# Size Theory as a Topological Tool for Computer Vision

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**Abstract.** In this paper we give an outline of Size Theory and its main results. The usefulness of such a theory in comparing shapes is highlighted by showing some examples. The robustness of Size Theory with respect to noise and occlusions is pointed out. In addition, an algebraic approach to the theory is presented.

*Keywords:* shape, Size Theory, size function, natural size distance.

## 1 Introduction

Comparing shapes of objects is a major task in Computer Vision. Size Theory is a new mathematical tool for dealing with this task, and it is now the subject of experimentation (cf., e.g., [2], [3], [4], [5], [21] and [22]). In this paper we shall give the main definitions and results in this theory, and show its main properties by displaying some examples. For the sake of conciseness we shall omit the proofs

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of our results: they can be found in [7], [8], [10], [13] and [14]. An extension of the theory appears in [10] and [16].

The fundamental ideas in Size Theory are the concepts of natural size distance and size function. Both of them provide a way of measuring the extent to which the shapes of two compact topological spaces resemble each other. They are modular concepts, in the sense that they depend on the arbitrary choice of particular functions (called measuring functions) which can be set in order to obtain invariance under those transformations that are required to preserve shape in each specific context. Furthermore these concepts are intrinsically related. We underline that Size Theory can be applied for studying all data that can be seen as a compact topological space, not only images (even though such a theory was originally conceived for use in Computer Vision).

After giving some mathematical results on natural size distances and size functions we shall deal with the problem of computing the latter. Next, as far as comparison of images is concerned, we shall exhibit some examples showing the properties of invariance, noise-resistance and occlusion-resistance of size functions (with respect to the choice of suitable measuring functions). Finally, we shall give an algebraic representation of size functions in terms of formal series and discuss its usefulness.

## 2 Natural Size Distances: Some Definitions and Results

In this section, we shall consider the set *Size* of all pairs $(\mathcal{M}, \varphi)$ (called *size pairs*) where $\mathcal{M}$ is a compact topological space and $\varphi$ is a continuous function from $\mathcal{M}$ to the set $\mathbb{R}$ of real numbers (called *measuring function*). In some cases, in order to obtain particular results, we shall assume that $\mathcal{M}$ is a sufficiently regular submanifold of some Euclidean space.

Our goal is to define a distance that allows us to measure the extent to which the shapes of $\mathcal{M}$ and $\mathcal{N}$ are similar to each other. We shall do so with respect to the continuous functions $\varphi$ and $\psi$, which have been chosen arbitrarily.

**Definition 1.** Let $(\mathcal{M}, \varphi)$, $(\mathcal{N}, \psi)$ be two size pairs and let $H(\mathcal{M}, \mathcal{N})$ be the set of homeomorphisms from $\mathcal{M}$ onto $\mathcal{N}$. Let us consider the function $\Theta$ that takes each homeomorphism $f \in H(\mathcal{M}, \mathcal{N})$ to the real number $\Theta(f) = \max_{P \in \mathcal{M}} |\varphi(P) - \psi(f(P))|$. We shall call $\Theta$ the *natural size measure* in $H(\mathcal{M}, \mathcal{N})$ with respect to the measuring functions $\varphi$ and $\psi$.

In plain words $\Theta$ measures how much $f$ changes the values taken by the measuring function.

**Proposition 1.** *The function $\Sigma : Size \times Size \to \mathbb{R} \cup \{+\infty\}$, defined by setting $\Sigma((\mathcal{M}, \varphi), (\mathcal{N}, \psi)) = \inf_{f \in H(\mathcal{M}, \mathcal{N})} \Theta(f)$ if $H(\mathcal{M}, \mathcal{N}) \neq \emptyset$ and $+\infty$ otherwise, is a pseudometric on Size.*

**Definition 2.** The metric $\sigma$ induced by the pseudometric $\Sigma$ will be called the *natural size distance in Size/* $\simeq$, where $\simeq$ denotes the equivalence relation defined by setting $(\mathcal{M}, \varphi) \simeq (\mathcal{N}, \psi)$ if and only if $\Sigma\left((\mathcal{M}, \varphi), (\mathcal{N}, \psi)\right) = 0$. The equivalence class of $(\mathcal{M}, \varphi)$ will be denoted by the symbol $[(\mathcal{M}, \varphi)]$.

More details on the passage from a pseudometric to a metric can be found in [1]. We have used the term "natural" because our manner of defining a pseudometric between compact topological spaces is a particular case of a more general method (cf. [6]).

Before proceeding we shall give a trivial example of natural size distance.

*Example 1.* In $\mathbb{R}^3$ consider the unit sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = 1$ and the ellipsoid $\mathcal{E}$ with equation $x^2 + 4y^2 + 9z^2 = 1$. On $\mathcal{S}$ and $\mathcal{E}$ consider respectively the measuring functions $\varphi$ and $\psi$ that take every point of $\mathcal{S}$ and $\mathcal{E}$ to the Gaussian curvature of the given manifold at that point. We have that $\sigma\left([(\mathcal{S}, \varphi)], [(\mathcal{E}, \psi)]\right) = 35$. In fact $\varphi(\mathcal{S}) = \{1\}$ while $\psi(\mathcal{E}) = \{r \in \mathbb{R} : \ 4/9 \leq r \leq 36\}$, and therefore for every $f \in H\left(\mathcal{S}, \mathcal{E}\right)$ we have $\Theta(f) = 35$.

**Definition 3.** We shall call *optimal in* $H\left(\mathcal{M}, \mathcal{N}\right)$ every homeomorphism $f \in H\left(\mathcal{M}, \mathcal{N}\right)$ such that $\sigma\left([(\mathcal{M}, \varphi)], [(\mathcal{N}, \psi)]\right) = \Theta(f)$.

