Filtrations induced by continuous functions

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Outline

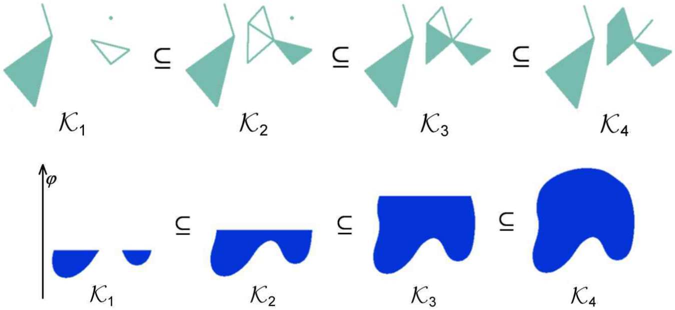
1. **Filtrations in Persistent Homology**
2. **The 1-dimensional case (filtrations depending on 1 parameter)**
3. **The n-dimensional case (filtrations depending on n parameters)**

Filtrations in Persistent Homology

Persistent Homology is based on the concept of filtration.
Filtrations are usually given

- by introducing a collection of sets that is ordered w.r.t. inclusion;
- by considering the sub-level sets of a continuous function from a topological space X to $\mathbb{R}^n$.

A natural question: Are these approaches equivalent?



A negative answer?

In general, the previously cited methods are NOT equivalent.

Example (A)

Let us consider this filtration:

$$\textit{Index set} = I = [0,1]$$
$$\textit{Filtered space} = K = [0,2]$$

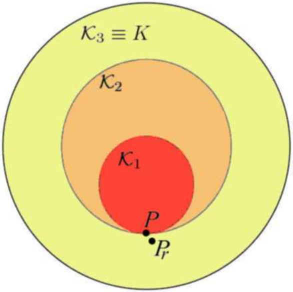
$$\mathcal{K}_i = \begin{cases} [0, i+1] & \textit{if } \ i > 0 \\ \{0\} & \textit{if } \ i = 0 \end{cases}$$

When the index i tends to 0, the compact sets $\mathcal{K}_i$ do not tend to $\mathcal{K}_0$.

This filtration cannot be induced by a continuous function $\varphi : K \to \mathbb{R}$. Indeed, if such a function existed, we would have $\varphi(P) \leq \epsilon$ for every $\epsilon > 0$ and every $P \in [0,1]$ (since $[0,1] \subseteq \mathcal{K}_\epsilon$). Therefore, φ would take a non-positive value at each point in $[0,1]$, against the equality $\mathcal{K}_0 = \{0\}$.

A negative answer?

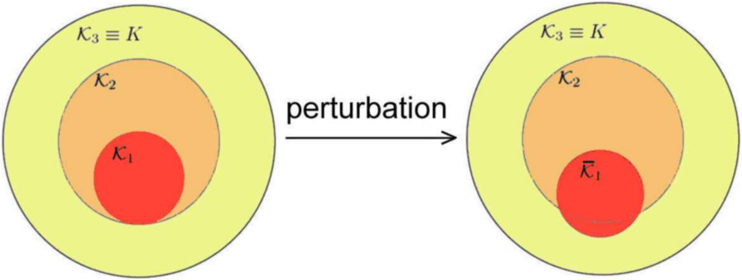
Example (B)



This filtration of the disk K cannot be induced by a continuous function $\varphi : K \to \mathbb{R}$ (if such a function existed, we should have $\varphi(P) \leq 1$ and $\varphi(P) = \lim_r \varphi(P_r) \geq 2$ at the same time, for every sequence (P_r) of points of $\mathcal{K}_3 \setminus \mathcal{K}_2$ converging to P).

Instability

However, we observe that the filtrations described in the previous examples are not stable. As concerns example B, an arbitrarily small perturbation can destroy its property of being a filtration:



Stability

Stability is important in applications. Without stability, approximated computations are useless.
What can we get if we assume stability of our data?
In this talk we shall present the following results:

Theorem

Every compact and stable 1*-dimensional filtration* $\{\mathcal{K}_i\}_{i\in I}$ *of a compact metric space* K *is induced by a continuous function* $\varphi : K \to \mathbb{R}$*.*

Theorem

Every compact, stable and complete n-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ *of a compact metric space* K *is induced by a continuous function* $\vec{\varphi} : K \to \mathbb{R}^n$*.*

We start from some definitions

Let (K, d) be a non-empty compact metric space.

