Projective normality of model wonderful varieties

Jacopo Gandini

(joint work with Paolo Bravi, Andrea Maffei)

Let G be a semisimple and connected complex algebraic group.

Definition. A G-variety M is called *wonderful* (of rank n) if it is smooth and projective and it satisfies the following conditions:

– M possesses an open orbit whose complement is a union of n smooth prime divisors (the *boundary divisors*) with non-empty transversal intersections;
– Any orbit closure in M equals the intersection of the boundary divisors which contain it.

Let M be a wonderful variety and let $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(M)$ be globally generated line bundles: is the multiplication of sections

$$m_{\mathcal{L},\mathcal{L}'} : \Gamma(M, \mathcal{L}) \times \Gamma(M, \mathcal{L}') \longrightarrow \Gamma(M, \mathcal{L} \otimes \mathcal{L}')$$

surjective? In particular, if these multiplications are all surjective, it follows that the complete linear system of any ample line bundle embeds M as a projectively normal variety. A trivial case is that of a flag variety: these are the wonderful varieties of rank zero and in this case the surjectivity of the multipication is an easy consequence of the irreducibility of the modules of sections.

An important class of wonderful varieties was introduced by De Concini and Procesi in the context of symmetric varieties [4]. In this case, it was shown by Chirivì and Maffei [3] that $m_{\mathcal{L},\mathcal{L}'}$ is surjective for every globally generated line bundles, giving in this way a positive answer to a question raised by Faltings [5].

Wonderful varieties were then considered in full generality by Luna, who proposed a general approach to attack the problem of their classification via combinatorial invariants [7]. Another remarkable class of wonderful varieties arises in the context of model varieties, which have been classified by Luna in [8], where it is shown that there exists a wonderful variety M_G^{mod} whose orbits parametrize the model varieties for G.

Let M be a wonderful G-variety and fix a maximal torus and a Borel subgroup $T \subset B \subset G$. Denote B^- the opposite Borel subgroup of B, let $z \in M$ be the unique B^- fixed point and denote $Y = Gz$ the unique closed orbit of M. Define Σ as the set of T-weights occurring in the T-module T_zM/T_zY: its elements are called the *spherical roots* of M and they naturally correspond to the local equations of the boundary divisors, which are G-stable. If $\sigma \in \Sigma$, we denote by M^σ the associated boundary divisor.

By the work of Brion [1], the Picard group of M is freely generated by the classes of the B-stable prime divisors which are not G-stable: such prime divisors are called the *colors* of M. Moreover, the semigroup of globally generated line bundles correspond to the free semigroup generated by the colors. If Δ denotes the set of colors of M, we get then a natural pairing $c : \Sigma \times \Delta \longrightarrow \mathbb{Z}$ (called the *Cartan pairing* of M), defined by the identity $[M^\sigma] = \sum_\Delta c(\sigma, D)[D]$ and which induces an embedding of $\mathbb{Z}[\Sigma]$ inside $\mathbb{Z}[\Delta]$.

The triple (Σ, Δ, c) is a main part of the combinatorial datum that Luna attached to a wonderful variety (the *spherical system* of M). If M is a wonderful symmetric variety (of non-exceptional type), then the situation is very nice: Σ is the set of simple roots of a root system Φ_Σ (the *restricted root system*), Δ is identified with the set of fundamental weights of Φ_Σ and c is the Cartan pairing of Φ_Σ. In the general case the situation can be more complicated, however by the work of Brion [2] and Knop [6] there always exists a root system Φ_Σ with basis Σ and we may think the triple (Σ, Δ, c) as a generalization of a root system.

As in the case of a root system, the semigroup $\mathbb{N}[\Delta]$ is naturally equipped with a partial order $\leqslant_\Sigma$, defined as follows: if $E, F \in \mathbb{N}[\Delta]$, then $E \leqslant_\Sigma F$ if and only if $F - E \in \mathbb{N}[\Sigma]$. In the case of a root system, this is the partial order on the semigroup of dominant weights studied by Stembridge [9]. The partial order $\leqslant_\Sigma$ is tightly related to the description of the sections of a line bundle on M. If $E \in \mathbb{N}[\Delta]$, denote $\Delta(E) = \{D \in \mathbb{N}[\Delta] \ : \ D \leqslant_\Sigma E\}$.

If $E \in \mathbb{N}[\Delta]$, denote $\mathcal{L}_E \in \mathrm{Pic}(M)$ the associated line bundle and $s_E \in \Gamma(M, \mathcal{L}_E)$ the canonical section, which is B semi-invariant, denote moreover $V_E = \langle G s_E \rangle \subset \Gamma(M, \mathcal{L}_E)$ the generated submodule. Similarly, if $\gamma = \sum a_\sigma \sigma \in \mathbb{N}[\Sigma]$, denote $\mathcal{L}_\gamma \in \mathrm{Pic}(M)$ the line bundle associated to $M^\gamma = \sum a_\sigma M^\sigma$ and denote $s^\gamma \in \Gamma(M, \mathcal{L}_\gamma)$ the canonical section, which is G invariant. Then it holds the following decomposition:

$$\Gamma(M, \mathcal{L}_E) = \bigoplus_{D \in \Delta(E)} s^{E-D} V_D.$$

Let $E, F \in \mathbb{N}[\Delta]$ and consider the multiplication of sections $m_{E,F} : \Gamma(M, \mathcal{L}_E) \times \Gamma(M, \mathcal{L}_F) \longrightarrow \Gamma(M, \mathcal{L}_{E+F})$. An easy inductive argument reduces the study of the surjectivity of $m_{E,F}$ to a particular set of triples.

