A LIOUVILLE-TYPE THEOREM ON HALF-SPACES FOR SUB-LAPLACIANS

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Abstract. Let $L$ be a sub-Laplacian on $\mathbb{L}^N$ and let $G = (\mathbb{L}^N, \circ, \delta_\lambda)$ be its related homogeneous Lie group. Let $E$ be a Euclidean subgroup of $\mathbb{L}^N$ such that the orthonormal projection $\pi : G \to E$ is a homomorphism of homogeneous groups, and let $\langle \ , \ \rangle$ be an inner product in $E$. Given $\alpha \in E$, $\alpha \neq 0$, define $\Omega(\alpha) := \{ x \in G : \langle \alpha, \pi(x) \rangle > 0 \}$. We prove the following Liouville-type theorem.

If $u$ is a nonnegative $L$-superharmonic function in $\Omega(\alpha)$ such that $u \in L^1(\Omega(\alpha))$, then $u \equiv 0$ in $\Omega(\alpha)$.

1. Introduction

In [14] F. Uguzzoni proved the following Liouville-type theorem.

Theorem A. Let $\Delta_{\mathbb{H}_n}$ be a sub-Laplacian on the Heisenberg group $\mathbb{H}_n$ and let $\Omega$ be a half-space of $\mathbb{H}_n$ whose boundary is parallel to the center of $\mathbb{H}_n$. If $u$ is a nonnegative $\Delta_{\mathbb{H}_n}$-superharmonic function such that $u \in L^1(\Omega)$, then $u \equiv 0$.

The aim of this note is to show that an analogous result holds in the general setting of the sub-Laplacians on $\mathbb{R}^N$.

Let $L$ be a sub-Laplacian in $\mathbb{R}^N$ whose related homogeneous Lie group is $(\mathbb{G}, \circ, \delta_\lambda)$. Let $E$ be an Euclidean subgroup of $\mathbb{R}^N$ such that the orthonormal projection

$$\pi : G \to E$$

is a homomorphism of homogeneous Lie groups, i.e.,

$$\pi(x \circ y^{-1}) = \pi(x) - \pi(y), \quad \pi(\delta_\lambda(x)) = \lambda \pi(x),$$

for every $x, y \in G$ and every $\lambda > 0$.

Let $\langle \ , \ \rangle$ be an inner product in $E$ and, for every $\alpha \in E$, $\alpha \neq 0$, define

$$\Omega(\alpha) := \{ x \in G : \langle \alpha, \pi(x) \rangle > 0 \}.$$

The main result of this paper is the following Liouville-type theorem.

Theorem 1.1. Let $u : \Omega(\alpha) \to (-\infty, \infty]$ be a $L$-superharmonic function in $\Omega(\alpha)$. If $u \geq 0$ and $u \in L^1(\Omega(\alpha))$, then

$$u \equiv 0 \text{ in } \Omega(\alpha).$$

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Liouville-type theorems in half-spaces for sub-Laplacian play a crucial role in looking for solutions to semilinear boundary value problems; see, e.g., [2], [1], [3], [7]. Liouville-type theorems in the whole space in a sub-Riemannian setting have received increasing attention in recent years; see, e.g., [4] (Section 5.8), [10], [11], [12], [13], the references therein, and the recent deep papers by D’Ambrosio and Mitidieri both for Riemannian and sub-Riemannian results ([8], [9]).

We would like to stress that to prove Theorem 1.1 we exploit a technique which is different with respect to the one used in the previous quoted papers. We follow the approach of Uguzzoni in [14] based on suitable mean value operators on the level set of the fundamental solution of $L$ and, moreover, a kind of invariance of $\Omega(\alpha)$ with respect to suitable left translations of $G$. For this last reason our method cannot work for half-spaces without this invariance property.

We would also like to stress that our result, in the case of the Heisenberg group $H_n$, gives back the result of Uguzzoni. As already noticed in [14], the assumption $u \in L^1(\Omega(\alpha))$ cannot be improved in the following sense.

**Proposition 1.2.** Let $p \in [1, +\infty]$ be fixed, and let $G$ be a Lie group whose homogeneous dimension $Q$ satisfies

$$\frac{Q}{2} > \frac{p}{p-1}.$$  

Then for every $\alpha \in \mathbb{E}$ there exists a strictly positive $\Delta_G$-harmonic function $u$ in $\Omega(\alpha)$ such that

$$\int_{\Omega(\alpha)} u^p \, dx < +\infty.$$  

In particular this statement holds for the classical Laplacian $\Delta$ in $\mathbb{R}^N$ if $\frac{N}{2} > \frac{p}{p-1}$.

