gamma_2(t) = (\cos t, \sin t, \sin t\cos t)$ for $0 \le t \le 2\pi$. If κ and τ denote respectively the curvature and the torsion of the curves, then take both $\boldsymbol{\varphi} : \mathcal{C}_1 \to \mathbb{R}^2$ and $\boldsymbol{\psi} : \mathcal{C}_2 \to \mathbb{R}^2$ as the pair (κ,τ). Since $\boldsymbol{\varphi}(\mathcal{C}_1) = \{(1,0)\}$, $\kappa(\mathcal{C}_2) = [1/2, \sqrt{5}]$, $\tau(\mathcal{C}_2) = [-3/4, 3/4]$ we have $\Theta(f) = \sqrt{5} - 1$, for every $f \in H(\mathcal{C}_1,\mathcal{C}_2)$, and therefore $\sigma([(\mathcal{C}_1,\boldsymbol{\varphi})],[(\mathcal{C}_2,\boldsymbol{\psi})]) = \sqrt{5} - 1$.

Before proceeding, we wish to point out that using natural size distances (possibly with a different choice of morphisms in the category) we can express several mathematical concepts. For example, if $\mathcal{M}$ and $\mathcal{N}$ are two closed C^0-submanifolds of a Euclidean space, the congruence relation between $\mathcal{M}$ and $\mathcal{N}$ can be expressed in terms of size distances. Let us take the size pairs $(\mathcal{M}\times\mathcal{M},\varphi)$, and $(\mathcal{N}\times\mathcal{N},\psi)$, where $\varphi(P_1,P_2)=\|P_1-P_2\|_2$ and $\psi(Q_1,Q_2)=\|Q_1-Q_2\|_2$. It is possible to prove that the natural size distance between $[(\mathcal{M}\times\mathcal{M},\varphi)]$, and $[(\mathcal{N}\times\mathcal{N},\psi)]$ vanishes if and only if $\mathcal{M}$ and $\mathcal{N}$ are congruent (here the "right" set of morphisms from $(\mathcal{M}\times\mathcal{M},\varphi)$ to $(\mathcal{N}\times\mathcal{N},\psi)$ is the set of homeomorphisms $f\times f$, where f is a homeomorphism from $\mathcal{M}$ to $\mathcal{N}$).

Analogously, using natural size distances we can express similarity or isometry between $\mathcal{M}$ and $\mathcal{N}$ or even simpler concepts, such as "having the same number of bumps", that can be used for applicative tasks.

Unfortunately, natural size distances are in general much more difficult to compute than in the previous examples. We observe that in each of these examples the computation of σ is trivial. In fact, we can find a morphism f such that $\Theta(f)$ equals the Hausdorff distance $d_H(\mathrm{Im}(\boldsymbol{\varphi}),\mathrm{Im}(\boldsymbol{\psi}))$ between the image sets. Since $\Theta(g)\geq d_H$, for every morphism g, it is obvious that $\sigma=\Theta(f)=d_H$. However, this is not generally the case and the computation of σ is much more difficult. We also point out that in Examples 2, 3 and 4 the images of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are different sets and therefore $\sigma>0$.

Since the direct computation of natural size distances is hard to perform, size functions have been introduced (see [5], [6], [7], [8] and [9]) to obtain useful lower bounds. In fact, they are easily computable, even from an algorithmic point of view. The approach to size functions arises from a direct generalization of the concept of homotopy, but it leads to "objects" lacking in algebraic structure. So, on the one hand, size functions do not correspond to the classical concept of homotopy groups while, on the other hand, their lack of algebraic structure reduces their ability to distinguish between "objects". For this purpose in the next Section we shall introduce the concept of the size homotopy group, an algebraic tool which allows us to obtain more efficient lower bounds for natural size distances.