Definition. Let $E, F \in \mathbb{N}[\Delta]$ and let $D \in \Delta(E+F)$. The triple (E, F, D) is called a *low triple* if the following condition is satisfied: if $E' \in \Delta(E)$ and $F' \in \Delta(F)$ are such that $D \in \Delta(E' + F')$, then $E' = E$ and $F' = F$.

The notion of low triple was introduced by Chirivì and Maffei in [3] in order to study the surjectivity of the multiplication map in the case of a symmetric wonderful variety however our definition is slightly more general than the original one. Notice that if (E, F, D) is a low triple and if $\gamma = E + F - D$, then $s^\gamma V_D \subset \Gamma(M, \mathcal{L}_E)\Gamma(M, \mathcal{L}_F)$ if and only if $s^\gamma V_D \subset V_E V_F$.

Suppose that $E, F \in \mathbb{N}[\Delta]$ are such that $F <_\Sigma E$ and suppose that F is maximal with this property: then we say that E *covers* F and we say that $E - F$ is a *covering relation* for (Σ, Δ, c). If $\gamma = \sum_\Delta n_D D$, define its *positive height* $\mathrm{ht}^+(\gamma) = \sum_{n_D > 0} n_D$. If (Σ, Δ, c) is identified with the triple of the root system Φ_Σ as in the case of a non-exceptional symmetric variety, then it is very easy to show that $\mathrm{ht}^+(\gamma) \leq 2$ for every covering relation $\gamma \in \mathbb{N}[\Sigma]$.

Theorem 1. *Let M be a wonderful variety with triple (Σ, Δ, c) and suppose that the following conditions are fulfilled:*

- *If (E, F, D) is a low triple with $E, F \in \Delta$, then $s^{E+F-D} V_D \subset V_E V_F$.*
- *If $\gamma \in \mathbb{N}[\Sigma]$ is a covering relation, then $\mathrm{ht}^+(\gamma) \leqslant 2$.*

Then the multiplication $m_{E,F}$ is surjective for every $E, F \in \mathbb{N}[\Delta]$.

We conjecture that the second condition of previous theorem is always satisfied. In particular, this would imply that the multiplication $m_{E,F}$ is surjective for every $E, F \in \mathbb{N}[\Delta]$ if and only if it is surjective for every $E, F \in \Delta$. On the other hand, following example shows that the first condition of previous theorem may not be fulfilled: in particular the multiplication $m_{E,F}$ may not be surjective.

Example. Let $G = \mathrm{SO}(9)$ and consider the model wonderful variety M_G^{mod}. Denote $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the simple roots of G and $\omega_1, \omega_2, \omega_3, \omega_4$ the fundamental weights of G, enumerated as usual. Then the restriction of line bundles to the closed orbit is injective and we may describe spherical roots and colors of M_G^{mod} as follows:

$$\Sigma = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, 2\alpha_4\}, \qquad \Delta = \{\omega_1, \omega_2, \omega_3, 2\omega_4\}.$$

Consider the low triple $(\omega_2, \omega_2, \omega_1)$: then $V(\omega_1) \not\subset V(\omega_2)^{\otimes 2}$, hence $s^{2\omega_2 - \omega_1} V_{\omega_1} \not\subset V_{\omega_2}^2$ and the multiplication m_{ω_2,ω_2} is not surjective.

In the case of a model wonderful variety, by classifying the covering relations and using the reduction of previous theorem, we proved the following theorem.

Theorem 2. *Let G be a semisimple connected group of classical type and consider the associated model wonderful variety M_G^{mod}. Then the multiplication of sections is surjective for any couple of globally generated line bundles on M_G^{mod} if and only if G has no adjoint factors of type B_r with $r \geq 4$.*

Actually, the counterexample given for the model wonderful variety of $\mathrm{SO}(9)$ does not express a lack of the multiplication, but rather a lack of the tensor product. Indeed $V(\omega_1) \not\subset V(\omega_2)^{\otimes 2}$ but $V(2\omega_1) \subset V(2\omega_2)^{\otimes 2}$: this expresses the fact that the saturation property does not hold for $\mathrm{SO}(9)$. Notice that if we assume that the multiplication of M is generic as much as possible, then the saturation property for the tensor product of G would imply an analogous saturation property for the multiplication of M.

References

[1] M. Brion, *Groupe de Picard et nombres caractéristiques des variétés sphériques*, Duke Math. J. **58** (1989), 397–424.

[2] M. Brion, *Vers une généralisation des espaces symétriques*, J. Algebra **134** (1990), 115–143.

[3] R. Chirivì and A. Maffei, *Projective normality of complete symmetric varieties*, Duke Math. J. **122** (2004), 93–123.

[4] C. De Concini and C. Procesi, *Complete symmetric varieties*, in Invariant theory (Montecatini, 1982), 1–44, Lecture Notes in Math. **996**, Springer, Berlin, 1983.

[5] G. Faltings, *Explicit resolution of local singularities of moduli-spaces*, J. Reine Angew. Math. **483** (1997), 183–196.

[6] F. Knop, *Weylgruppe und Momentabbildung*, Invent. Math. **99** (1990), 1–23.

[7] D. Luna, *Variétés sphériques de type* A, Publ. Math. Inst. Hautes Études Sci. **94** (2001), 161–226.

[8] D. Luna, *La variété magnifique modèle*, J. Algebra **313** (2007), 292–319.

[9] J.R. Stembridge, *The partial order of dominant weights*, Adv. Math. **136** (1998), 340–364.