On the use of the extended Pareto grid in 2-parameter persistent homology

Patrizio Frosini

Department of Mathematics, University of Bologna
patrizio.frosini@unibo.it

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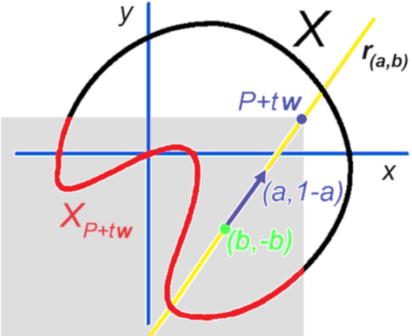
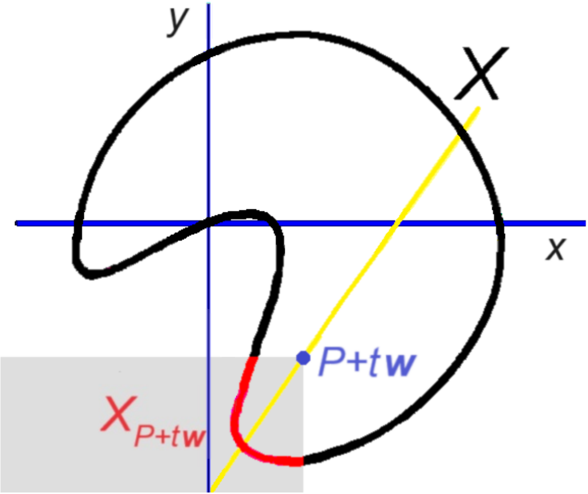
Bifiltrations can be seen as families of monofiltrations

Let us take a continuous function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) : X \to \mathbb{R}^2$ and consider the bifiltration $X_{(u_1,u_2)} = \{p \in X : \varphi_1(p) \leq u_1, \varphi_2(p) \leq u_2\}$ with (u_1, u_2) varying in $\mathbb{R}^2$. This bifiltration is equivalent to the family of monofiltrations that we get by assuming that the point (u_1, u_2) varies on a positive slope line $r_{(a,b)} = \{P + t\mathbf{w} : t \in \mathbb{R}\}$, where $P = (b, -b)$ and $\mathbf{w} = (a, 1-a)$, for $a \in]0,1[$ and $b \in \mathbb{R}$.



Bifiltrations can be seen as families of monofiltrations



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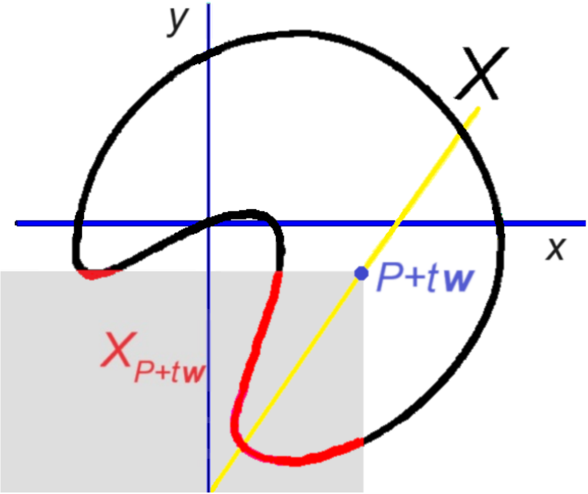
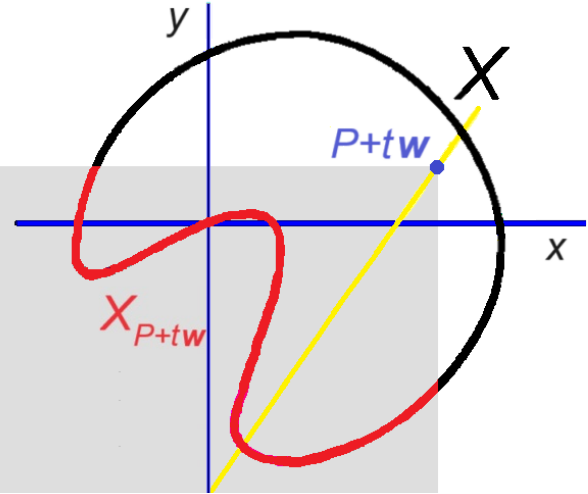