We point out that an optimal homeomorphism does not generally exist, even in cases when $\mathcal{M}$ and $\mathcal{N}$ are regular, compact and without boundary manifolds and $\varphi$, $\psi$ are regular measuring functions (cf. [10]). However, if we assume such hypotheses (in particular that $\mathcal{M}$

and $\mathcal{N}$ are manifolds of class $C^2$ and $\varphi$, $\psi$ are measuring functions of class $C^1$) we have the following theorem.

**Theorem 1.** *Let us assume that an optimal homeomorphism exists in $H(\mathcal{M}, \mathcal{N})$. Then the natural size distance between $[(\mathcal{M}, \varphi)]$ and $[(\mathcal{N}, \psi)]$ is the Euclidean distance between a critical value of $\varphi$ and a critical value of $\psi$.*

The above theorem no longer holds if we drop the hypothesis of existence of an optimal homeomorphism (cf. [10]). Such a theorem makes the computation of natural size distances less difficult.

A survey on natural size distances can be found in [11].

## 3 Size Functions: Definitions and Properties

In general, natural size distances are difficult to compute, as they involve the study of all homeomorphisms between two compact topological spaces. On the other hand, they can compare compact topological spaces with respect to given measuring functions in a very powerful manner, and quantify the difference. Thus we need a tool to easily obtain information on natural size distances without computing them directly: the concept of size function is such a tool. In addition, size functions are useful for comparison of shapes even independently of natural size distances. In the following definitions, we shall assume a size pair $(\mathcal{M}, \varphi)$ is given.

**Definition 4.** For every $y \in \mathbb{R}$ we define a relation $\cong_{\varphi \leq y}$ in $\mathcal{M}$ by setting $P \cong_{\varphi \leq y} Q$ $(P, Q \in \mathcal{M})$ if and only if either $P = Q$ or there exists a continuous path $\gamma : [0,1] \to \mathcal{M}$ such that $\gamma(0) = P$, $\gamma(1) = Q$ and $\varphi(\gamma(\tau)) \leq y$ for every $\tau \in [0,1]$. In this second case we shall say that $P$ and $Q$ are $(\varphi \leq y)$- *homotopic* and call $\gamma$ a *$(\varphi \leq y)$-homotopy from $P$ to $Q$.*

*Remark 1.* It is easy to show that $\cong_{\varphi \leq y}$ is an equivalence relation on $\mathcal{M}$ for every $y \in \mathbb{R}$.

**Definition 5.** For every $x \in \mathbb{R}$ we shall denote by $\mathcal{M}\langle\varphi \leq x\rangle$ the set $\{P \in \mathcal{M} : \ \varphi(P) \leq x\}$.

**Definition 6.** Consider the function $\ell_{(\mathcal{M},\varphi)} : \mathbb{R} \times \mathbb{R} \to \mathbb{N} \cup \{\infty\}$ defined by setting $\ell_{(\mathcal{M},\varphi)}(x,y)$ equal to the (finite or infinite) number of equivalence classes in which $\mathcal{M}\langle\varphi \leq x\rangle$ is divided by the equivalence relation $\cong_{\varphi \leq y}$. Such a function will be called the *size function associated with the size pair* $(\mathcal{M}, \varphi)$.

*Remark 2.* When $x \leq y$ size functions have a simple geometric interpretation: in such a case $\ell_{(\mathcal{M},\varphi)}(x,y)$ is equal to the number of arcwise connected components of $\mathcal{M}\langle\varphi \leq y\rangle$ containing at least one point of $\mathcal{M}\langle\varphi \leq x\rangle$.

Now we shall give some simple examples of size functions.

*Example 2.* In Fig. 1 we show the size function of an ellipse $\mathcal{E}$ with respect to the measuring function that takes each point to its distance from the barycentre of $\mathcal{E}$. In every highlighted region the constant value taken by the size function is given by the number displayed.

This size function has been computed by means of a computer and therefore some errors due to the necessary discretization appear in the form of small triangles near the diagonal.

*Remark 3.* We shall often display only the part of a size function inside a square $(x\min, x\max) \times (y\min, y\max)$ with $x\min = y\min$ and $x\max = y\max$. When this is the case the values $x\min$, $y\min$, $x\max$, $y\max$ will be shown in each figure.