3 Size homotopy groups

Consider a size pair $(\mathcal{M},\boldsymbol{\varphi})$ and $\boldsymbol{\xi}\preceq\boldsymbol{\eta}\in\mathbb{R}^k$, such that $\mathcal{M}_{\boldsymbol{\xi}}\neq\emptyset$. For every fixed point $P\in\mathcal{M}_{\boldsymbol{\xi}}$, let $L_P(\boldsymbol{\xi})$ be the set of loops in $\mathcal{M}_{\boldsymbol{\xi}}$ based at P. Two loops $\alpha,\beta\in L_P(\boldsymbol{\xi})$ are said to be $(\boldsymbol{\varphi}\preceq\boldsymbol{\eta})$ -homotopic if and only if there exists a homotopy from α to β, pointed at P, in the space $\mathcal{M}_{\boldsymbol{\eta}}$ (such a homotopy will be called a $(\boldsymbol{\varphi}\preceq\boldsymbol{\eta})$-*homotopy*).

Definition 4. The quotient of the space of loops $L_P(\boldsymbol{\xi})$ modulo the equivalence relation of $(\boldsymbol{\varphi}\preceq\boldsymbol{\eta})$-homotopy admits a natural structure of group. We shall call it the *first size homotopy group of* $(\mathcal{M},\boldsymbol{\varphi})$*, based at* P *and associated to* $(\boldsymbol{\xi},\boldsymbol{\eta})$. We shall denote this group by $\pi_1((\mathcal{M},\boldsymbol{\varphi}),P,(\boldsymbol{\xi},\boldsymbol{\eta}))$ (or simply by $\pi_1(\boldsymbol{\xi},\boldsymbol{\eta})$, when the size pair and the base point are clearly specified).

Remark 2. The previous definition could be naturally extended in order to define the concept of *i-th size homotopy group*. It is interesting to point out that, for $i = 0$, we obtain a set $\pi_0(\boldsymbol{\xi}, \boldsymbol{\eta})$, whose cardinality equals the value of the size function $\ell_{(\mathcal{M},\boldsymbol{\varphi})}$ at $(\boldsymbol{\xi}, \boldsymbol{\eta})$ (cf. [9]).

As a simple example of size homotopy groups we give the following:

Example 5. Consider the sphere $\mathcal{S} \subseteq \mathbb{R}^3$ of equation $x^2 + y^2 + z^2 = 1$ and set $\varphi(x, y, z) = |z|$ for every $(x, y, z) \in \mathcal{S}$. Moreover, choose $P = (1, 0, 0)$ as base point. The following statements hold when $0 \leq \xi \leq \eta$:

1. if $\eta < 1$, then $\pi_1((\mathcal{S}, \varphi), P, (\xi, \eta)) \cong \mathbb{Z}$;
2. if $\eta \geq 1$, then $\pi_1((\mathcal{S}, \varphi), P, (\xi, \eta))$ is a trivial group.

Remark 3. If $\mathcal{M}_{\boldsymbol{\xi}} \neq \emptyset$, then $\pi_1((\mathcal{M}, \boldsymbol{\varphi}), P, (\boldsymbol{\xi}, \boldsymbol{\eta}))$ is naturally isomorphic to the subgroup $i_{\#}(\pi_1(\mathcal{M}_{\boldsymbol{\xi}}, P))$ of $\pi_1(\mathcal{M}_{\boldsymbol{\eta}}, P)$, where $i_{\#}$ is the homomorphism between fundamental groups, canonically induced by the inclusion map $i : \mathcal{M}_{\boldsymbol{\xi}} \to \mathcal{M}_{\boldsymbol{\eta}}$. In particular, for every $\boldsymbol{\eta} \in \mathbb{R}^k$ and for a fixed base point, the size homotopy group $\pi_1(\boldsymbol{\eta}, \boldsymbol{\eta})$ coincides with the fundamental group of the space $\mathcal{M}_{\boldsymbol{\eta}}$.

The natural isomorphism between size homotopy groups and subgroups of the fundamental group of a suitable space prompts us to introduce the following notations.