In Remark 3.1 we will recognize also that the assumption $u \geq 0$ cannot be removed from Theorem 1.1.

We close this introduction by showing some explicit examples of applications of our Theorem 1.1.

**Example 1.3.** In $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$, whose point is denoted by $(x, t), x \in \mathbb{R}^m, t \in \mathbb{R}^n$, consider the linear second order partial differential operator (PDO)

$$\mathcal{L} = \Delta_x + \frac{1}{4} |x|^2 \Delta_t + \sum_{k=1}^{n} \langle B^{(k)} x, \nabla x \rangle \partial_{t_k},$$

where $\Delta_x = \sum_{j=1}^{m} \partial_{x_j}^2$ and $\Delta_t = \sum_{j=1}^{n} \partial_{t_j}^2$ are the usual Laplace operator in $\mathbb{R}^m$ and in $\mathbb{R}^n$, respectively. $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_m})$ and $B^{(1)}, \ldots, B^{(m)}$ are $m \times m$ matrices having the following properties:

(i) $B^{(k)}$ is skew-symmetric and orthogonal, $k = 1, \ldots, m$;
(ii) $B^{(i)}B^{(j)} = -B^{(j)}B^{(i)}$ for every $i, j \in \{j = 1, \ldots, m\}, i \neq j$.

Then $\mathcal{L}$ in (1.1) is a sub-Laplacian on a group of Heisenberg type $\mathbb{H}$, and the map $\pi : \mathbb{H} \rightarrow \mathbb{R}^m, \pi(x, t) = x$ is a homomorphism of homogeneous groups (see [6, Section 3.6]).

For every fixed $\alpha \in \mathbb{R}^m, \alpha \neq 0$,

$$\Omega(\alpha) := \{ x \in G : \langle \alpha, \pi(x) \rangle > 0 \},$$

is a half-space to which our Liouville-type Theorem 1.1 applies.
Example 1.4. In \( \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \), whose point is denoted by \((x, y, t), x, y \in \mathbb{R}^n, t \in \mathbb{R}\), consider the linear second order PDO
\[
L = \Delta_x + (x \cdot \nabla_y - \partial_t)^2.
\]
This operator is a sub-Laplacian on a group \( K \) named in [6] of Kolmogorov-type. Taking into account the composition law and the dilations on \( K \) defined in [6, Section 4.3.4], one immediately recognizes that the half-spaces to which our Liouville-type Theorem 1.1 applies are of the kind
\[
\{(x, y, t) \in \mathbb{R}^N : \langle \alpha, x \rangle + \beta t > 0\},
\]
where \(|\alpha|^2 + \beta^2 > 0\).

Our paper is organized as follows.

The next section is devoted to the notation, definitions, and results needed in the note.

In section 3 we will prove Theorem 1.1, Proposition 1.2, and Remark 3.1.

2. Sub-Laplacians and related sub-harmonic functions

We call a sub-Laplacian on \( \mathbb{R}^N \) any linear second order partial differential operator \( \mathcal{L} \) of the kind
\[
\mathcal{L} = \sum_{j=1}^{m} X_j^2
\]
where the \( X_j \)'s are smooth vector fields (i.e. linear partial differential operator of order one and smooth coefficients) satisfying the following conditions:

(H1) the Lie algebra
\[
a := \text{Lie}\{X_1, \ldots, X_m\}
\]
is a vector space of dimension \( N \); moreover,
\[
\text{rank } a(x) = N \text{ at any point } x \in \mathbb{R}^N;
\]

(H2) there exists a group of dilations \((\delta_\lambda)_{\lambda > 0}\) in \( \mathbb{R}^N \) such that every \( X_j \) is \( \delta_\lambda \)-homogeneous of degree one.

A group of dilations in \( \mathbb{R}^N \) is a family of diagonal linear functions \((\delta_\lambda)_{\lambda > 0}\) of the kind
\[
\delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N
\]
where \( \sigma_1 = 1 \leq \sigma_2 \leq \cdots \leq \sigma_N, \sigma_j \in \mathbb{N} \).

