

B-orbits on spherical homogeneous spaces

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Let G be a reductive algebraic group over an algebraically closed field k and fix a Borel subgroup $B \subset G$. A subgroup $H \subset G$ is called *spherical* if B acts with finitely many orbits on G/H , or equivalently if H acts with finitely many orbits on the flag variety G/B . We denote by $\mathcal{B}(G/H)$ the set of the B -orbits on G/H , the talk surveyed some of the main results concerning this set.

The set $\mathcal{B}(G/H)$ comes naturally endowed with the *Bruhat order*, namely the partial order \leq induced by the inclusion of orbit closures. For instance, if $H = B$ and if T is a maximal torus contained in it, then there is a bijection between $\mathcal{B}(G/H)$ and the Weyl group $W = N/T$ (where N denotes the normalizer of T in G), and the partial order \leq coincides with the classical Bruhat order. When H is a symmetric subgroup of G (namely the set of points fixed by an algebraic involution of G), the partially ordered set $\mathcal{B}(G/H)$ was studied by R. W. Richardson and T. A. Springer in [6].

Let $H \subset G$ be a spherical subgroup. Fix a maximal torus $T \subset B$, let W be the Weyl group of T and let $R \supset S$ resp. be the attached sets of roots and of simple roots. The *Richardson-Springer monoid* is the monoid W^* generated by the simple reflection s_α with the relations $s_\alpha^2 = s_\alpha$ for all $\alpha \in S$ and the braid relations. As a set, W^* is the Weyl group W of G but with a different multiplication. An action of W^* on $\mathcal{B}(G/H)$ was defined by Richardson and Springer in [6] as follows: if $w \in W^*$ and $\mathcal{O} \in \mathcal{B}(G/H)$, then $w * \mathcal{O}$ is the unique open B -orbit contained in the B -stable subset $Bw\mathcal{O}$. The *weak order* is the partial order \preceq on $\mathcal{B}(G/H)$ induced by the action of W^* : if $\mathcal{O}, \mathcal{O}' \in \mathcal{B}(G/H)$, then $\mathcal{O} \preceq \mathcal{O}'$ if and only if $\mathcal{O}' = w * \mathcal{O}$ for some $w \in W^*$. The Bruhat order is compatible with the W^* -action and with the dimension function, namely the following properties hold for all $\alpha \in S$ and for all $\mathcal{O}, \mathcal{O}' \in \mathcal{B}(G/H)$:

- i) $\mathcal{O} \leq s_\alpha * \mathcal{O}$,
- ii) If $\mathcal{O} \leq \mathcal{O}'$, then $s_\alpha * \mathcal{O} \leq s_\alpha * \mathcal{O}'$,
- iii) If $\mathcal{O} \leq \mathcal{O}'$ and if $\dim(\mathcal{O}) = \dim(\mathcal{O}')$, then $\mathcal{O} = \mathcal{O}'$.

Theorem 1 ([6], [7]). *Suppose that H is a symmetric subgroup. Then the Bruhat order is the weakest partial order on $\mathcal{B}(G/H)$ which is compatible with the W^* -action and with the dimension function.*

Let $\text{rk}(H)$ the dimension of a maximal torus of H . Given a B -variety Z , denote by $\mathcal{X}(Z) = \{\text{weights of } B\text{-eigenfunctions } f \in k(Z)\}$ the *weight lattice* of Z and define the *rank* of Z as the rank $\mathcal{X}(Z)$. These are invariants of Z under birational B -morphisms. If $\mathcal{O} \in \mathcal{B}(G/H)$, then we have the inequalities

$$\text{rk}(G) - \text{rk}(H) \leq \text{rk}(\mathcal{O}) \leq \text{rk}(G/H) :$$

while the latter is the rank of the open B -orbit, the first one coincide with the rank of any closed orbit. If $\alpha \in S$ and $\mathcal{O} \in \mathcal{B}(G/H)$, then we have $\text{rk}(\mathcal{O}) \leq \text{rk}(s_\alpha * \mathcal{O}) \leq \text{rk}(\mathcal{O}) + 1$. More precisely, if $P_\alpha \supset B$ is the minimal parabolic subgroup associated to α and if $x \in G/H$, then we have the following possibilities:

Type G: $P_\alpha x = Bx$. Then we also set $s_\alpha \cdot Bx = Bx$.

Type U: $P_\alpha x = Bx_0 \sqcup Bx_\infty$, with $\dim(Bx_0) = \dim(Bx_\infty) + 1$ and $\text{rk}(Bx_0) = \text{rk}(Bx_\infty)$. Then we also set $s_\alpha \cdot Bx_1 = Bx_0$ and $s_\alpha \cdot Bx_0 = Bx_1$.