Let $\mathcal{M}_1 \subseteq \mathcal{M}_2$ be non-empty topological spaces and $P \in \mathcal{M}_1$. If $i : \mathcal{M}_1 \to \mathcal{M}_2$ is the inclusion map, we define the group $\tilde{\pi}_1(\mathcal{M}_1, \mathcal{M}_2, P) \stackrel{def}{=} i_{\#}(\pi_1(\mathcal{M}_1, P))$. Obviously $\tilde{\pi}_1(\mathcal{M}_1, \mathcal{M}_2, P) \leq \pi_1(\mathcal{M}_2, P)$ (the symbol $\leq$ means "subgroup of") and, by the first homomorphism theorem, $\tilde{\pi}_1(\mathcal{M}_1, \mathcal{M}_2, P) \cong \pi_1(\mathcal{M}_1, P)/\ker(i_{\#})$.

The above notation provides a different way of looking at size homotopy groups. In fact, if $(\mathcal{M}, \boldsymbol{\varphi})$ is a size pair and $\boldsymbol{\xi} \preceq \boldsymbol{\eta} \in \mathbb{R}^k$, then $\mathcal{M}_{\boldsymbol{\xi}} \subseteq \mathcal{M}_{\boldsymbol{\eta}}$. Therefore, for each $P \in \mathcal{M}_{\boldsymbol{\xi}}$, we have $\pi_1((\mathcal{M}, \boldsymbol{\varphi}), P, (\boldsymbol{\xi}, \boldsymbol{\eta})) \cong \tilde{\pi}_1(\mathcal{M}_{\boldsymbol{\xi}}, \mathcal{M}_{\boldsymbol{\eta}}, P)$.

Lemma 5. *If $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \mathcal{M}_4$ are topological spaces, then the group $\tilde{\pi}_1(\mathcal{M}_1, \mathcal{M}_4, P)$ is a subgroup of a quotient of $\tilde{\pi}_1(\mathcal{M}_2, \mathcal{M}_3, P)$, for each $P \in \mathcal{M}_1$.*

Proof. Let $i_{hk} : \mathcal{M}_h \to \mathcal{M}_k$ be the inclusion maps (for $1 \leq h \leq k \leq 4$). We easily obtain :

$$\begin{aligned}\tilde{\pi}_1(\mathcal{M}_1, \mathcal{M}_4, P) &= i_{14\#}(\pi_1(\mathcal{M}_1, P)) = i_{24\#} \circ i_{12\#}(\pi_1(\mathcal{M}_1, P)) \\ &= i_{24\#}(\tilde{\pi}_1(\mathcal{M}_1, \mathcal{M}_2, P)) \leq i_{24\#}(\pi_1(\mathcal{M}_2, P)) = \tilde{\pi}_1(\mathcal{M}_2, \mathcal{M}_4, P).\end{aligned}$$

Since

$$\begin{aligned}\tilde{\pi}_1(\mathcal{M}_2, \mathcal{M}_4, P) &= i_{34\#} \circ i_{23\#}(\pi_1(\mathcal{M}_2, P)) \\ &= i_{34\#}(\tilde{\pi}_1(\mathcal{M}_2, \mathcal{M}_3, P)) \cong \tilde{\pi}_1(\mathcal{M}_2, \mathcal{M}_3, P)/\ker(j),\end{aligned}$$

where $j = i_{34\#|\tilde{\pi}_1(\mathcal{M}_2,\mathcal{M}_3,P)}$, the result is obtained. ∎

The main property of size homotopy groups is given by the following:

Proposition 6. *Assume that* $\boldsymbol{h} = (h_1, h_2, \ldots, h_k) \in \mathbb{R}^k$ *verifies the condition* $\sigma([(\mathcal{M}, \boldsymbol{\varphi})], [(\mathcal{N}, \boldsymbol{\psi})]) < h_i$ *for every index* i. *Then for every* $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^k$ *such that* $\boldsymbol{\xi} + \boldsymbol{h} \preceq \boldsymbol{\eta} - \boldsymbol{h}$ *and for every point* $P \in \mathcal{M}_{\boldsymbol{\xi}}$ *the following statement holds:*