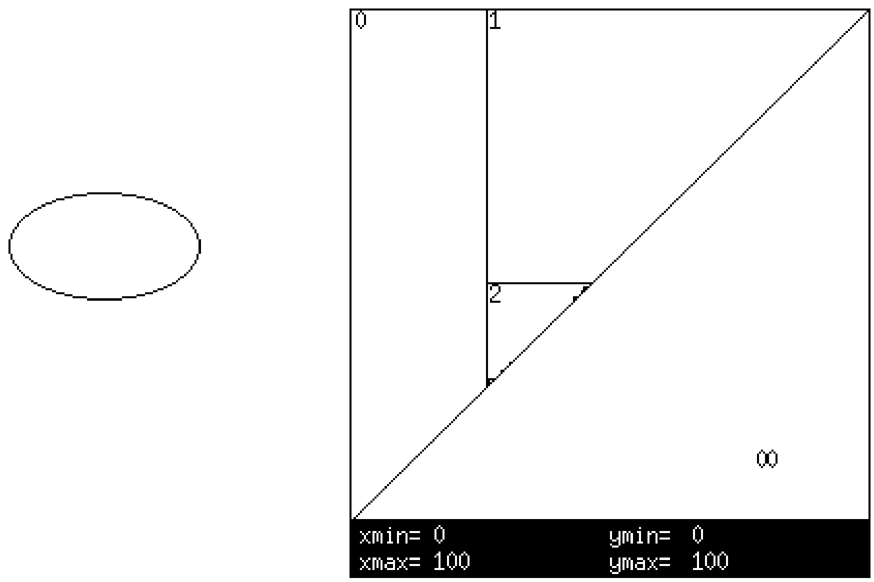


**Fig. 1.** Size function of an ellipse with respect to the distance from the barycentre.

*Example 3.* Consider the size pair $\left(\mathcal{E}_{(a,b,c)}, \psi\right)$ where $\mathcal{E}_{(a,b,c)}$ is the ellipsoid with equation $ax^2 + by^2 + cz^2 = 1$ in $\mathrm{I\!R}^3$ $(a, b, c > 0)$ and $\psi$ is the 1-dimensional measuring function that takes each point of $\mathcal{E}_{(a,b,c)}$ to the Gaussian curvature of the ellipsoid at that point. In

Fig. 2 we show the function $\ell_{\left(\mathcal{E}_{(a,b,c)},\psi\right)} : \mathbb{R} \times \mathbb{R} \to \mathbb{N} \cup \{\infty\}$: in every highlighted region the constant value taken by the size function is given for the case $a < b < c$.

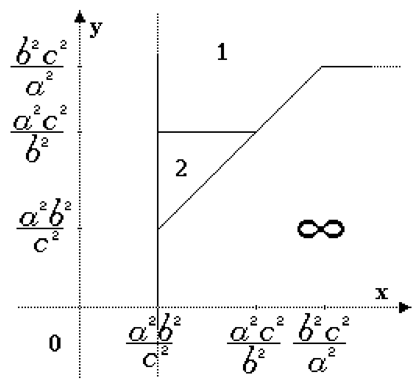


**Fig. 2.** Size function of the ellipsoid with equation $ax^2 + by^2 + cz^2 = 1$ ($c > b > a > 0$) with respect to the Gaussian curvature.

*Example 4.* Consider the size pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is the curve depicted in Fig. 3, left, and $\varphi$ is the 1-dimensional measuring function that takes each point of $\mathcal{M}$ to its distance from the barycentre of $\mathcal{M}$. In Fig. 3, right, we show the function $\ell_{(\mathcal{M},\varphi)} : \mathbb{R} \times \mathbb{R} \to \mathbb{N} \cup \{\infty\}$: in every highlighted region the constant value taken by the size function is given.

### 3.1 Main Properties of the Size Function $\ell_{(\mathcal{M},\varphi)}$

1. $\ell_{(\mathcal{M},\varphi)}(x, y)$ is non-decreasing in $x$ and non-increasing in $y$.
2. $\ell_{(\mathcal{M},\varphi)}(x, y)$ is finite for $x < y$ (under the hypothesis that $\mathcal{M}$ is not only compact but also locally arcwise connected).

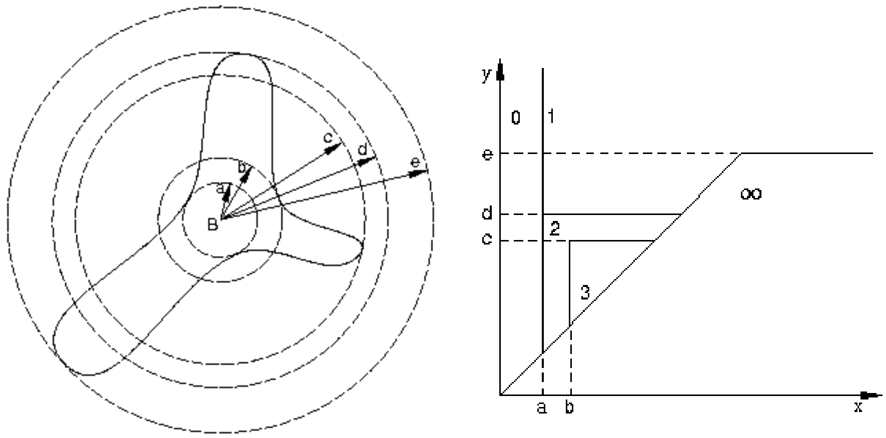


**Fig. 3.** Size function of the curve depicted on the left with respect to the distance from the barycentre used as measuring function.

3. $\ell_{(\mathcal{M},\varphi)}(x, y) = 0$ for every $x < \min_{P \in \mathcal{M}} \varphi(P)$.
4. $\ell_{(\mathcal{M},\varphi)}(x, y)$ is equal to the number of the arcwise connected components of $\mathcal{M}$ for every $x, y \geq \max_{P \in \mathcal{M}} \varphi(P)$.
5. $\ell_{(\mathcal{M},\varphi)}(x, y) = \infty$ for every $x, y$ such that there exists a non-isolated point $Q \in \mathcal{M}$ for which $x > \varphi(Q)$ and $y < \varphi(Q)$.

We complete this summary of properties of size functions by giving a useful theorem, which helps us in localizing the discontinuity points of the size function $\ell_{(\mathcal{M},\varphi)}$ in cases when $\mathcal{M}$ is a closed (i.e. compact and without boundary) submanifold of some Euclidean space.

**Theorem 2.** *Let us assume that $\mathcal{M}$ is $C^2$ and the measuring function $\varphi$ is $C^1$. In such a case, if $(x, y)$ is a discontinuity point for the size function $\ell_{(\mathcal{M},\varphi)}$ and $x < y$ then either $x$ or $y$ or both are critical values for $\varphi$.*

This theorem can be checked with the size function of Example 3, where if $(x, y)$ is a discontinuity point and $x < y$ then either $x$ is a minimum value or $y$ is a saddle value of the measuring function or both. Analogously, in Examples 2 and 4 if $(x, y)$ is a discontinuity point and $x < y$ then either $x$ is a minimum value or $y$ is a maximum value of the measuring function or both.

