Projective normality of model wonderful varieties

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(joint with Paolo Bravi and Andrea Maffei)

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The problem

G is a semisimple connected algebraic group over $\mathbb{C}$.

Definition

A G-variety M is **wonderful** *(of rank n) if*

i) *M is smooth and projective,*

ii) *M possesses an open orbit whose complement is the union of n smooth prime divisors (the* **boundary divisors***) with normal crossings and non-empty intersection,*

iii) *every orbit closure in M equals the intersection of the boundary divisors which contain it.*

Question. Let M be a wonderful variety and let $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(M)$ be globally generated. Is the multiplication of sections

$$m_{\mathcal{L},\mathcal{L}'} : \Gamma(M, \mathcal{L}) \times \Gamma(M, \mathcal{L}') \longrightarrow \Gamma(M, \mathcal{L} \otimes \mathcal{L}')$$

surjective?

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Two examples

Symmetric wondeful varieties. Let $\theta : G \to G$ be an algebraic involution and denote G^θ the set of fixed points: then $G/N_G(G^\theta)$ admits a wonderful compactification [De Concini-Procesi 83].

In this case the question was raised in [Faltings 98] and a positive answer was given in [Chirivì-Maffei 04]: $m_{\mathcal{L},\mathcal{L}'}$ is surjective for every $\mathcal{L}, \mathcal{L}'$.

Model wondeful varieties. Let $K \subset G$ be such that G/K is quasi affine and $\mathbb{C}[G/K] \simeq \bigoplus_{Irr(G)} V$: then $G/N_G(K)$ admits a wonderful compactification.

There exists a unique model wonderful variety M_G^{mod} of maximal dimension: every G-stable subvariety of M_G^{mod} is a model wonderful variety for G and every model wonderful variety for G is isomorphic to a G-stable subvariety of M_G^{mod} [Luna 07].

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Colors

Let M be a wonderful variety. Fix $T \subset B$ a maximal torus and a Borel subgroup.

Definition

*The **colors** of M are the elements of the set*

$$\Delta = \{B\text{-stable prime divisors of } M \text{ which are not } G\text{-stable}\}$$

The classes of colors form a basis of the Picard group [Brion 89]:

$$\mathrm{Pic}(M) \simeq \mathbb{Z}[\Delta], \qquad \mathrm{Pic}(M)_{\geqslant 0} \simeq \mathbb{N}[\Delta]$$

where $\mathrm{Pic}(M)_{\geqslant 0}$ is the monoid of globally generated line bundles.

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Spherical roots

Let $z \in M$ the B^- fixed point and let $Y = Gz$ the closed orbit.

Definition

The **spherical roots** *of M are the elements of the set*

$$\Sigma = \{T\text{-weights of the } T\text{-module } T_zM/T_zY\}.$$

i) There is a natural bijection

$$\begin{array}{ccc} \Sigma & \longleftrightarrow & \{\text{boundary divisors of } M\} \\ \sigma & \longmapsto & M^\sigma \;:\; T_zM/T_zM^\sigma \simeq \mathbb{C}_\sigma \end{array}$$

This induces a pairing $c : \Sigma \times \Delta \to \mathbb{Z}$ defined by

$$[M^\sigma] = \sum_{D \in \Delta} c(\sigma, D)[D]$$

By identifying σ with $[M^\sigma]$ we get an inclusion $\mathbb{Z}[\Sigma] \subset \mathbb{Z}[\Delta]$.

ii) Σ is the basis of a reduced root system Φ_Σ [Brion 90, Knop 90].

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The triple of a wonderful variety

The triple (Σ, Δ, c) was introduced by Luna to classify wonderful varieties via their invariants [Luna 01] and it is as a big part of the combinatorial datum of M (the *spherical system*).

Example. In the case of the wonderful compactification of G_{ad} regarded as a symmetric $G \times G$ variety, then the following identifications hold:

$$\begin{array}{ccc} \Sigma & \leadsto & \{\text{simple roots of } G \text{ w.r.t. } T \subset B\} \\ \Delta & \leadsto & \{\text{fund. weights of } G \text{ w.r.t. } T \subset B\} \\ c & \leadsto & \text{Cartan matrix of } G \text{ w.r.t. } T \subset B \end{array}$$

More generally, whenever M is a (non-exceptional) symmetric wonderful variety, the triple (Σ, Δ, c) is identified with the triple of the root system Φ_Σ.

This is false in general and the situation can be pretty much more complicated.