($*$) *There exists a point* $Q \in \mathcal{N}_{\boldsymbol{\xi}+\boldsymbol{h}}$, *with* $|\psi_i(Q) - \varphi_i(P)| < h_i$ *for every index* i, *such that the group* $\pi_1((\mathcal{M}, \boldsymbol{\varphi}), P, (\boldsymbol{\xi}, \boldsymbol{\eta}))$ *is isomorphic to a subgroup of a quotient of* $\pi_1((\mathcal{N}, \boldsymbol{\psi}), Q, (\boldsymbol{\xi} + \boldsymbol{h}, \boldsymbol{\eta} - \boldsymbol{h}))$.

Proof. We can take a homeomorphism $f \in H(\mathcal{M}, \mathcal{N})$, such that $\Theta(f) < h_i$ for every index i. Moreover, $\mathcal{M}_{\boldsymbol{\xi}} \cong f(\mathcal{M}_{\boldsymbol{\xi}}) \subseteq \mathcal{N}_{\boldsymbol{\xi}+\boldsymbol{h}} \subseteq \mathcal{N}_{\boldsymbol{\eta}-\boldsymbol{h}} \subseteq f(\mathcal{M}_{\boldsymbol{\eta}}) \cong \mathcal{M}_{\boldsymbol{\eta}}$ and, since f is a homeomorphism,

$$\pi_1((\mathcal{M}, \boldsymbol{\varphi}), P, (\boldsymbol{\xi}, \boldsymbol{\eta})) \cong \tilde{\pi}_1(\mathcal{M}_{\boldsymbol{\xi}}, \mathcal{M}_{\boldsymbol{\eta}}, P) \cong \tilde{\pi}_1(f(\mathcal{M}_{\boldsymbol{\xi}}), f(\mathcal{M}_{\boldsymbol{\eta}}), Q),$$

where $Q = f(P)$. By Lemma 5, the group $\tilde{\pi}_1(f(\mathcal{M}_{\boldsymbol{\xi}}), f(\mathcal{M}_{\boldsymbol{\eta}}), Q)$ is isomorphic to a subgroup of a quotient of $\tilde{\pi}_1(\mathcal{N}_{\boldsymbol{\xi}+\boldsymbol{h}}, \mathcal{N}_{\boldsymbol{\eta}-\boldsymbol{h}}, Q) \cong \pi_1((\mathcal{N}, \boldsymbol{\psi}), Q, (\boldsymbol{\xi} + \boldsymbol{h}, \boldsymbol{\eta} - \boldsymbol{h}))$. ∎

In order to make the assertion of this proposition clear, we provide an example.

Example 6. In $\mathbb{R}^3$ consider the 2-spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ of equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$, respectively. Then consider the model of the projective plane $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) obtained as the set of pairs of antipodal points in $\mathcal{S}_1$ (resp. $\mathcal{S}_2$), denoted by $[(x, y, z)]$. We define the measuring functions $\varphi : \mathcal{P}_1 \to \mathbb{R}$ and $\psi : \mathcal{P}_2 \to \mathbb{R}$ by $[(x, y, z)] \mapsto z^2$. We obtain $\sigma([(\mathcal{P}_1, \varphi)], [(\mathcal{P}_2, \psi)]) = 5$, that is the Hausdorff distance between $\varphi(\mathcal{P}_1)$ and $\psi(\mathcal{P}_2)$. For $0 \le \xi \le \eta$ the following statements hold (assume $P = [(2, 0, 0)]$ and $Q = [(3, 0, 0)]$):

$$\pi_1((\mathcal{P}_1, \varphi), P, (\xi, \eta)) \cong \begin{cases} \mathbb{Z} & \text{if } \eta < 4 \\ \mathbb{Z}_2 & \text{if } \eta \ge 4 \end{cases} ; \quad \pi_1((\mathcal{P}_2, \varphi), Q, (\xi, \eta)) \cong \begin{cases} \mathbb{Z} & \text{if } \eta < 9 \\ \mathbb{Z}_2 & \text{if } \eta \ge 9 \end{cases} .$$