For further details about size functions (both from the applicational and theoretical point of view) we refer the reader to [9], [12], [18], [19] and [20].

### 3.2 Invariance Under Transformation Groups

The usefulness of size functions in comparing shapes is mostly given by the fact that they inherit invariance, under certain classes of transformations, from the measuring functions. In plain words, if we need to consider two planar shapes equivalent when they are isometric, it is sufficient to take a function invariant under isometry as a measuring function. For example, we can use the distance from the barycentre of the "object". If we are interested in planar affine invariance we can compute the measuring function at a point $P$ by taking, for instance, the ratio between the smallest area of an ellipse containing the planar shape $\mathcal{M}$ and the smallest area of an ellipse containing $\mathcal{M}$ and with centre at $P$, and so on for invariance under other transformations. Obviously there is usually an infinite number of suitable choices.

It is important to point out that Size Theory is a modular theory: we need only change the measuring function in order to obtain the invariance we require. In Fig. 4 we offer an example of invariance of size functions under isometry, obtained by using the distance from the barycentre as a measuring function.

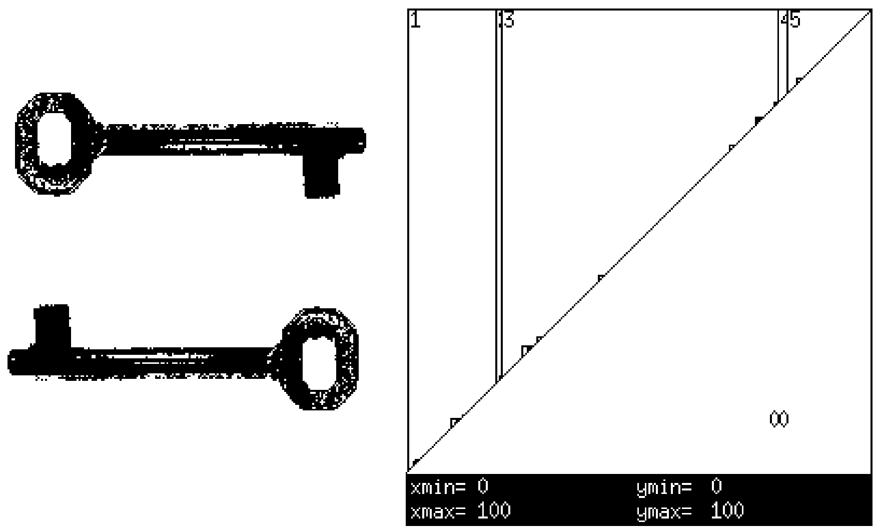


**Fig. 4.** Two isometric images have the same size function with respect to the distance from the barycentre.

### 3.3 Comparing Size Functions

There are many ways to compare two size functions $\ell_{(\mathcal{M},\varphi)}$ and $\ell_{(\mathcal{N},\psi)}$ corresponding to different shapes. Perhaps the simplest one is to compute the integral of $|\ell_{(\mathcal{M},\varphi)} - \ell_{(\mathcal{N},\psi)}|$ on a given domain, i.e. to consider the $L^1$-norm of the function $\ell_{(\mathcal{M},\varphi)} - \ell_{(\mathcal{N},\psi)}$, but obviously an $L^p$-norm with $p \neq 1$ can also be used. Another way is to consider the set $\mathcal{D}$ of the discontinuity points (including the diagonal $y = x$), for each size function. Then we can define a pseudodistance between

$\ell_{(\mathcal{M},\varphi)}$ and $\ell_{(\mathcal{N},\psi)}$ as the Hausdorff distance between $\mathcal{D}\left(\ell_{(\mathcal{M},\varphi)}\right)$ and $\mathcal{D}\left(\ell_{(\mathcal{N},\psi)}\right)$.

It is clear that many choices are possible. The important thing to note is that Size Theory changes the problem of comparing shapes into the mathematical problem of comparing functions from the real plane to the extended natural numbers, i.e. a much simpler task.

### 3.4 The Link between Size Functions and Natural Size Distances

The key fact in Size Theory is that natural size distances and size functions are strongly related. This allows us to obtain information about the former (powerful but intrinsically difficult to compute) by studying the latter (easily computable). This statement is a consequence of the following two theorems.

**Theorem 3.** *If* $\sigma\left(\left[(\mathcal{M},\varphi)\right],\left[(\mathcal{N},\psi)\right]\right) < \epsilon$ *then for every* $x, y \in \mathbb{R}$ *and every* $h \geq \epsilon$ *the following statement holds:*

$$\ell_{(\mathcal{M},\varphi)}\left(x-h, y+h\right) \leq \ell_{(\mathcal{N},\psi)}\left(x, y\right) \leq \ell_{(\mathcal{M},\varphi)}\left(x+h, y-h\right).$$

**Theorem 4.** *Let us assume that there exist* $\bar{x}, \bar{y}, \tilde{x}, \tilde{y} \in \mathbb{R}$ *for which* $\ell_{(\mathcal{N},\psi)}\left(\bar{x}, \bar{y}\right) > \ell_{(\mathcal{M},\varphi)}\left(\tilde{x}, \tilde{y}\right)$. *Then it holds that*

$$\sigma\left(\left[(\mathcal{M},\varphi)\right],\left[(\mathcal{N},\psi)\right]\right) \geq \min\left\{\tilde{x}-\bar{x}, \bar{y}-\tilde{y}\right\}.$$

We point out that the latter theorem allows us to obtain a lower bound for the natural size distance from knowledge of the size functions at two points. Since the direct computation of $\sigma\left(\left[(\mathcal{M},\varphi)\right],\left[(\mathcal{N},\psi)\right]\right)$ requires the study of all homeomorphisms from

$\mathcal{M}$ to $\mathcal{N}$, it is clear that this new method is very useful. Moreover, as we shall see in the following section, size functions are not difficult to compute.