Condition (H1) implies the hypoellipticity of \( \mathcal{L} \): in particular, the \( \mathcal{L} \)-harmonic functions, i.e., the solution to \( \mathcal{L}u = 0 \), are smooth. Moreover, conditions (H1) and (H2) imply the existence of a group law \( \circ \) in \( \mathbb{R}^N \) such that \( G = (\mathbb{R}^N, \circ, \delta_\lambda) \) is a homogeneous Lie group on which the vector fields \( X'_j \)'s are left translation invariant and \( \delta_\lambda \)-homogeneous of degree one (see [4]). The natural number
\[
Q = \sigma_1 + \cdots + \sigma_N
\]
is called the homogeneous dimension of \( G \). Throughout the paper we always assume \( Q \geq 3 \) (if \( Q = 2 \), then \( G \) is the Euclidean group). Then there exists a
continuous function $d : \mathbb{G} \to \mathbb{R}$, smooth and strictly positive outside the origin, $\delta_\lambda$-homogeneous of degree one and such that

$$\gamma(x) := \left( \frac{1}{d(x)} \right)^{Q-2}$$

is $\mathcal{L}$-harmonic in $\mathbb{R}^N \setminus \{0\}$ (see [6, Section 5.4]). This function $d$ is called an $\mathcal{L}$-gauge and for $\mathcal{L}$ plays a role analogous to the one played by the Euclidean norm with respect to the classical Laplacian. In particular, the $d$-balls

$$B_d(x, r) := \{ y \in \mathbb{G} : d(x^{-1} \circ y) < r \}$$

support averaging operators characterizing the $\mathcal{L}$-harmonicity. To be precise, define

$$\psi := |\nabla_L d|^2, \quad \nabla_L = (X_1, \ldots, X_m),$$

$$M_r u(x) := \frac{1}{c_d r^Q} \int_{B_d(x, r)} \psi(x^{-1} \circ y) u(y) \, dy$$

and

$$N_r(\mathcal{L}u)(x) = \frac{1}{(Q-2)c_d r^Q} \int_0^r \rho^{Q-1} \left( \int_{B_d(x, \rho)} \mathcal{L}u(y) \left( d(x^{-1} \circ y)^{2-Q} - \rho^{2-Q} \right) dy \right) \, d\rho$$

where $c_d = \int_{B_d(0, 1)} \psi \, dy$.

Then, if $\Omega$ is an open subset of $\mathbb{G}$, $u \in C^2(\Omega)$ and $B_d(x, r) \subseteq \Omega$, (2.1)

$$u(x) = M_r u(x) - N_r(\mathcal{L}u)(x)$$

(see [6, Theorem 5.6.1]).

We stress that $\psi$ is smooth outside the origin, $\delta_\lambda$-homogeneous of degree zero, and nonconstant unless $\mathbb{G}$ is the Euclidean group (see [5]; see also [6, Proposition 9.8.9]). In some particular important cases, such as, e.g., the group of Heisenberg type, explicit expressions of $\psi$ are known (see [6, Example 5.5.3]). In any case it is known that $\psi > 0$ in a dense open subset of $\mathbb{R}^N$ (see [6, page 262]).

With these mean value operators, one can prove a version of the Gauss-Koebbe Theorem in our setting (see [6, Section 5.6]):

**Theorem 2.1** (Gauss-Koebbe-type Theorem). If $\Omega \subseteq \mathbb{R}^N$ is open and $u : \Omega \to \mathbb{R}$ is $\mathcal{L}$-harmonic, then

(2.2)

$$u(x) = M_r u(x)$$

for every $x \in \Omega$ and $r > 0$ such that $B_d(x, r) \subseteq \Omega$.

Vice versa, if $u$ is merely continuous in $\Omega$ and satisfies (2.2), then $u$ is $C^\infty$ and $\mathcal{L}$-harmonic in $\Omega$.

The average operator $M_r$ can also be used to fix the notion of $\mathcal{L}$-superharmonic function.

A lower semicontinuous function $u : \Omega \to [-\infty, \infty]$ is called $\mathcal{L}$-superharmonic if $u$ is finite in a dense subset of $\Omega$ and

$$u(x) \geq M_r u(x)$$

for every $x \in \Omega$ and $r > 0$ such that $B_d(x, r) \subseteq \Omega$.

A quite exhaustive theory of $\mathcal{L}$-subharmonic functions is presented in the monograph [6, Chapter 8]. In particular, there it is proved that every $\mathcal{L}$-subharmonic
function is $L^1_{loc}$ and that if $u$ is of class $C^2$, then $u$ is $\mathcal{L}$-subharmonic if and only if $\mathcal{L}u \geq 0$.