Bifiltrations can be seen as families of monofiltrations



The normalized function $\boldsymbol{\varphi}^*_{(a,b)}$

If, for any $(a,b) \in]0,1[\times\mathbb{R}$, we define on X the function

$$\boldsymbol{\varphi}_{(a,b)}(p) := \max\left\{\frac{\varphi_1(p)-b}{a}, \frac{\varphi_2(p)+b}{1-a}\right\}$$

we can express the set

$$X_{P+t\mathbf{w}} = \{p \in X : \varphi_1(p) \le at+b, \varphi_2(p) \le (1-a)t-b\}$$

as the set

$$\{p \in X : \boldsymbol{\varphi}_{(a,b)}(p) \le t\}.$$

As a consequence, the monofiltration $\{X_{P+t\mathbf{w}}\}_{t\in\mathbb{R}}$ of X is associated with the persistence diagram $\mathrm{Dgm}(\boldsymbol{\varphi}_{(a,b)})$ of the function $\boldsymbol{\varphi}_{(a,b)}$.

To get a stability theorem we have to normalize $\boldsymbol{\varphi}_{(a,b)}$ by setting

$$\boldsymbol{\varphi}^*_{(a,b)}(p) := \min\{a, 1-a\} \cdot \boldsymbol{\varphi}_{(a,b)}(p).$$

The matching distance $\mathscr{D}_{\mathrm{match}}$ and its stability

If $\boldsymbol{\varphi}, \boldsymbol{\psi}$ are two continuous functions from X to $\mathbb{R}^2$, we can define the matching distance $\mathscr{D}_{\mathrm{match}}(\boldsymbol{\varphi}, \boldsymbol{\psi})$ by setting

$$\mathscr{D}_{\mathrm{match}}(\boldsymbol{\varphi}, \boldsymbol{\psi}) := \sup_{(a,b) \in]0,1[\times \mathbb{R}} d_{\mathrm{B}}\left(\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)}), \mathrm{Dgm}(\boldsymbol{\psi}^*_{(a,b)})\right)$$

where d_{B} is the usual bottleneck distance.

We recall that the matching distance $\mathscr{D}_{\mathrm{match}}$ is stable:

Theorem (Stability Theorem)

$\mathscr{D}_{\mathrm{match}}(\boldsymbol{\varphi}, \boldsymbol{\psi}) \leq \|\boldsymbol{\varphi} - \boldsymbol{\psi}\|_\infty$.

Computation of the matching distance

For any arbitrary precision, the matching distance can be approximated in polynomial time:

- *S. Biasotti, A. Cerri, P. Frosini, D. Giorgi, A new algorithm for computing the 2-dimensional matching distance between size functions, Pattern Recognition Letters, vol. 32 (2011), n. 14, 1735-1746.*
- *A. Cerri, P. Frosini, A new approximation algorithm for the matching distance in multidimensional persistence, Journal of Computational Mathematics, vol. 38 (2020), n. 2, 291-309.*

From bifiltrations to monofiltrations

Extended Pareto grid

Position theorem

Studying the matching distance via the extended Pareto grid

Some technical assumptions

To define the extended Pareto grid, we need some technical assumptions.

First, we assume that the topological space X is a closed smooth manifold M of dimension $r \geq 2$.
Then, we assume that $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ is a smooth map from M to the real plane $\mathbb{R}^2$. We choose a Riemannian metric on M so that we can define gradients for φ_1 and φ_2.

The Jacobi set $\mathbb{J}(\boldsymbol{\varphi})$ is the set of all points $p \in M$ at which the gradients of φ_1 and φ_2 are linearly dependent.

If $p \in \mathbb{J}(\boldsymbol{\varphi})$ and $\nabla\varphi_1(p) \cdot \nabla\varphi_2(p) \leq 0$, we say that the point p is a critical Pareto point for $\boldsymbol{\varphi}$. The set of all critical Pareto points of $\boldsymbol{\varphi}$ is denoted by $\mathbb{J}_P(\boldsymbol{\varphi})$.