It is important to point out that Theorem 4 gives a lower bound, not an upper bound for the natural size distance.

### 3.5 Noise-resistance

Size functions have a good resistance to noise, as has been verified by means of experimentation (cf., e.g., [5] and [23]). The mathematical reason for this is the above Theorem 3, which says that small changes in shape by means of homeomorphisms give small changes in size functions. Indeed, even when we perform a transformation that changes the shape non-homeomorphically, we often observe that some discontinuities of the size function are preserved (in the sense that they are moved slightly). Hence we can use Size Theory even when the topological type of the shape is changed. In order to clarify these facts we shall display two examples, which show the effect on the size function of a perturbation of the shape. The first perturbation does not change the topology of the shape (see Fig. 5), the second one does (see Fig. 6). We can see that the "main" discontinuities appearing in Fig. 1 are preserved in Fig. 5 and Fig. 6, in the sense that they appear just slightly shifted: hence such discontinuities give noise-resistant information about the shape.

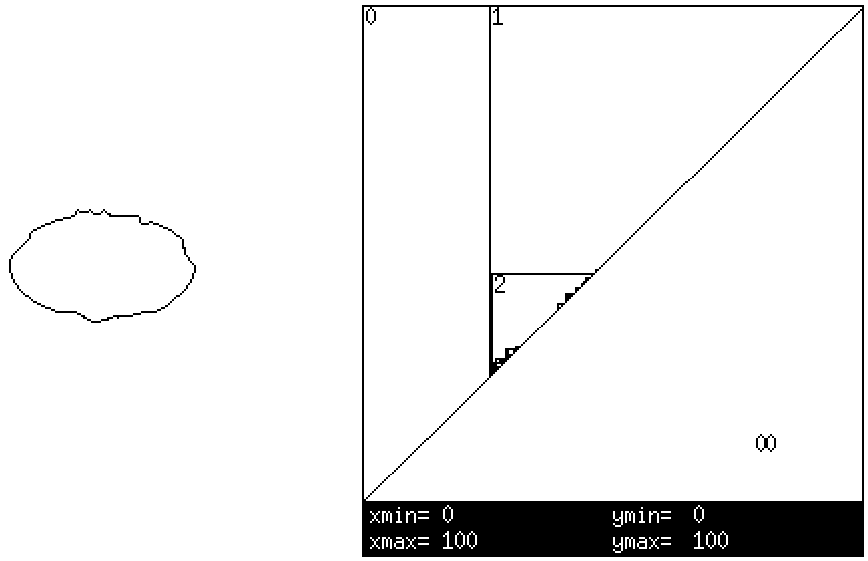


**Fig. 5.** Size function of a deformed ellipse with respect to the distance from the barycentre.

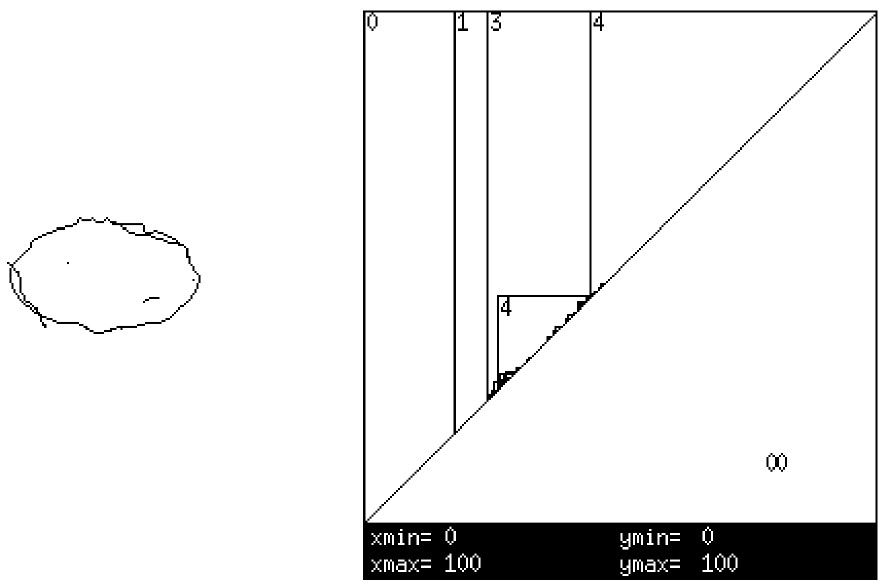


**Fig. 6.** An ellipse with some added noise and its size function with respect to the distance from the barycentre.

### 3.6 What About Occlusions?

Here we want to highlight the capacity of Size Theory to successfully address the problem of occlusion. The key fact has already been pointed out in the previous section: when we perturb an image (now by adding an occlusion) we often observe that some discontinuities of the corresponding size function are preserved (in the sense that they are moved slightly). Hence we can also use size functions in presence of occlusions. Obviously it is important to choose a suitable measuring function, i.e. a function capable, in some sense, of not seeing the occlusion. Here we can give a clarifying example. In Fig. 7 $(a, b, c)$ a wrench is displayed both without and with occlusions. Now, for each image $\mathcal{M}$ let us consider the subset of $\mathbb{R}^2 \times \mathbb{R}^2$ whose elements are the pairs $(P, Q)$ with $P, Q \in \mathcal{M}$, i.e. $\mathcal{M} \times \mathcal{M}$. As a measuring function on $\mathcal{M} \times \mathcal{M}$ we shall use the map that takes each pair $(P, Q)$ to the number $-\|P - Q\|$. In Fig. 8 the size functions corresponding in this manner to the images shown in Fig. 7 are given (the images really used in computation are the small ones).