3. PROOF OF THEOREM 1.1 PROPOSITION 1.2 AND REMARK 3.1

The most important part of this section is the

Proof of Theorem 1.1 Let $\alpha \in \mathbb{E}$, $\alpha \neq 0$, be fixed and let 

$$\Omega(\alpha) := \{x \in \mathbb{G} : \langle \alpha, \pi(x) \rangle > 0\}.$$ 

For every $x \in \Omega(\alpha)$ we define 

$$r(x) := \varepsilon \langle \alpha, \pi(x) \rangle,$$

where $\varepsilon > 0$ is fixed in such a way that 

$$(3.1) \quad B(x, r(x)) \subseteq \Omega(\alpha) \quad \forall x \in \Omega(\alpha).$$

We will show in a moment the existence of a suitable $\varepsilon > 0$ satisfying (3.1).

For a function $u \in L^1_{loc}(\Omega(\alpha))$ we let 

$$T(u) : \Omega(\alpha) \rightarrow \mathbb{R}, \quad T(u)(x) := M_{r(x)}(u)(x).$$

Hence, 

$$T(u)(x) = \int_{\Omega(\alpha)} K(x, y)u(y) \, dy, \quad x \in \Omega(\alpha),$$

where 

$$(3.2) \quad K(x, y) = \frac{1}{c_d(r(x))Q} \psi(x^{-1} \circ y) \chi_{B_z}(y).$$

In what follows we also use the following notation: 

$$A_x := \{y \in \Omega(\alpha) \mid d(y^{-1} \circ x) < r(y)\}.$$ 

With this notation, we have 

$$\chi_{B_z}(y) = \chi_{A_y}(x).$$ 

Indeed 

$$y \in B_x \iff d(x^{-1} \circ y) < r(x) \iff x \in A_y.$$ 

Let us now show (3.1). We first remark that $\mathbb{E} \ni e \mapsto d(e) \in \mathbb{R}$ is homogeneous of degree one with respect to the Euclidean dilation $e \mapsto \lambda e$. As a consequence, by a suitable constant $c > 0$, we have 

$$d(e) \geq c|e| \quad \forall e \in \mathbb{E}, \quad |\cdot| = \text{Euclidean norm}.$$ 

Moreover, we can also assume that 

$$d(x) \geq c|\pi(x)| \quad \forall x \in \mathbb{G}.$$ 

Then, if $x \in \Omega(\alpha)$, for every $z \in B_d(x, r(x))$, we have $r(x) > d(z, x) \geq c|\pi(z) - \pi(x)|$.

Hence 

$$\langle \alpha, \pi(z) \rangle = \langle \alpha, \pi(x) \rangle + \langle \alpha, \pi(z) - \pi(x) \rangle \geq \langle \alpha, \pi(x) \rangle - |\alpha||\pi(z) - \pi(x)|$$

$$\geq \langle \alpha, \pi(x) \rangle - \left|\frac{|\alpha|}{c} r(x) = \langle \alpha, \pi(x) \rangle \left(1 - \frac{|\alpha|}{c} \varepsilon\right)\right.$$ 

Thus, if $0 < \varepsilon < \frac{c}{|\alpha|}$, we get $\langle \alpha, \pi(z) \rangle > 0$; i.e., $z \in \Omega(\alpha)$ and (3.1) is proved.

The proof of Theorem 1.1 will immediately follow from the next lemma.
Main Lemma.

(i) \( K(x,y) \geq 0 \) for every \( x, y \in \Omega(\alpha) \);
(ii) \( \int_{\Omega(\alpha)} K(x,y) \, dy = 1 \) for every \( x \in \Omega(\alpha) \);
(iii) \( \int_{\Omega(\alpha)} K(x,y) \, dx = \int_{\Omega(\alpha)} K(x,\alpha) \, dx \) for every \( y \in \Omega(\alpha) \);
(iv) \( c^* := \int_{\Omega(\alpha)} K(x,\alpha) \, dx > 1 \).

Proof of the Main Lemma.

(i) It straightforwardly follows from (3.2).

(ii) By the Gauss-Koebe-type Theorem 2.1 for \( L \)-harmonic functions, if \( u \) is \( L \)-harmonic in \( \Omega(\alpha) \), then \( T(u) = u \). In particular \( T(1) = 1 \), that is, \( 1 = \int_{\Omega(\alpha)} K(x,y) \, dy \) for every \( x \in \Omega(\alpha) \).

(iii) This is the crucial part of the Main Lemma. We start by proving the following property of \( \Omega(\alpha) \): \( \forall y \in \Omega(\alpha) \) there exists \( \lambda = \lambda(y) > 0 \) such that

\[
\delta_\lambda(x) \circ y^{-1} \circ x \in \Omega(\alpha) \text{ and } r(\delta_\lambda(x) \circ y^{-1} \circ x) = r(x)
\]

for every \( x \in \Omega(\alpha) \).