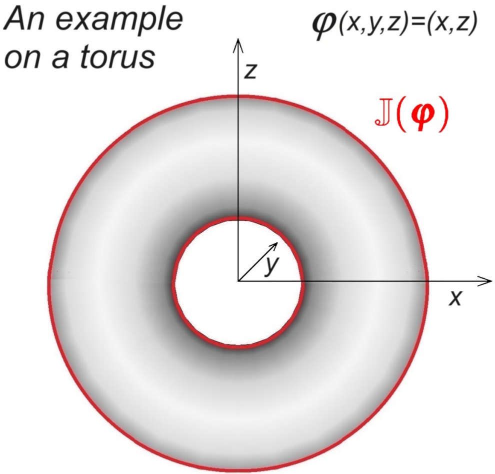
Some technical assumptions

If we assume that $\boldsymbol{\varphi} : M \to \mathbb{R}^2$ is regular enough in a suitable sense (here we skip the technical details), then the Jacobi set is a smooth 1-submanifold of M, consisting of finitely many components, each one diffeomorphic to a circle.

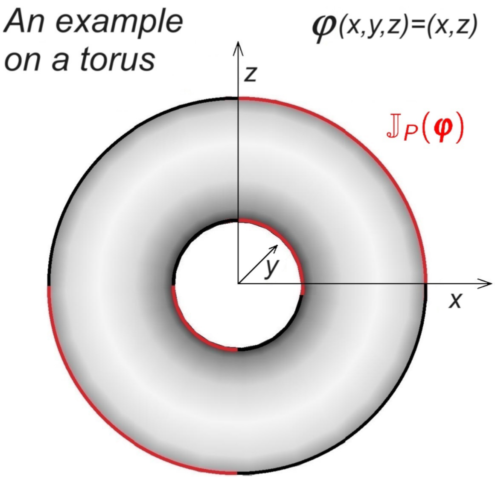
Furthermore, the set of critical Pareto points at which the gradients of φ_1 and φ_2 are not orthogonal to the Jacobi set is made of a finite family $\{\alpha_i\}$ of arcs. Along these arcs, one of φ_1 and φ_2 is strictly increasing and the other is strictly decreasing. Each arc α_i can meet critical points for φ_1, φ_2 only at its endpoints.

For more details: *[Y.H. Wan, Morse theory for two functions, Topology 14 (1975), no. 3, 217-228.]*

The Jacobi set



The set of critical Pareto points



The extended Pareto grid $\Gamma(\boldsymbol{\varphi})$

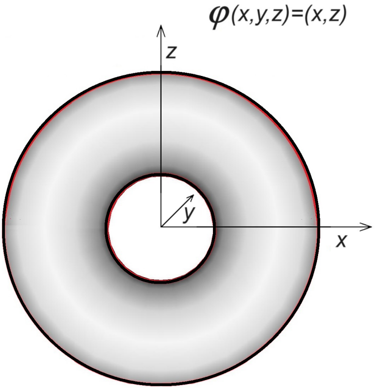
Our purpose is to establish a formal link between the position of points of $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$ for a function $\boldsymbol{\varphi}$ and the intersections of the positive slope line $r_{(a,b)}$ with a particular subset of the plane $\mathbb{R}^2$, called the extended Pareto grid of $\boldsymbol{\varphi}$.

The extended Pareto grid $\Gamma(\boldsymbol{\varphi})$ of $\boldsymbol{\varphi}$ is the union of the image by $\boldsymbol{\varphi}$ of the set $\mathbb{J}_P(\boldsymbol{\varphi})$ of all critical Pareto points with

1. The vertical upward half-lines starting from $\boldsymbol{\varphi}(p_i)$, where $p_1, \ldots, p_h$ are the critical points of φ_1;
2. The horizontal rightward half-lines starting from $\boldsymbol{\varphi}(q_j)$, where $q_1, \ldots, q_k$ are the critical points of φ_2.

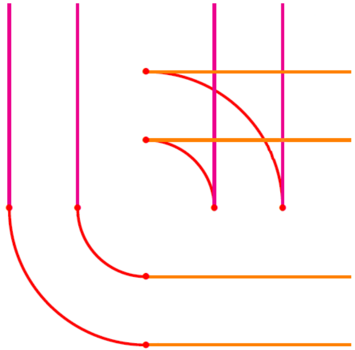
We assume that $\{p_1, \ldots, p_h\} \cap \{q_1, \ldots, q_k\} = \emptyset$.

The extended Pareto grid: An example



The torus endowed with the filtering function $\boldsymbol{\varphi}(p) := (x(p), z(p))$.

