An example of monodromy in multidimensional persistence

The phenomenon of monodromy in multidimensional persistence can be illustrated by this example. Let us consider the function $\varphi = (\varphi_1, \varphi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined on the real plane in the following way: $\varphi_1(x, y) = x$, and

$$\varphi_2(x,y) = \begin{cases} -x & \text{if } y = 0 \\ -x+1 & \text{if } y = 1 \\ -2x & \text{if } y = 2 \\ -2x + \frac{5}{4} & \text{if } y = 3 \end{cases},$$

$\varphi_2(x, y)$ then being extended linearly for every x on the segment joining $(x, 0)$ with $(x, 1)$, $(x, 1)$ with $(x, 2)$, and $(x, 2)$ to $(x, 3)$. On the half-lines $\{(x, y) \in \mathbb{R}^2 : y < 0\}$ and $\{(x, y) \in \mathbb{R}^2 : y > 3\}$, φ_2 is then being taken with constant slope -1 in the variable y. The function φ_2 is shown plotted in Figure 1.

For any line $r_{(a,b)}$ of equation $x = at + b, y = (1 - a)t - b$ in the real plane with $0 < a < 1$ and $b \in \mathbb{R}$, we can consider the filtration of $\mathbb{R}^2$ given by the sets $L_t := \{(x, y) \in \mathbb{R}^2 : \varphi_1(x, y) \le at + b, \varphi_2(x, y) \le (1 - a)t - b\}$, $t \in \mathbb{R}$. Therefore, for any pair (a, b) we get a persistence diagram $D_{(a,b)}$. We observe that the persistence diagram $D_{(1/4,0)}$ contains a point with multiplicity 2.

Now, let us choose a closed path $\gamma : [0, 1] \to (0, 1) \times \mathbb{R}$ turning around the point $(1/4, 0)$ in the parameter space $(0, 1) \times \mathbb{R}$. You can see that two points in the persistent diagram $D_{\gamma(\tau)}$ exchange their position, when τ varies from 0 to 1. In other words, the loop γ around the singular point $(1/4, 0)$ induces a permutation on the persistence diagram.

For a short movie made by Marc Ethier (Jagiellonian University - Kraków) visualizing the previous example please click on the following link:

http://www.dm.unibo.it/∼frosini/movies/monodromy.mov

As for the movie, on the left side of the screen we can see the point (a, b) moving along a loop around $(1/4, 0)$ in the parameter space. In the middle, the corresponding leading line $r_{(a,b)}$ is displayed. On the right side of the screen we can see the persistence diagram corresponding to the chosen line. We observe that if the point (a, b) runs round $(1/4, 0)$ then the red point and the blue point exchange their position in the persistence diagram. For more info click **here**.

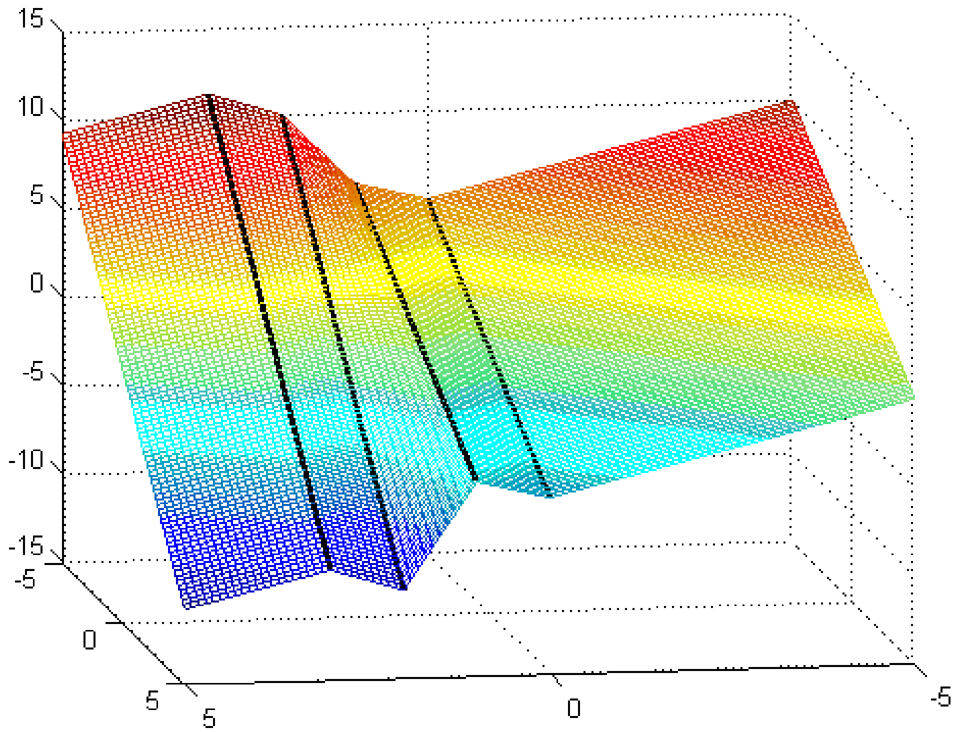

Figure 1: Function φ_2. Depth is x, width is y.