We can see that the changes in the size functions depending on the occlusions leave some discontinuities almost unmodified (these are highlighted in the figure by using bold lines): such "discontinuity structures" can be considered the "fingerprint" of the wrench. The size function of the nutcracker allows a comparison. Naturally, this does not mean that every size function is resistant to every type of occlusion, but it is clear that no theory can exhibit good behaviour in presence of arbitrarily large occlusions.

The aim of this section is to point out that even though Size Theory uses a global approach to the problem of comparison of shapes, it preserves local information by distributing it in the real plane, thereby also allowing successful management of uncertainty due to presence of occlusions in the image.

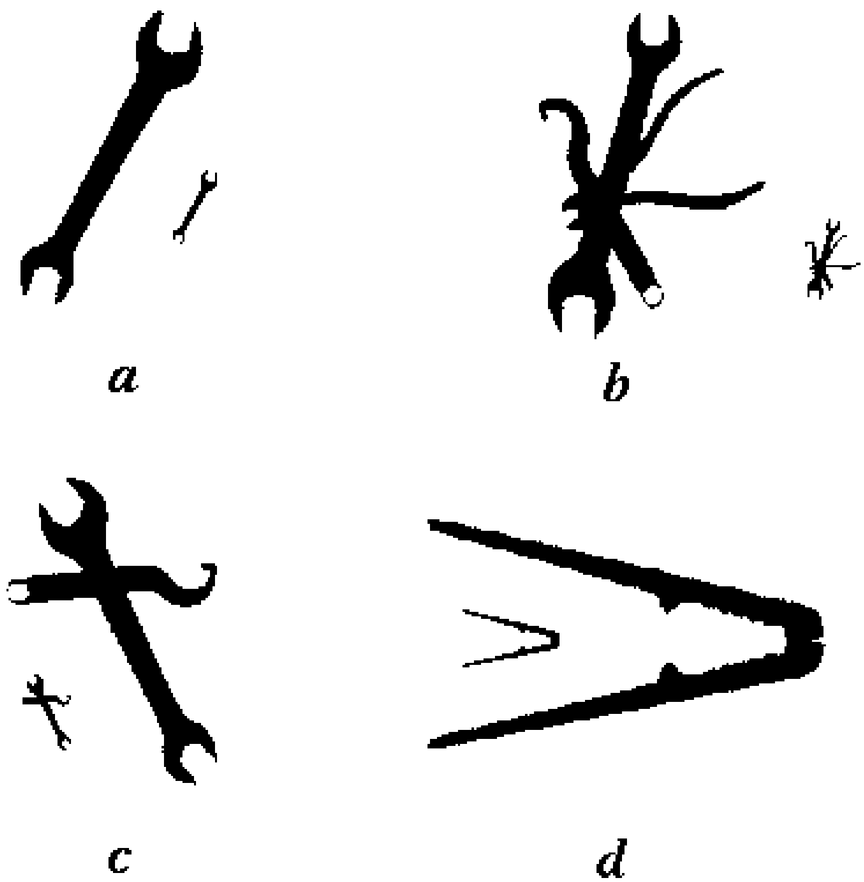


**Fig. 7.** Some images displaying a wrench, a partially occluded wrench and a nutcracker.

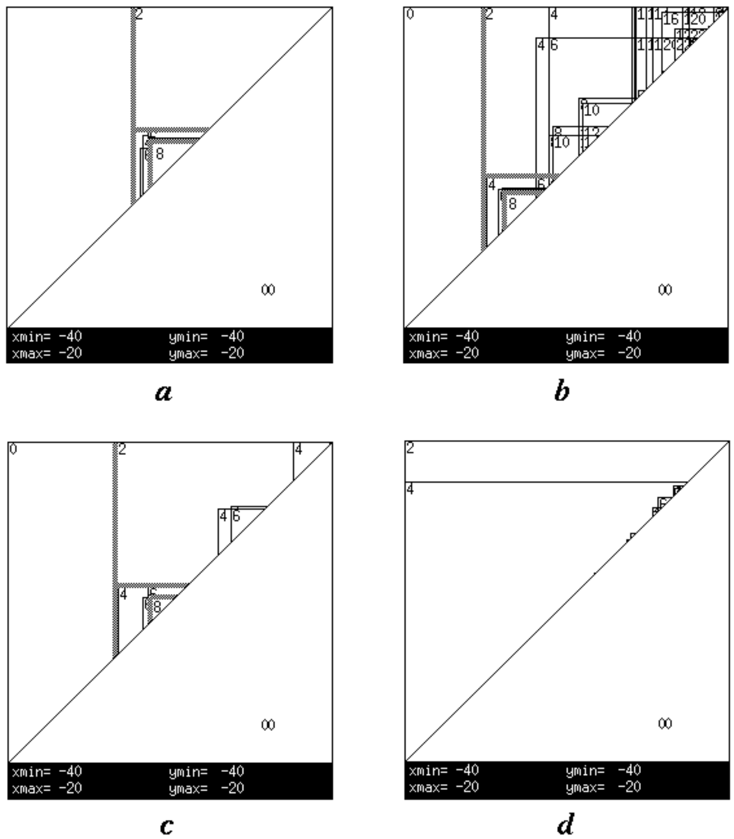


**Fig. 8.** Size functions corresponding to the images shown in Fig. 7 (see text).

### 3.7 Computation of Size Functions

In this section we shall give a technique to compute the function $\ell_{(\mathcal{M},\varphi)}(x,y)$ (a different method can be found in [8]). From now on we shall assume that $\mathcal{M}$ is a compact and locally arcwise connected subset of $\mathbb{R}^m$ and that the measuring function $\varphi$ is the restriction to $\mathcal{M}$ of a continuous function $\bar{\varphi}:\mathbb{R}^m \to \mathbb{R}$. In the following we shall denote by $\omega(\delta)$ the modulus of continuity of the measuring function $\bar{\varphi}$ (that is $\omega(\delta) = \sup\{|\bar{\varphi}(P) - \bar{\varphi}(Q)| : P, Q \in \mathbb{R}^m, \|P-Q\| < \delta\}$ for every $\delta > 0$). Our purpose is that of "approximating" the considered set $\mathcal{M}$ and the corresponding function $\ell_{(\mathcal{M},\varphi)}(x,y)$ with a finite set $\mathcal{P}$ and a function $\ell_{approx}$ respectively. The function $\ell_{approx}$ will be related to the function $\ell_{(\mathcal{M},\varphi)}(x,y)$ but will be much simpler to compute. From now on $\delta$ will be a positive real number.

**Definition 7.** Let $\mathcal{P} = \{P_0, P_1, \ldots, P_h\}$ be a finite set of points of $\mathbb{R}^m$ and let us denote by $\mathcal{B}_\delta$ the set of the $h+1$ open balls $B(P_i,\delta)$ of radius $\delta$ with centre at the points of $\mathcal{P}$. Let us assume that $\mathcal{B}_\delta$ verifies the following properties:

1. $\mathcal{M}$ is contained in $\cup_{i=0}^{h} B(P_i,\delta)$
2. for every index $i$, $0 \le i \le h$, $B(P_i,\delta) \cap \mathcal{M}$ is a non-empty arcwise connected set.