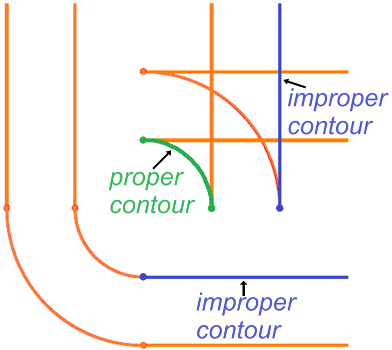
The extended Pareto grid: An example



The extended Pareto grid for the torus endowed with the filtering function $\boldsymbol{\varphi}(p) := (x(p), z(p))$.

Contours

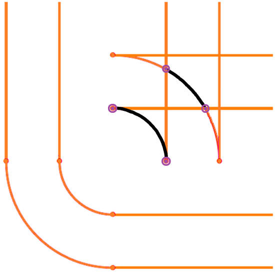
The closures of the images of the previously cited arcs α_i will be called proper contours of $\boldsymbol{\varphi}$, while the half-lines will be called improper contours of $\boldsymbol{\varphi}$. We observe that every contour is a closed set.



Contour-arcs

We can endow the points of $\Gamma(\boldsymbol{\varphi})$ with a suitable concept of multiplicity.

Let $\mathscr{D}(\boldsymbol{\varphi})$ be the set of double points in $\Gamma(\boldsymbol{\varphi})$. Each connected component of $\Gamma(\boldsymbol{\varphi}) \setminus \mathscr{D}(\boldsymbol{\varphi})$ is called a contour-arc of $\boldsymbol{\varphi}$.



Two contour-arcs are displayed in black.

The Position Theorem

With the concept of extended Pareto grid at hand, we can state and prove the following result, which gives a necessary condition for D to be a point of $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$.
We recall that

$$\boldsymbol{\varphi}^*_{(a,b)} := \max\left\{\frac{\min\{a,1-a\}}{a}\cdot(\varphi_1-b), \frac{\min\{a,1-a\}}{1-a}\cdot(\varphi_2+b)\right\}.$$

We set $\Delta := \{(u,v)\in\mathbb{R}^2 : u=v\}$.

Theorem (Position Theorem)

Let $(a,b)\in]0,1[\times\mathbb{R}$, $D\in\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})\setminus\Delta$. Then, for each finite coordinate c of D a point $(x,y)\in r_{(a,b)}\cap\Gamma(\boldsymbol{\varphi})$ exists, such that $c=\frac{\min\{a,1-a\}}{a}\cdot(x-b)=\frac{\min\{a,1-a\}}{1-a}\cdot(y+b)$.

Using the extended Pareto grid

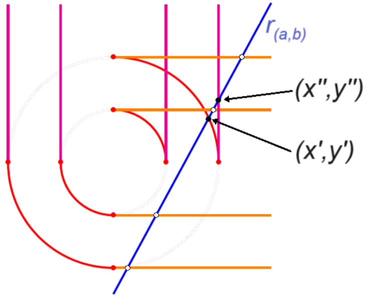
The Position Theorem suggests a way to find the possible positions for points of $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$. It consists in drawing the extended Pareto grid $\Gamma(\boldsymbol{\varphi})$ and considering its intersections $(x_1, y_1), \ldots, (x_\ell, y_\ell)$ with the positive slope line $r_{(a,b)}$. If $(u, v) \in \mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$ and $u < v$, then

$$u, v \in \left\{ \frac{\min\{a, 1-a\}}{a} \cdot (x_i - b) = \frac{\min\{a, 1-a\}}{1-a} \cdot (y_i + b) \right\}_{1 \le i \le \ell} \cup \{\infty\}.$$

In other words, the Position Theorem allows us to follow the movements of the points in the persistence diagram $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$, varying (a, b), by following the intersection points of the line $r_{(a,b)}$ with the extended Pareto grid of $\boldsymbol{\varphi}$.

An example

Let us consider the case $a < \frac{1}{2}$ (i.e., slope of $r_{(a,b)}$ greater than 1).