In particular $\pi_1\Big((\mathcal{P}_1, \varphi), P, (0, 12)\Big) \cong \mathbb{Z}_2$ and $\pi_1\Big((\mathcal{P}_2, \psi), Q, (6, 6)\Big) \cong \mathbb{Z}$. Compare this fact with the assertion of Proposition 6 (set $\mathcal{M} = \mathcal{P}_1$, $\mathcal{N} = \mathcal{P}_2$, $\xi = 0$, $\eta = 12$ and $h = 6$).

Now we state the main result of this work.

Theorem 7. *Assume there exist* $\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}}, \widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}} \in \mathbb{R}^k$ *with* $\bar{\boldsymbol{\xi}} \preceq \bar{\boldsymbol{\eta}}$, $\widehat{\boldsymbol{\xi}} \preceq \widehat{\boldsymbol{\eta}}$ *and a point* $P \in \mathcal{M}_{\bar{\boldsymbol{\xi}}}$ *such that the following property holds:*

($\otimes$) *For each point* $Q \in \mathcal{N}_{\widehat{\boldsymbol{\xi}}}$, *such that* $|\psi_i(Q) - \varphi_i(P)| < h_i \overset{def}{=} \min\{\widehat{\xi}_i - \bar{\xi}_i, \bar{\eta}_i - \widehat{\eta}_i\}$ *for every index* i, *the group* $\pi_1((\mathcal{M}, \boldsymbol{\varphi}), P, (\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}}))$ *is not isomorphic to a subgroup of any quotient of* $\pi_1((\mathcal{N}, \boldsymbol{\psi}), Q, (\widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}}))$.

Then $\sigma_k([(\mathcal{M}, \boldsymbol{\varphi})], [(\mathcal{N}, \boldsymbol{\psi})]) \ge \min_{1 \le i \le k} h_i$.

Proof. On the contrary, suppose $\sigma_k([(\mathcal{M},\boldsymbol{\varphi})],[(\mathcal{N},\boldsymbol{\psi})]) < h = \min_{1\le i\le k} h_i$. Therefore, we can take a homeomorphism $f : \mathcal{M} \to \mathcal{N}$ such that $\Theta_k(f) < h$. If we define $\boldsymbol{h} = (h_1, h_2, \ldots, h_k)$, then $\widehat{\boldsymbol{\xi}} \succeq \bar{\boldsymbol{\xi}} + \boldsymbol{h}$, $\bar{\boldsymbol{\eta}} - \boldsymbol{h} \succeq \widehat{\boldsymbol{\eta}}$ and we have $f(\mathcal{M}_{\bar{\boldsymbol{\xi}}}) \subseteq \mathcal{N}_{\bar{\boldsymbol{\xi}}+\boldsymbol{h}} \subseteq \mathcal{N}_{\widehat{\boldsymbol{\xi}}} \subseteq \mathcal{N}_{\widehat{\boldsymbol{\eta}}} \subseteq \mathcal{N}_{\bar{\boldsymbol{\eta}}-\boldsymbol{h}} \subseteq f(\mathcal{M}_{\bar{\boldsymbol{\eta}}})$. So, the point $Q = f(P)$ belongs to $\mathcal{N}_{\widehat{\boldsymbol{\xi}}}$, with $|\psi_i(Q) - \varphi_i(P)| < h_i$ for every index i. By Lemma 5 the group $\tilde{\pi}_1(f(\mathcal{M}_{\bar{\boldsymbol{\xi}}}), f(\mathcal{M}_{\bar{\boldsymbol{\eta}}}), Q) \cong \tilde{\pi}_1(\mathcal{M}_{\bar{\boldsymbol{\xi}}}, \mathcal{M}_{\bar{\boldsymbol{\eta}}}, P) \cong \pi_1((\mathcal{M},\boldsymbol{\varphi}), P, (\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}}))$ is isomorphic to a subgroup of a quotient of $\tilde{\pi}_1(\mathcal{N}_{\widehat{\boldsymbol{\xi}}}, \mathcal{N}_{\widehat{\boldsymbol{\eta}}}, Q) \cong \pi_1((\mathcal{N},\boldsymbol{\psi}), Q, (\widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}}))$ and this is a contradiction. ■

