Spherical orbit closures in simple projective spaces

JACOPO GANDINI

(partially joint work with Paolo Bravi, Andrea Maffei and Alessandro Ruzzi)

Let G be a semisimple simply connected algebraic group over $\mathbb{C}$, fix a maximal torus T and a Borel subgroup $B \supset T$. Denote R the root system of G associated to T and $S \subset R$ the basis associated to B. If $G_i \subset G$ is a simple factor, denote $S_i \subset S$ the corresponding subset of simple roots. If λ is a dominant weight, denote V_λ the associated simple module and define its *support* as follows

$$\mathrm{Supp}(\lambda) = \{\alpha \in S : \langle \alpha^\vee, \lambda \rangle \neq 0\}.$$

Suppose that $Gx_0 \subset \mathbb{P}(V_\lambda)$ is a *spherical orbit*: this means that B has an open orbit in Gx_0. Then we are interested in its closure $X = \overline{Gx_0}$, and in particular in the normality of X.

Particular cases are that of the adjoint group $G_{\mathrm{ad}} \simeq (G \times G)/N_G(\mathrm{diag}(G))$, regarded as a $(G \times G)$-space, which is spherical because of the Bruhat decomposition, and more generally that of a symmetric space, i. e. of the shape $G/N_G(G^\sigma)$, where $\sigma : G \to G$ is an algebraic involution, which is spherical because of the Iwasawa decomposition.

1. **The case of the adjoint group.** If $\mathrm{Supp}(\lambda) \cap S_i \neq \varnothing$ for every i, then G_{ad} is identified with the orbit of the identity line in $\mathbb{P}(\mathrm{End}(V_\lambda))$; since $\mathrm{End}(V_\lambda)$ is a simple $(G \times G)$-module, the situation is the one considered above. In joint work with P. Bravi, A. Maffei and A. Ruzzi, we gave a complete classification of the normality of the of the associated compactification $X_\lambda = \overline{(G \times G)[\mathrm{Id}]}$, proving the following theorem:

Theorem 1 (see [1]). *The variety X_λ is normal if and only if λ satisfies the following condition, for every connected component $S_i \subset S$:*

(N) *If $\mathrm{Supp}(\lambda) \cap S_i$ contains a long root, then it contains also the short simple root that is adjacent to a long simple root.*

A main tool in the proof of Theorem 1 is the multiplication map between sections of globally generated line bundles on the wonderful completion of G_{ad}: such completion coincides with the variety associated as above to any regular dominant weight and it was studied by C. De Concini and C. Procesi in [5] in the more general setting of a symmetric space. Unlike the general case of a wonderful variety, in the case of the group such map is explicitly described; moreover it was proved to be surjective by S. Kannan in [7] and more generally by R. Chirivì and A. Maffei in [4] in the case of a wonderful symmetric variety. These facts allow to describe a set of generators of the projective coordinate ring of the normalization of X_λ and they allow to give a criterion of normality which turns out to be equivalent to condition (N).

Moreover we gave an explicit characterization of the smoothness of X_λ, proving the following theorem:

Theorem 2 (see [1])**.** *The variety X_λ is smooth if and only if, for every connected component $S_i \subset S$, λ satisfies condition* (N) *of Theorem 1 together with the following conditions:*

(QF1) $\mathrm{Supp}(\lambda) \cap S_i$ *is connected and, in case it contains a unique element, then this element is an extreme of S_i;*

(QF2) $\mathrm{Supp}(\lambda) \cap S_i$ *contains every simple root which is adjacent to three other simple roots and at least two of the latter ones.*

(S) $S \smallsetminus \mathrm{Supp}(\lambda)$ *is of type* A.

While conditions (QF1) and (QF2) characterize $\mathbb{Q}$-factoriality following a theorem given by M. Brion in [2] which holds for a general spherical variety, condition (S) follows by a theorem given by D. Timashev in [10] which holds for a projective group embedding. However Theorem 2 holds in a similar way for any simple normal completion of a symmetric space (see [1]).

Even if X_λ is non-normal, actually it is homeomorphic to its normalization. This follows considering the more general case of a symmetric orbit, which was considered by A. Maffei in [9], where it is proved that the corresponding orbit closure X is always homeomorphic to its normalization.