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Partial order in $\mathbb{N}[\Delta]$ and covering relations

If $E, F \in \mathbb{N}[\Delta]$, we write $F \leqslant_\Sigma E$ if $E - F \in \mathbb{N}[\Sigma]$. Denote

$$\Delta(E) = \{D \in \mathbb{N}[\Delta] \ : \ D \leqslant_\Sigma E\}.$$

Definition

*Let $E, F \in \mathbb{N}[\Delta]$. We say that E **covers** F if $F <_\Sigma E$ and F is maximal with this property. If this the case, we say that the difference $E - F \in \mathbb{N}[\Sigma]$ is a **covering relation** for (Σ, Δ, c).*

Remark 1. Let $\gamma \in \mathbb{N}[\Sigma] \subset \mathbb{Z}[\Delta]$ and write $\gamma = \gamma^+ - \gamma^-$, where $\gamma^+, \gamma^- \in \mathbb{N}[\Delta]$ have no common support. Then γ is a covering relation if and only if γ^+ covers γ^-.

Remark 2. If (Σ, Δ, c) is the triple of the root system Φ_Σ, then $\leqslant_\Sigma$ is the usual partial order between the dominant weights of Φ_Σ.

In this case, covering relations were classified in [Stembridge 98], where it is shown that a covering relation is always a positive root.

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Section of line bundles and low triples

If $E \in \mathbb{N}[\Delta]$, denote $s_E \in \Gamma(M, \mathcal{L}_E)^{(B)}$ the canonical section and denote $V_E = \langle G s_E \rangle$. If $\gamma \in \mathbb{N}[\Sigma]$, denote $s^\gamma \in \Gamma(M, \mathcal{L}_\gamma)^G$ the canonical section. Then for $E \in \mathbb{N}[\Delta]$ we have

$$\Gamma(M, \mathcal{L}_E) = \bigoplus_{F \in \Delta(E)} s^{E-F} V_F.$$

Definition

Let $E, F \in \mathbb{N}[\Delta]$ and $D \in \Delta(E+F)$. We say that (E, F, D) is a **low triple** *if the following condition holds:*

If $E' \in \Delta(E)$ and $F' \in \Delta(F)$ are such that $D \in \Delta(E' + F')$, then $E' = E$ and $F' = F$.

Remark. Let (E, F, D) be a low triple and denote $\gamma = E + F - D$. Then $s^\gamma V_D \subset \Gamma(M, \mathcal{L}_{E+F})$ and by the decomposition above we get

$$s^\gamma V_D \subset \Gamma(M, \mathcal{L}_E)\Gamma(M, \mathcal{L}_F) \iff s^\gamma V_D \subset V_E V_F$$

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Reduction theorem

If $T = \sum n_D D \in \mathbb{Z}[\Delta]$, denote $\mathrm{ht}(T) = \sum n_D$.

Theorem (Bravi, G., Maffei)

Let M be a wonderful variety and suppose that

i) *For every low triple (E, F, D) with $E, F \in \Delta$, it holds $s^{E+F-D} V_D \subset V_E V_F$.*

ii) *For every covering relation $\gamma \in \mathbb{N}[\Sigma]$, it holds $\mathrm{ht}(\gamma^+) \leqslant 2$.*

Then the multiplication $m_{E,F}$ is surjective for every $E, F \in \mathbb{N}[\Delta]$.

In case (Σ, Δ, c) is the triple associated to Φ_Σ, then it is very easy to show that $\mathrm{ht}(\gamma^+) \leqslant 2$ for every covering relation $\gamma \in \mathbb{N}[\Sigma]$.

Conjecture

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For every covering relation $\gamma \in \mathbb{N}[\Sigma]$, it holds $\mathrm{ht}(\gamma^+) \leqslant 2$.

The case of a model wonderful variety

On the contrary, condition i) may not be satisfied.

Example. Let $G = \mathrm{SO}(9)$ and let $M = M_G^{\mathrm{mod}}$. Then the restriction to the closed orbit induces an inclusion $\mathrm{Pic}(M) \subset \mathcal{X}(B)$:

$$\Sigma = \{\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4, 2\alpha_4\}$$

$$\Delta = \{\omega_1, \omega_2, \omega_3, 2\omega_4\}$$

Then $(\omega_2, \omega_2, \omega_1)$ is a low triple and $V(\omega_1) \not\subset V(\omega_2)^{\otimes 2}$: it follows that $s^{2\omega_2-\omega_1} V_{\omega_1} \not\subset V_{\omega_2}^2$ and m_{ω_2,ω_2} is not surjective.

Theorem (Bravi, G., Maffei)

Let G be semisimple and connected of classical type and let $M = M_G^{\mathrm{mod}}$. Then the multiplication

$$m_{E,F} : \Gamma(M, \mathcal{L}_E) \times \Gamma(M, \mathcal{L}_F) \longrightarrow \Gamma(M, \mathcal{L}_{E+F})$$

is surjective for every $E, F \in \mathbb{N}[\Delta]$ if and only if G has no adjoint factors of type B_r with $r \geqslant 4$.

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Thank you