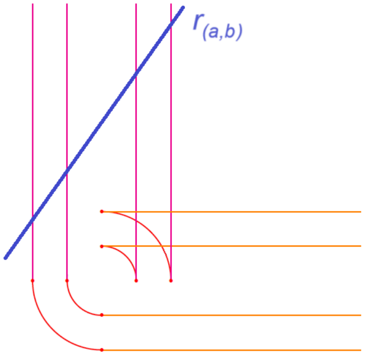


If $(u, v) \in \mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$ and $u < v < \infty$, then there exist $(x', y'), (x'', y'') \in r_{(a,b)} \cap \Gamma(\boldsymbol{\varphi})$, such that $u = x' - b$ and $v = x'' - b$.

Using the extended Pareto grid

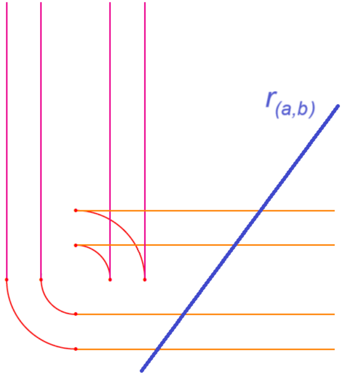
Note that when $b < 0$ and $|b|$ is sufficiently large, the positive slope line $r_{(a,b)}$ may intersect $\Gamma(\boldsymbol{\varphi})$ only at the vertical half-lines.

In this case, $\boldsymbol{\varphi}^*_{(a,b)} := \frac{\min\{a,1-a\}}{a} \cdot (\varphi_1 - b)$, and the values $x_1, \ldots, x_\ell$ are the critical values of φ_1.



Using the extended Pareto grid

Similarly, when $b > 0$ and $|b|$ is large enough, $r_{(a,b)}$ intersects $\Gamma(\boldsymbol{\varphi})$ only at the horizontal half-lines. Then $\boldsymbol{\varphi}^*_{(a,b)} := \frac{\min\{a,1-a\}}{1-a} \cdot (\varphi_2 + b)$, and the values $y_1, \ldots, y_\ell$ are the critical values of φ_2.

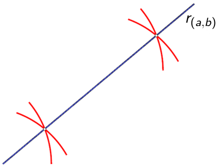


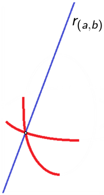
Using the extended Pareto grid

The Position Theorem also allows us to find the pairs (a,b) for which $\mathrm{Dgm}\left(\boldsymbol{\varphi}^*_{(a,b)}\right)$ can contain a proper multiple point.

Proposition

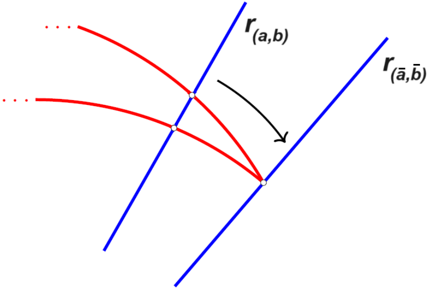
If $\mathrm{Dgm}\left(\boldsymbol{\varphi}^*_{(a,b)}\right)$ *contains a proper multiple point, then* $r_{(a,b)}$ *must contain two points of* $\mathscr{D}(\boldsymbol{\varphi})$*. If* $\mathrm{Dgm}\left(\boldsymbol{\varphi}^*_{(a,b)}\right)$ *contains an improper multiple point, then* $r_{(a,b)}$ *must contain at least one point of* $\mathscr{D}(\boldsymbol{\varphi})$*.*





Destruction of points in $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$

The Position Theorem allows us to find the pairs (a, b) where points of $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$ can disappear.



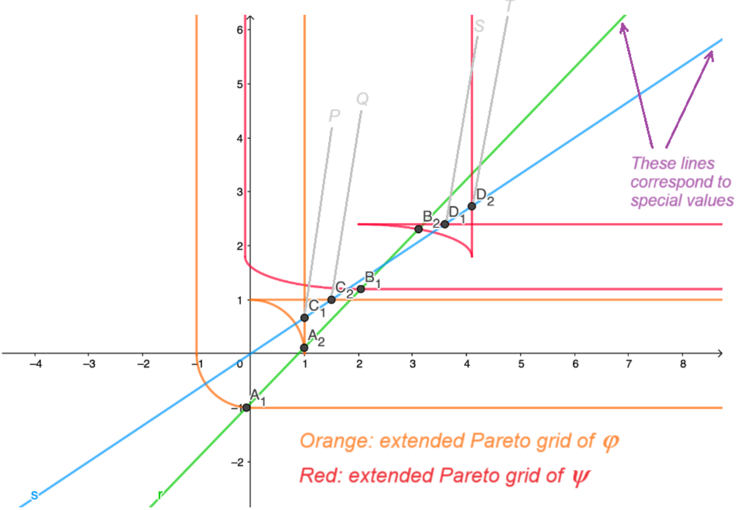
In this configuration, a point of $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$ reaches the diagonal Δ and disappears.