2. **The model case.** A very different behaviour, somehow opposite to the one which occurs in the symmetric case, occurs in the model case, i. e. if the considered orbit is of the shape $G/N_G(H)$, where G/H is a model space: a model space for G is an homogeneous space G/H such that every simple G-module occurs with multiplicity one in $\mathbb{C}[G/H]$. Model spaces were classified by D. Luna in [8], where it is defined a wonderful variety M_G^{mod} (called the *wonderful model variety* of G) whose orbits naturally parametrize up to isomorphism the model spaces for G: more precisely any orbit in M_G^{mod} is of the shape $G/N_G(H)$ where G/H is a model space, and this correspondence gives a bijection up to isomorphism. This construction highlights a *principal model space*, namely the model space which dominates the open orbit in M_G^{mod}.

If $G_i \subset G$ is a simple factor of type B or C, number the simple roots in $S_i = \{\alpha^i_1, \dots \alpha^i_{r(i)}\}$ starting from the extreme of the Dynkin diagram of G_i where the double link is; define moreover $S_i^{\mathrm{even}}, S_i^{\mathrm{odd}} \subset S_i$ as the subsets whose element index is respectively even and odd; set

$$N_i^{\mathrm{even}}(\lambda) = \min\{k \leqslant r(i) \,:\, \alpha^i_k \in \mathrm{Supp}(\lambda) \cap S_i^{\mathrm{even}}\},$$

$$N_i^{\mathrm{odd}}(\lambda) = \min\{k \leqslant r(i) \,:\, \alpha^i_k \in \mathrm{Supp}(\lambda) \cap S_i^{\mathrm{odd}}\}.$$

Finally, if G_i is of type F_4, number the simple roots in $S_i = \{\alpha^i_1, \alpha^i_2, \alpha^i_3, \alpha^i_4\}$ starting from the extreme of the Dynkin diagram which contains a long root. Then we proved the following theorem:

Theorem 3 (see [6])**.** *Let $x_0 \in \mathbb{P}(V_\lambda)$ be such that* $\mathrm{Stab}(x_0) = N_G(H)$*, where G/H is the principal model space of G. Then X is homeomorphic to its normalization if and only if following conditions are fulfilled, for every connected component $S_i \subset S$:*

(i) *If S_i is of type* B, *then either* $\alpha_1^i \in \mathrm{Supp}(\lambda)$ *or* $\mathrm{Supp}(\lambda) \cap S_i^{\mathrm{even}} = \varnothing$*;*
(ii) *If S_i is of type* C, *then* $N_i^{\mathrm{odd}}(\lambda) \geqslant N_i^{\mathrm{even}}(\lambda) - 1$*;*
(iii) *If S_i is of type* F_4 *and* $\alpha_2^i \in \mathrm{Supp}(\lambda)$, *then* $\alpha_3^i \in \mathrm{Supp}(\lambda)$ *as well.*

3. **The strict case.** Let's go back to a generic spherical orbit $Gx_0 \subset \mathbb{P}(V_\lambda)$ and set $H = \mathrm{Stab}(x_0)$. It has been shown by P. Bravi and D. Luna in [3] that such an orbit admits a wonderful completion M; this allows us to describe the orbits of X and those of its normalization from a combinatorial point of view in terms of their *spherical system*, which is a triple of combinatorial invariants that D. Luna attached to a spherical homogeneous space which admits a wonderful completion and which uniquely determines it.

Suppose moreover that M is *strict*, i. e. that the stabilizer of any point $x \in M$ is self-normalizing: this includes the symmetric case as well as the model case. Then, following the description of the orbits of X and of those of its normalization, we get a complete classification of the simple modules V_λ endowed with an embedding $G/H \hookrightarrow \mathbb{P}(V_\lambda)$ (which, if it exists, it is unique) which gives rise to an orbit closure homeomorphic to its normalization (Theorem 6.9 in [6]). The classification is based on a combinatorial condition on $\mathrm{Supp}(\lambda)$ which is easily read off by the *spherical diagram* of G/H, which is a very useful tool to represent its spherical system starting by the Dynkin diagram of G. Such condition of bijectivity is substantially deduced from the model case, where the classification is expressed by Theorem 3, whereas it is always fulfilled if H is a symmetric subgroup or if G is simply laced.

References

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