Now let $(\mathcal{N},\boldsymbol{\psi})$ be a size pair such that $\mathcal{N}$ is a C^1-submanifold of a Euclidean space and $\boldsymbol{\psi} \in C^1(\mathcal{N}, \mathbb{R}^k)$. A point $P \in \mathcal{N}$ will be called *pseudo-critical* for $\boldsymbol{\psi}$ if the convex hull of the set $\{\nabla\psi_1, \nabla\psi_2, \ldots, \nabla\psi_k\}$ contains the null vector of the tangent space of $\mathcal{N}$ at P. We shall denote the set of pseudo-critical points of $\mathcal{N}$ by $\boldsymbol{C}(\mathcal{N},\boldsymbol{\psi})$.

Notice that, when $k = 1$, all pseudo-critical points are genuine critical points for ψ.

Lemma 8. *If $P \notin \boldsymbol{C}(\mathcal{N},\boldsymbol{\psi})$ then there exists a path $\lambda : [0,1] \to \mathcal{N}$, with $\lambda(0) = P$ and $\boldsymbol{\psi}(\lambda(t)) \preceq \boldsymbol{\psi}(P)$ for each t, such that $\lambda(1) \in \boldsymbol{C}(\mathcal{N},\boldsymbol{\psi})$.*

Proof. First of all, define $\boldsymbol{\xi} = \boldsymbol{\psi}(P)$ and for each $\delta > 0$, let $\boldsymbol{\delta} = (\delta, \delta, \ldots, \delta)$ (k times). If $P \notin \boldsymbol{C}(\mathcal{N},\boldsymbol{\psi})$, then there exists a vector v belonging to the tangent space of $\mathcal{N}$ at P, such that $\nabla\psi_i \cdot v < 0$ for every i. So, there exist $\delta > 0$, a point $P' \in \mathcal{N}_{\boldsymbol{\xi}-\boldsymbol{\delta}}$ and a path γ from P to P', such that each $\psi_i \circ \gamma$ is strictly decreasing. Now, let Γ be the arc-connected component of P' in $\mathcal{N}_{\boldsymbol{\xi}}$ and define $\Gamma_\delta = \Gamma \cap \mathcal{N}_{\boldsymbol{\xi}-\boldsymbol{\delta}}$. We show that Γ_δ is a closed subspace of $\mathcal{N}$ and therefore is compact. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of points of Γ_δ converging to a point R. Of course, $R \in \mathcal{N}_{\boldsymbol{\xi}-\boldsymbol{\delta}}$ and, furthermore, there exists an (arc-connected) open neighborhood U_R of R, such that $U_R \subseteq \mathcal{N}_{\boldsymbol{\xi}}$. For a suitable $\bar{n} \in \mathbb{N}$, the point $R_{\bar{n}}$ belongs to U_R and it is easy to take a path in $\mathcal{N}_{\boldsymbol{\xi}}$ from P' to R, via $R_{\bar{n}}$. Therefore, R also belongs to Γ and, as a consequence, Γ_δ is compact. Now, if Q' is a global minimum point for the restriction of ψ_1 to Γ_δ, then the path $\lambda = \gamma * \gamma'$, where γ' is any path in $\mathcal{N}_{\boldsymbol{\xi}}$ from P' to Q', gives the statement. For, if $Q' \notin \boldsymbol{C}(\mathcal{N},\boldsymbol{\psi})$, then we can take a path $\rho : [0,1] \to \mathcal{N}$ such that $\rho(0) = Q'$ and each $\psi_i \circ \rho$ is strictly decreasing. Obviously, $\rho(t) \in \mathcal{N}_{\boldsymbol{\xi}-\boldsymbol{\delta}}$ for every $t \in [0,1]$ and, since $\psi_1(\rho(1)) < \psi_1(Q')$, we get the contradiction. ■