Special set of a pair of functions

Definition

Let $\mathrm{Ctr}(\boldsymbol{\varphi}, \boldsymbol{\psi})$ be the set of all curves that are contours of $\boldsymbol{\varphi}$ or $\boldsymbol{\psi}$. Set $\overline{C} = \max\{\|\boldsymbol{\varphi}\|_\infty, \|\boldsymbol{\psi}\|_\infty\}$. The *special set* of $(\boldsymbol{\varphi}, \boldsymbol{\psi})$, denoted by $\mathrm{Sp}(\boldsymbol{\varphi}, \boldsymbol{\psi})$, is the collection of all (a, b) in $]0, 1[\times[-\overline{C}, \overline{C}]$ for which two distinct pairs $\{\alpha_p, \alpha_q\}$, $\{\alpha_s, \alpha_t\}$ of contours in $\mathrm{Ctr}(\boldsymbol{\varphi}, \boldsymbol{\psi})$ intersecting $r_{(a,b)}$ exist, such that $\boxed{\|P - Q\| = \|S - T\| \text{ or } \|P - Q\| = 2\|S - T\|}$, where $P = r_{(a,b)} \cap \alpha_p$, $Q = r_{(a,b)} \cap \alpha_q$, $S = r_{(a,b)} \cap \alpha_s$ and $T = r_{(a,b)} \cap \alpha_t$. An element of the special set $\mathrm{Sp}(\boldsymbol{\varphi}, \boldsymbol{\psi})$ is called a *special value* of the pair $(\boldsymbol{\varphi}, \boldsymbol{\psi})$.

Special set of a pair of functions



Special set of a pair of functions

The special values are the pairs $(a,b) \in]0,1[\times[-\overline{C},\overline{C}]$ at which the optimal matching between $\mathrm{Dgm}(\boldsymbol{\varphi}^*_{(a,b)})$ and $\mathrm{Dgm}(\boldsymbol{\psi}^*_{(a,b)})$ may abruptly change.

The following statement holds.

Theorem

A point $(\bar{a},\bar{b}) \in]0,1[\times\mathbb{R}$ *exists, such that*

$$D_{\mathrm{match}}(\boldsymbol{\varphi},\boldsymbol{\psi}) = d_B\left(\mathrm{Dgm}\left(\boldsymbol{\varphi}^*_{(\bar{a},\bar{b})}\right),\mathrm{Dgm}\left(\boldsymbol{\psi}^*_{(\bar{a},\bar{b})}\right)\right).$$

and

$$\bar{a} = \frac{1}{2} \text{ or } (\bar{a},\bar{b}) \in \mathrm{Sp}(\boldsymbol{\varphi},\boldsymbol{\psi}).$$

References

The previous result is based on these papers:

- *A. Cerri, M. Ethier, P. Frosini, On the geometrical properties of the coherent matching distance in 2D persistent homology, Journal of Applied and Computational Topology, vol. 3 (2019), n. 4, 381–422.*
- *M. Ethier, P. Frosini, N. Quercioli, F. Tombari, Geometry of the matching distance for 2D filtering functions, Journal of Applied and Computational Topology, vol. 7 (2023), 815–830.*
- *P. Frosini, E. Mósig García, N. Quercioli, F. Tombari, Matching distance via the extended Pareto grid, https://arxiv.org/pdf/2312.04201 (2024).*

Some collaborators in the research on using extended Pareto grids in biparameter persistent homology

Andrea Cerri
FST srl - FOM Software Technology

Nicola Quercioli
DEI-UniBo and WiLab-CNIT

Francesca Tombari
Max Planck Institute for Mathematics in the Sciences

Sara Scaramuccia
University of Rome Tor Vergata

Marc Ethier
Université du Québec en Abitibi-Témiscamingue

Eloy Mósig García
University of Pisa

Ulderico Fugacci
CNR - IMATI

Thanks for your attention!

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