We shall call $\mathcal{B}_\delta$ a *$\delta$-covering* of $\mathcal{M}$. The set $\mathcal{P}$ will be called *the set of the centres* of $\mathcal{B}_\delta$ . We shall denote by $\sim$ the following relation on the set $\mathcal{P}$: $P_i \sim P_j$ if $(B(P_i,\delta) \cup B(P_j,\delta)) \cap \mathcal{M}$ is an arcwise connected set.

*Remark 4.* Obviously $\sim$ is a reflexive and symmetric relation.

In the rest of this section we shall assume that a $\delta$-covering $\mathcal{B}_\delta$ of $\mathcal{M}$ is given and denote by $\mathcal{P}$ the set $\{P_0, P_1, \ldots, P_h\}$ of the centres of $\mathcal{B}_\delta$.

**Definition 8.** For every $x$,$y \in \mathbb{R}$ let us denote by $\mathcal{P}\langle\bar{\varphi} \leq x\rangle$ the set of the elements of $\mathcal{P}$ at which the function $\bar{\varphi}$ takes a value not greater than $x$ and by $\approx_{\bar{\varphi}\leq y}$ the equivalence relation on $\mathcal{P}$ defined as follows: if $P_a, P_b \in \mathcal{P}$ we write $P_a \approx_{\bar{\varphi}\leq y} P_b$ if either $P_a = P_b$ or there exists a finite sequence $\left(P_{s(i)}\right)_{i=0,\ldots,r}$ of points in $\mathcal{P}\langle\bar{\varphi} \leq y\rangle$ such that $P_{s(0)} = P_a$, $P_{s(r)} = P_b$ and for every index $i$ with $0 \leq i \leq r-1$ we have $P_{s(i)} \sim P_{s(i+1)}$. This equivalence relation will be called $(\bar{\varphi} \leq y)$-*equivalence with respect to* $\mathcal{B}_\delta$. We shall denote by $\ell_{approx}(x, y)$ the number of equivalence classes in which $\mathcal{P}\langle\bar{\varphi} \leq x\rangle$ is divided by $(\bar{\varphi} \leq y)$-equivalence.

Now we can give two results that allow us to compute size functions easily.

**Theorem 5.** *For every $x, y \in \mathbb{R}$ and every $\bar{\omega} \geq \omega(\delta)$ with $x + \bar{\omega} \leq y - \bar{\omega}$ we have $\ell_{approx}(x - \bar{\omega}, y + \bar{\omega}) \leq \ell_{(\mathcal{M},\varphi)}(x, y) \leq \ell_{approx}(x + \bar{\omega}, y - \bar{\omega})$ and $\ell_{(\mathcal{M},\varphi)}(x - \bar{\omega}, y + \bar{\omega}) \leq \ell_{approx}(x, y) \leq \ell_{(\mathcal{M},\varphi)}(x + \bar{\omega}, y - \bar{\omega})$.*

**Corollary 1.** *Let us assume that $\bar{x}, \bar{y}, b, c \in \mathbb{R}$ with $b, c \geq 0$ and $\bar{\omega} \geq \omega(\delta)$ with $\bar{x} + \bar{\omega} \leq \bar{y} - \bar{\omega}$. If the function $\ell_{approx}$ takes the same value $v$ at the points $(\bar{x} + \bar{\omega}, \bar{y} - \bar{\omega})$ and $(\bar{x} - b - \bar{\omega}, \bar{y} + c +$*

$\bar{\omega}$) *then we have* $\ell_{(\mathcal{M},\varphi)}(x,y) = v$ *for every* $(x,y)$ *in the rectangle* $\left\{(x,y) \in \mathbb{R}^2 : \bar{x} - b \leq x \leq \bar{x}, \bar{y} \leq y \leq \bar{y} + c\right\}$.