Type T: $P_\alpha x = Bx_1 \sqcup Bx_0 \sqcup Bx_\infty$, with $\dim(Bx_1) = \dim(Bx_0) + 1 = \dim(Bx_\infty) + 1$ and $\text{rk}(Bx_1) = \text{rk}(Bx_0) + 1 = \text{rk}(Bx_\infty) + 1$. Then we also set $s_\alpha \cdot Bx_1 = Bx_1$, $s_\alpha \cdot Bx_0 = Bx_\infty$ and $s_\alpha \cdot Bx_\infty = Bx_0$.

Type N: $P_\alpha x = Bx_1 \sqcup Bx_0$, with $\dim(Bx_1) = \dim(Bx_0) + 1$ and $\text{rk}(Bx_1) = \text{rk}(Bx_0) + 1$. Then we also set $s_\alpha \cdot Bx_1 = Bx_1$ and $s_\alpha \cdot Bx_0 = Bx_0$.

These cases follow by analysing the action of the subgroups of $\text{PGL}(2) \simeq \text{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^1 \simeq P_\alpha/B$ acting with finitely many orbits (see [2], [6]). Given $\alpha \in S$ and $x \in G/H$, the map $(s_\alpha, \mathcal{O}) \mapsto s_\alpha \cdot \mathcal{O}$ defines an action of s_α on the set of B -orbits contained in $P_\alpha x$, hence we get an action of s_α on the whole set $\mathcal{B}(G/H)$. F. Knop showed that these actions of the simple reflections glue together to an action of the Weyl group.

Theorem 2 ([2]). *The actions of the simple reflections defined above induce an action of the Weyl group W on $\mathcal{B}(G/H)$.*

One can define an action of the Hecke algebra attached to W on a module which is tightly related to the set $\mathcal{B}(G/H)$. This module, which was constructed by G. Lusztig and D. A. Vogan in the case of a symmetric homogeneous space in [3], is a main tool in the proof of previous theorem (see also [4], [8]).

If $\mathcal{O} \in \mathcal{B}(G/H)$ denote $\mathcal{O}_\leq = \{\mathcal{O}' \in \mathcal{B}(G/H) : \mathcal{O}' \leq \mathcal{O}\}$. A property which links the actions of W and of W^* with the Bruhat order is the *one-step property*:

$$\text{if } s_\alpha * \mathcal{O} \neq \mathcal{O}, \text{ then } (s_\alpha * \mathcal{O})_\leq = \bigcup_{\mathcal{O}' \leq \mathcal{O}} \{\mathcal{O}, s_\alpha \cdot \mathcal{O}, s_\alpha * \mathcal{O}\}.$$

This reduces the description of the Bruhat order on $\mathcal{B}(G/H)$ to the description of the sets \mathcal{O}_\leq when the orbit \mathcal{O} is minimal w.r.t. the weak order. In the case of a symmetric subgroup these orbits are always closed, this is false however in the general case.

As it follows by the definition of the action of the simple reflections, we have that the action of W on $\mathcal{B}(G/H)$ preserves the rank of the orbits. In two special cases, namely in the maximal and in the minimal rank case, the rank determines uniquely the W -orbit in $\mathcal{B}(G/H)$.

Denote $(G/H)^\circ \subset G/H$ the open B -orbit and denote by $P(G/H)$ the stabilizer of $(G/H)^\circ$ in G , namely $P(G/H) = \{g \in G : g(G/H)^\circ = (G/H)^\circ\}$. Let $W_{P(G/H)}$ the Weyl group of the Levi of $P(G/H)$. If the characteristic of k is zero, then Knop showed that one can recover the little Weyl group $W_{G/H}$ of G/H from the stabilizer of $(G/H)^\circ$ w.r.t the action of W .

Theorem 3 ([2]). *The set of elements in $\mathcal{B}(G/H)$ of maximal rank is an orbit under W . If moreover $\text{char}(k) = 0$, then the stabilizer of $(G/H)^\circ$ respect to the action of W is described as follows:*

$$W_{(G/H)^\circ} = W_{G/H} \rtimes W_{P(G/H)}.$$

Denote W_H the Weyl group of H , namely the quotient $N_H(T_H)/C_H(T_H)$ where T_H is a maximal torus in H and where $N_H(T_H)$ and $C_H(T_H)$ are resp. its normalizer and its centralizer in H . N. Ressayre showed that W_H is recovered from the stabilizer in W of an orbit of minimal rank.

Theorem 4 ([5]). *The set of elements in $\mathcal{B}(G/H)$ of minimal rank is an orbit under W . The stabilizer of any such an element is isomorphic to W_H .*

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