As a consequence we obtain the following:

Proposition 9. *Assume that $\boldsymbol{h} = (h_1, h_2, \ldots, h_k) \in \mathbb{R}^k$ verifies the condition $h_i > \sigma([(\mathcal{M},\boldsymbol{\varphi})],[(\mathcal{N},\boldsymbol{\psi})])$ for every index i. Then for every $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^k$ with $\boldsymbol{\xi} + \boldsymbol{h} \preceq \boldsymbol{\eta} - \boldsymbol{h}$ and every point $P \in \mathcal{M}_{\boldsymbol{\xi}}$ the following statement holds:*

($\diamond$) *There exists a point $R \in \mathcal{N}_{\boldsymbol{\xi}+\boldsymbol{h}} \cap \boldsymbol{C}(\mathcal{N},\boldsymbol{\psi})$ such that the group*

$$\pi_1((\mathcal{M},\boldsymbol{\varphi}), P, (\boldsymbol{\xi}, \boldsymbol{\eta}))$$

is isomorphic to a subgroup of a quotient of $\pi_1((\mathcal{N},\boldsymbol{\psi}), R, (\boldsymbol{\xi}+\boldsymbol{h}, \boldsymbol{\eta}-\boldsymbol{h}))$.

Proof. Let Q be the point of Proposition 6. If $Q \in \boldsymbol{C}(\mathcal{N}, \boldsymbol{\psi})$ the thesis is proved by setting $R = Q$. If $Q \notin \boldsymbol{C}(\mathcal{N}, \boldsymbol{\psi})$, then there exists, by the previous lemma, a continuous path λ in $\mathcal{N}_{\boldsymbol{\xi}+\boldsymbol{h}}$ from Q to a pseudo-critical point Q'. We take $R = Q'$ and, since the homomorphism $\lambda_{\#} : \pi_1((\mathcal{N}, \boldsymbol{\psi}), Q, (\boldsymbol{\xi}+\boldsymbol{h}, \boldsymbol{\eta}-\boldsymbol{h})) \to \pi_1((\mathcal{N}, \boldsymbol{\psi}), Q', (\boldsymbol{\xi}+\boldsymbol{h}, \boldsymbol{\eta}-\boldsymbol{h}))$ defined by $\lambda_{\#}([\alpha]) = [\lambda^{-1} * \alpha * \lambda]$ is an isomorphism, the statement follows from Proposition 6. ∎

One can prove the following, in the same way as Theorem 7.

Theorem 10. *Assume there exist $\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}}, \widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}} \in \mathbb{R}^k$ with $\bar{\boldsymbol{\xi}} \preceq \bar{\boldsymbol{\eta}}$, $\widehat{\boldsymbol{\xi}} \preceq \widehat{\boldsymbol{\eta}}$ and a point $P \in \mathcal{M}_{\bar{\boldsymbol{\xi}}}$ such that the following property holds:*

($\oplus$) *For each point $R \in \mathcal{N}_{\widehat{\boldsymbol{\xi}}} \cap \boldsymbol{C}(\mathcal{N}, \boldsymbol{\psi})$, the group $\pi_1((\mathcal{M}, \boldsymbol{\varphi}), P, (\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}}))$ is not isomorphic to a subgroup of any quotient of $\pi_1((\mathcal{N}, \boldsymbol{\psi}), R, (\widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}}))$.*

Then $\sigma_k([(\mathcal{M}, \boldsymbol{\varphi})], [(\mathcal{N}, \boldsymbol{\psi})]) \geq \min_{1 \leq i \leq k} \left\{ \min\{\widehat{\xi}_i - \bar{\xi}_i, \bar{\eta}_i - \widehat{\eta}_i\} \right\}$.