Let us denote by $Comp(K)$ the set $\{\mathcal{K} : \mathcal{K}$ compact in $K\}$ and consider the Hausdorff distance d_H on $Comp(K) \setminus \{\emptyset\}$. Moreover, let I be a non-empty subset of $\mathbb{R}^n$ such that $I = I_1 \times I_2 \times \ldots \times I_n$.

The following relation $\preceq$ is defined in I: for $i = (i_1, \ldots, i_n), i' = (i'_1, \ldots, i'_n) \in I$, we say $i \preceq i'$ if and only if $i_r \leq\ i'_r$ for every $r = 1, \ldots, n$.

Definition

An n-dimensional filtration of K is a family $\{\mathcal{K}_i \in Comp(K)\}_{i \in I}$ such that, $\emptyset, K \in \{\mathcal{K}_i\}_{i \in I}$, and $\mathcal{K}_i \subseteq \mathcal{K}_{i'}$ for every $i, i' \in I$, with $i \preceq i'$.

(cf. G. Carlsson, A. Zomorodian, *The theory of multidimensional persistence*, SCG07)

We start from some definitions

Definition

An n-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ of K is induced by a function $\vec{\varphi} : K \to \mathbb{R}^n$ if $\mathcal{K}_i = \{P \in K : \vec{\varphi}(P) \preceq i\}$ for every $i \in I$.

Definition

We shall call compact or finite any filtration $\{\mathcal{K}_i\}_{i\in I}$ with $I = I_1 \times I_2 \times \ldots \times I_n$ a compact or finite subset of $\mathbb{R}^n$, respectively.

Stability in the 1-dimensional case

Definition

We shall say that a 1*-dimensional compact filtration* $\{\mathcal{K}_i\}_{i\in I}$ *of* K *is stable with respect to the metric d if the following statements hold:*

(*a*) *If* $(i_m \in I)_{m\in\mathbb{N}}$ *is a sequence converging to* $\bar{\imath} \in I$*, the sequence* $(\mathcal{K}_{i_m})$ *converges to* $\mathcal{K}_{\bar{\imath}}$*, and the sequence* $(\overline{\mathcal{K}_{i_m}^c})$ *converges to* $(\overline{\mathcal{K}_{\bar{\imath}}^c})$ *with respect to the Hausdorff distance* d_H *(in other words, we require that the maps* $i \mapsto \mathcal{K}_i$ *and* $i \mapsto \overline{\mathcal{K}_i^c}$ *are continuous).*

(*b*) *If* $i < j$ *then* $\mathcal{K}_i \subseteq int(\mathcal{K}_j)$*.*

Hence, a 1-dimensional filtration is stable if it resists small perturbations both of the indexes and of the compact sets in the filtration.

Remark

Let us note that in the case $\{\mathcal{K}_i\}_{i\in I}$ is a finite 1-dimensional filtration of K, previous definition reduces to property (b).

Our first result

Theorem

Every compact and stable 1*-dimensional filtration* $\{\mathcal{K}_i\}_{i \in I}$ *of a compact metric space* K *is induced by a continuous function* $\varphi : K \to \mathbb{R}$*.*

In other words, any compact 1-dimensional filtration that resists small perturbations of the indexes and of the compact sets is induced by a continuous function.

Sketch of the proof

Let $\{\mathcal{K}_i\}_{i\in I}$ be a 1-dimensional compact and stable filtration of K.
For every $P \in K$, we set

$$A(P) = \{i \in I, P \in \overline{\mathcal{K}_i^c}\} = \{i \in I, P \notin int(\mathcal{K}_i)\}$$

and

$$B(P) = \{i \in I, P \in \mathcal{K}_i\}.$$

Then we prove that $A(P)$ admits a maximum and $B(P)$ admits a minimum. Hence we can define $\alpha(P) = \max A(P)$ and $\beta(P) = \min B(P)$.