*Remark 5.* Corollary 1 leads naturally to a procedure for computing $\ell_{(\mathcal{M},\varphi)}$. We can arbitrarily choose a real number $\bar{\omega} \geq \omega(\delta)$ and compute the function $\ell_{approx}$ in the set $S_{\bar{\omega}} = \left\{(x,y) \in \mathbb{R}^2 : x = i\bar{\omega},\right.$ $\left. y = j\bar{\omega}, i, j \in \mathbb{Z}, j \geq i\right\}$. Every time that we find the same value $v$ at two points $(i\bar{\omega}, j\bar{\omega})$ and $((i-p)\bar{\omega}, (j+q)\bar{\omega})$ of $S_{\bar{\omega}}$ with $p, q \geq 2$, we can say that in the closed rectangle defined by the vertices $((i-1)\bar{\omega}, (j+1)\bar{\omega})$ and $((i-p+1)\bar{\omega}, (j+q-1)\bar{\omega})$ the value of $\ell_{(\mathcal{M},\varphi)}$ is $v$. Obviously, if the shape of $\mathcal{M}$ is complicated and the size function $\ell_{(\mathcal{M},\varphi)}$ has many discontinuity points in a small region, in order to usefully apply our procedure we shall have to choose a $\delta$-covering constructed by using a very small $\delta$ and a value $\bar{\omega}$ not far away from $\omega(\delta)$.

It is interesting to point out that the approximation of $\mathcal{M}$ has some influence on knowledge of the size function $\ell_{(\mathcal{M},\varphi)}$ (recall that every size function takes a discrete value at each point of the real plane). What happens is that our procedure yields regions in which $\ell_{(\mathcal{M},\varphi)}$ is perfectly computed (i.e. without errors) and regions in which we have only lower and upper bounds for the size function. In other words, approximation produces uncertainty as to the exact location of the discontinuities of $\ell_{(\mathcal{M},\varphi)}$. Such an uncertainty increases with the parameter $\delta$ and, when $\delta$ becomes too large (i.e. the approximation of $\mathcal{M}$ is bad), uncertainty becomes infinite and we cannot learn anything about the size function.

Methods for making the computation of size functions faster can be found in [17].

### 3.8 Algebraic Representation of Size Functions

We have already pointed out in the sections on noise and occlusion resistance, as well as in the section on comparing size functions, that discontinuities of size functions play a considerable role in the recognition process. Indeed, it turns out that discontinuity points convey almost all the considerable information about the object under study contained in a size function. The key idea is then that of capturing the information concerning discontinuities of size functions in a more portable object. This can be achieved because it can be proven (cf. [14]) that if $\mathcal{M}$ is a finite union of compact and locally arcwise connected subsets of some Euclidean space, then the discontinuity points of a size function belonging to the region $\mathcal{S}_0 = \{(x, y) \in \mathbb{R}^2 : x < y\}$ divide the part of the domain of a size function which lies above the diagonal $\{(x, y) \in \mathbb{R}^2 : x = y\}$ into overlapping triangular regions with a side on the diagonal. Some triangles may have infinite area. As an example, it can be seen that in the size function of Fig. 3 there are three overlapping triangles. One is of infinite area with an unbounded side on the diagonal and another unbounded side on the line $x = a$. Another triangle is of finite area and has one side on the diagonal and the opposite vertex at the point $(a, d)$, and the last one is again of finite area and has one side on the diagonal and opposite vertex at the point $(b, c)$.

In general, we shall identify each triangle of finite area by a point and each triangle of infinite area by a vertical line. More precisely, we call *cornerpoint* any point $p = (x, y)$, with $x < y$, satisfying the following property: if we denote by $\mu_{\alpha,\beta}$ the number

$$\ell_{(\mathcal{M},\varphi)}(x+\alpha, y-\beta) - \ell_{(\mathcal{M},\varphi)}(x+\alpha, y+\beta) -$$
$$\ell_{(\mathcal{M},\varphi)}(x-\alpha, y-\beta) + \ell_{(\mathcal{M},\varphi)}(x-\alpha, y+\beta),$$

then $\mu(p) \stackrel{\text{def}}{=} \min\{\mu_{\alpha,\beta} : \alpha > 0, \beta > 0, x + \alpha < y - \beta\} > 0$ must hold. We shall call $\mu(p)$ the *multiplicity of* $p$.

In Fig. 3 the only cornerpoints for the size function are the points with coordinates $(a, d)$ and $(b, c)$ and both have multiplicity equal to 1. These points can be used to identify the triangles of finite area in which discontinuities of the size function divide the region above the diagonal.

Analogously, we call *cornerline* any vertical line $r : x = k$ $(k \in \mathbb{R})$ for which we have

$$\mu(r) \stackrel{\text{def}}{=} \min_{\alpha>0, k+\alpha<y} \ell_{(\mathcal{M},\varphi)}(k+\alpha, y) - \ell_{(\mathcal{M},\varphi)}(k-\alpha, y) > 0.$$

We shall call $\mu(r)$ the *multiplicity of* $r$.

In Fig. 3 the only cornerline for the size function is the line with equation $x = a$ and it has multiplicity equal to 1. This line can be used to identify the unbounded triangle above the diagonal. Let us point out that it is easy to exhibit examples of size functions with cornerpoints and cornerlines with multiplicities greater than 1.

The importance of cornerpoints and cornerlines of size functions lies in the fact that the value of a size function at almost every point above the diagonal is equal to the sum of the multiplicities of cornerpoints and cornerlines identifying triangles containing that point. Therefore, a size function can be represented by the collection of all its cornerlines and cornerpoints together with their multiplicities. In other words, a size function can be given by a formal series of points and lines of the plane.

This representation of size functions makes it possible to reduce the complexity necessary to work with size functions. Moreover, it allows all problems concerning size functions to be translated into problems regarding formal series, i.e. into algebraic problems (cf., e.g., [15]).

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