We point out that Theorems 7 and 10 allow us to obtain lower bounds for natural size distances, from the knowledge of the size homotopy groups $\pi_1((\mathcal{N}, \boldsymbol{\psi}), Q, (\widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}}))$ and $\pi_1((\mathcal{M}, \boldsymbol{\varphi}), P, (\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\eta}}))$, by varying Q and P. Since the direct computation of $\sigma([(\mathcal{M}, \boldsymbol{\varphi})], [(\mathcal{N}, \boldsymbol{\psi})])$ requires the study of all homeomorphisms from $\mathcal{M}$ to $\mathcal{N}$, the usefulness of our approach is clear. Now we shall give an example of this procedure.

Example 7. Consider the two tori $\mathcal{T}, \mathcal{T}' \subset \mathbb{R}^3$ generated by the rotation around the y-axis of the circles lying in the plane yz and with centers $A = (0, 0, 3)$ and $B = (0, 0, 4)$, and radii 2 and 1, respectively (see Figure 1). As measuring function φ (resp. φ') on $\mathcal{T}$ (resp. on $\mathcal{T}'$) we take the restriction to $\mathcal{T}$ (resp. to $\mathcal{T}'$) of the function $\zeta : \mathbb{R}^3 \to \mathbb{R}$, $\zeta(x, y, z) = z$. We point out that, for both $\mathcal{T}$ and $\mathcal{T}'$, the image of the measuring function is the closed interval $[-5, 5]$. We want to prove that the natural size distance between $[(\mathcal{T}, \varphi)]$ and $[(\mathcal{T}', \varphi')]$ is 2. In order to do that, let us consider the homeomorphism f, that takes each point of the former torus to the point having the same toroidal coordinates in the latter. We can easily verify that $\Theta(f) = 2$. So we have only to prove that $\sigma([(\mathcal{T}, \varphi)], [(\mathcal{T}', \varphi')]) \geq 2$. This inequality follows from Theorem 7 by choosing $P = (0, 0, -5)$, $\bar{\xi} = 1$, $\bar{\eta} = 5 - \delta$, $\widehat{\xi} = \widehat{\eta} = 3 - \delta$ and observing that if δ is any small enough positive number, then $\pi_1((\mathcal{T}, \varphi), P, (1,\ 5 - \delta)) \cong \mathbb{Z} * \mathbb{Z}$ and $\pi_1((\mathcal{T}', \varphi'), R, (3 - \delta, 3 - \delta)) \cong \mathbb{Z}$ for each $R \in \mathcal{T}'_{3-\delta}$. From Theorem 7 we obtain $\sigma([(\mathcal{T}, \varphi)], [(\mathcal{T}', \varphi')]) \geq \min\{(3 - \delta) - 1, (5 - \delta) - (3 - \delta)\} = 2 - \delta$. This implies the desired inequality.

Observe that, in Example 7, the natural size distance is strictly greater than the (vanishing) Hausdorff distance between the images of the two measuring functions.

It is also interesting to point out that, in our example, size homotopy groups give more information than size functions. In fact, the size functions of $(\mathcal{T}, \varphi)$ and $(\mathcal{T}', \varphi')$ are both trivial and therefore do not give any positive lower bound for the natural size distance.

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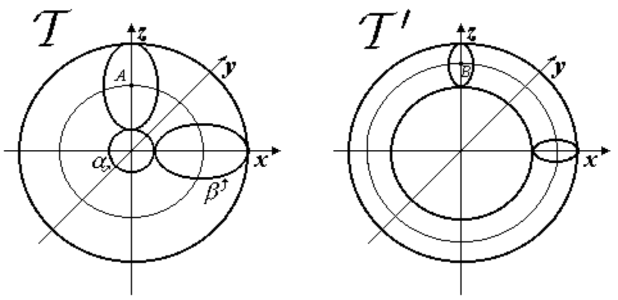

Figure 1: Theorem 7 allows us to prove that $\sigma([(\mathcal{T},\varphi)],[(\mathcal{T}',\varphi')]) = 2$.

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