Sketch of the proof

Lemma

Let $\{\mathcal{K}_i\}_{i\in I}$ be a 1-dimensional compact and stable filtration of K. Then the following statements hold:

1. $\alpha(P) \leq \beta(P)$ *for every* $P \in K$.
2. *If* $P, Q \in K$ *and* $\alpha(P) < \alpha(Q)$, *then* $\beta(P) \leq \alpha(Q)$.
3. *If* $P, Q \in K$ *and* $\beta(P) < \beta(Q)$, *then* $\beta(P) \leq \alpha(Q)$.

Lemma

Let $\{\mathcal{K}_i\}_{i\in I}$ be a 1-dimensional compact and stable filtration of K. Then the following statements hold:

1. *The function* α *is everywhere upper semi-continuous.*
2. *The function* β *is everywhere lower semi-continuous.*

Sketch of the proof

By the previous lemmas we can prove the previously cited theorem:

Theorem

Every compact and stable 1*-dimensional filtration* $\{\mathcal{K}_i\}_{i \in I}$ *of a compact metric space* K *is induced by a continuous function* $\varphi : K \to \mathbb{R}$.

To prove this result, we use the function $\varphi(P) =$

$$\begin{cases} \beta(P) & \text{if } i_{\min} = \alpha(P) \\ \dfrac{\alpha(P) \cdot d_H\left(\{P\}, \overline{\mathcal{K}^c_{\beta(P)}}\right) + \beta(P) \cdot d_H\left(\{P\}, \mathcal{K}_{\alpha(P)}\right)}{d_H\left(\{P\}, \overline{\mathcal{K}^c_{\beta(P)}}\right) + d_H\left(\{P\}, \mathcal{K}_{\alpha(P)}\right)} & \text{if } \begin{array}{c} i_{\min} < \alpha(P) \\ \leq \beta(P) < i_{\max} \end{array} \\ \dfrac{\alpha(P) \cdot d_H(\{P\}, \{R\}) + \beta(P) \cdot d_H\left(\{P\}, \mathcal{K}_{\alpha(P)}\right)}{d_H(\{P\}, \{R\}) + d_H\left(\{P\}, \mathcal{K}_{\alpha(P)}\right)} & \text{if } \beta(P) = i_{\max} \end{cases}$$

Here R is a suitable fixed point in $\mathcal{K}_{i_{\max}}$.
We can prove that φ is continuous and induces the wanted filtration.

1. Filtrations in Persistent Homology

2. The 1-dimensional case (filtrations depending on 1 parameter)

3. **The n-dimensional case (filtrations depending on n parameters)**

Some definitions

Let $I = I_1 \times I_2 \times \ldots \times I_n$ be a compact subset of $\mathbb{R}^n$. For every j with $1 \le j \le n$ and every fixed $h \in I_j$, let us set

$$\mathcal{K}^j_h = \mathcal{K}_{(\max I_1, \ldots, \max I_{j-1}, h, \max I_{j+1}, \ldots, \max I_n)}$$

We observe that $\{\mathcal{K}^j_h\}_{h \in I_j}$ is a 1-dimensional filtration of K.

Stability in the n-dimensional case

Let d be the metric we consider on the compact metric space K.

Definition

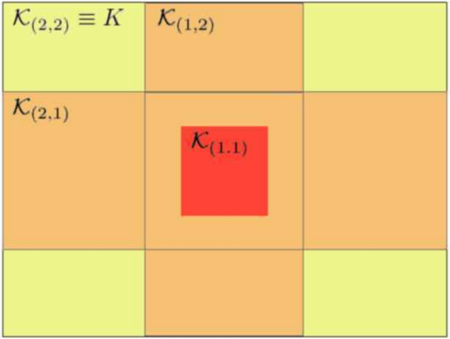
We shall say that a n-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ in K is stable with respect to d if the 1-dimensional filtrations $\{\mathcal{K}^1_{i_1}\}_{i_1\in I_1}$, $\{\mathcal{K}^2_{i_2}\}_{i_2\in I_2}$, …, $\{\mathcal{K}^n_{i_n}\}_{i_n\in I_n}$ are stable with respect to d.

Completeness

Definition

A n-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ *of* K *will be said to be* *complete* *if, for every* $i = (i_1, \ldots, i_n) \in I$, $\mathcal{K}_i = \mathcal{K}^1_{i_1} \cap \mathcal{K}^2_{i_2} \cap \ldots \cap \mathcal{K}^n_{i_n}$.

The following filtration is NOT complete:



Our result for the n-dimensional case

Theorem

Every compact, stable and complete n-dimensional filtration $\{\mathcal{K}_i\}_{i \in I}$ of a compact metric space K is induced by a continuous function $\vec{\varphi} : K \to \mathbb{R}^n$.

Proof of the theorem for the n-dimensional case

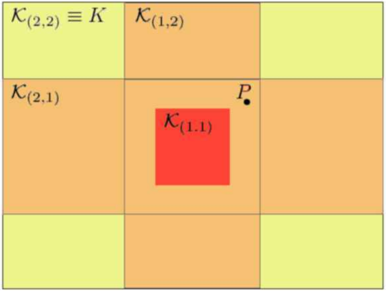
The completeness of $\{\mathcal{K}_i\}_{i\in I}$ implies that, for every $i=(i_1,i_2,\ldots,i_n)\in I$, $\mathcal{K}_i$ is equal to $\mathcal{K}^1_{i_1}\cap\mathcal{K}^2_{i_2}\cap\ldots\cap\mathcal{K}^n_{i_n}$. Moreover, the stability of $\{\mathcal{K}_i\}_{i\in I}$ implies the stability of the 1-dimensional filtrations $\{\mathcal{K}^1_{i_1}\}_{i_1\in I_1}$, $\{\mathcal{K}^2_{i_2}\}_{i_2\in I_2}$, …, $\{\mathcal{K}^n_{i_n}\}_{i_n\in I_n}$. Then, by the theorem for the 1-dimensional case, for every $\{\mathcal{K}^j_{i_j}\}_{i_j\in I_j}$, $j=1,\ldots,n$, there exists a continuous function $\varphi_j:K\to\mathbb{R}$ such that $\mathcal{K}^j_{i_j}=\{P\in K:\varphi_j(P)\le i_j\}$ for every $i_j\in I_j$. Hence,

$$
\begin{aligned}
&\mathcal{K}_{(i_1,i_2,\ldots,i_n)}=\mathcal{K}^1_{i_1}\cap\mathcal{K}^2_{i_2}\cap\ldots\cap\mathcal{K}^n_{i_n}\\
&=\{P\in K:\varphi_1(P)\le i_1\}\cap\ldots\cap\{P\in K:\varphi_n(P)\le i_n\}\\
&=\{P\in K:\vec{\varphi}(P)=(\varphi_1,\varphi_2,\ldots,\varphi_n)(P)\preceq(i_1,i_2,\ldots,i_n)\}.
\end{aligned}
$$

Therefore, the function $\vec{\varphi}:K\to\mathbb{R}^n$ induces $\{\mathcal{K}_i\}_{i\in I}$. Moreover, $\vec{\varphi}$ is continuous since its components $\varphi_1,\varphi_2,\ldots,\varphi_n:K\to\mathbb{R}$ are continuous.

The assumption of completeness is important

We cannot drop the assumption of completeness.



There is no continuous function $\vec{\varphi} : K \to \mathbb{R}^2$, inducing the previous filtration. Indeed, since $P \in \mathcal{K}_{(1,2)} \cap \mathcal{K}_{(2,1)}$, we should have that $\vec{\varphi}(P) \preceq (1,2)$ and $\vec{\varphi}(P) \preceq (2,1)$. Therefore $\vec{\varphi}(P) \preceq (1,1)$. This would contradict the fact that $P \notin \mathcal{K}_{(1,1)}$.

Conclusions

In this talk we have shown that

- Every compact and stable 1-dimensional filtration can be induced by a continuous function.
- This statement remains true in the *n*-dimensional case, if we also assume completeness.

In other words, in most of the problems relevant for applications it does not seem to be restrictive to assume that the considered filtrations are induced by continuous functions.

Some related results have been obtained in a joint work with Tomasz Kaczynski, Marc Ethier and Claudia Landi. Claudia will present these results in the next talk.