Indeed, let \( y, x \in \Omega(\alpha) \) and consider

\[
\langle \alpha, \pi(\delta_\lambda(x) \circ y^{-1} \circ x) \rangle = \langle \alpha, \pi(\delta_\lambda(x)) \rangle + \langle \alpha, \pi(y^{-1}) \rangle + \langle \alpha, \pi(x) \rangle = \langle \alpha, \pi(x) \rangle + \lambda \langle \alpha, \alpha \rangle - \langle \alpha, \pi(y) \rangle.
\]

Then, if we choose \( \lambda = \frac{\langle \alpha, \pi(y) \rangle}{|\alpha|^2} \) we have \( \lambda > 0 \) and

\[
\langle \alpha, \pi(\delta_\lambda(x) \circ y^{-1} \circ x) \rangle > 0, \quad r(\delta_\lambda(x) \circ y^{-1} \circ x) = r(x).
\]

This completes the proof of the stated property of \( \Omega(\alpha) \).

In what follows we also use a homogeneity property of \( x \mapsto r(x) \), precisely

\[
r(\delta_\lambda(x)) = \lambda r(x) \text{ for every } x \in \Omega(\alpha) \text{ and } \lambda > 0.
\]

Indeed

\[
r(\delta_\lambda(x)) = \varepsilon \langle \alpha, \pi(\delta_\lambda(x)) \rangle = \varepsilon \langle \alpha, \lambda \pi(x) \rangle = \lambda r(x).
\]
Let us now fix \( y \in \Omega(\alpha) \) and compute
\[
\int_{\Omega(\alpha)} K(x, \alpha) \, dx = \frac{1}{c_d} \int_{\Omega(\alpha)} \left( \frac{1}{r(x)} \right)^Q \psi(x^{-1} \circ \alpha) \, A_{B_x}(\alpha) \, dx
\]
(letting \( \hat{\psi}(z) = \psi(z^{-1}) \))
\[
= \frac{1}{c_d} \int_{A_{\alpha}} \left( \frac{1}{r(x)} \right)^Q \hat{\psi}(\alpha^{-1} \circ x) A_{A_{\alpha}}(x) \, dx
\]
(using the change of variables \( x = \delta_{x}(\xi) \) and noticing
\[
\text{that } r\left( \delta_{x}^{1}(\xi) \right) = \frac{1}{\lambda} r(\xi) \text{ and that } dx = \lambda^{-Q} d\xi
\]
\[
= \frac{1}{c_d} \int_{\delta_{x}(A_{\alpha})} \left( \frac{1}{r(\xi)} \right)^{-Q} \hat{\psi}(\alpha^{-1} \circ \delta_{x}^{1}(\xi)) \, d\xi
\]
(keeping in mind that \( \hat{\psi} \) is \( \delta_{\lambda} \)-homogeneous of degree zero)
\[
= \frac{1}{c_d} \int_{\delta_{x}(A_{\alpha})} \left( \frac{1}{r(\xi)} \right)^Q \psi^{-1}(\delta_{\lambda}(\alpha^{-1}) \circ \xi) \, d\xi.
\]
We now choose \( \lambda = \lambda(y) > 0 \) such that \( r(\delta_{\lambda}(\alpha) \circ y^{-1} \circ x) = r(x) \) for every \( x \in \Omega(\alpha) \) and use the change of variable
\[
\xi = \delta_{\lambda}(\alpha) \circ y^{-1} \circ x.
\]
We obtain
\[
\int_{\Omega(\alpha)} K(x, \alpha) \, dx = \frac{1}{c_d} \int_{y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha})} \left( \frac{1}{r(x)} \right)^Q \psi^{-1}(y^{-1} \circ x) \, dx.
\]
On the other hand, as we will recognize in a moment,
\[
y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha}) = A_{y}.
\]
Then
\[
\int_{\Omega(\alpha)} K(x, \alpha) \, dx = \frac{1}{c_d} \int_{A_{y}} \left( \frac{1}{r(x)} \right)^Q \psi(y^{-1} \circ x) \, dx = \int_{\Omega(\alpha)} K(x, y) \, dx,
\]
and (iii) is proved.

We are left to prove (3.3). One has
\[
x \in y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha}) \iff \exists z := \alpha \circ \delta_{x}^{1}(y^{-1} \circ x) \in A_{\alpha}
\]
\[
\iff d(z, \alpha) < r(z).
\]
We know that \( r(z) = \frac{1}{\lambda} r(\delta_{\lambda}(\alpha) \circ y^{-1} \circ x) = \frac{1}{\lambda} r(x) \), while
\[
d(z, \alpha) = d(\alpha^{-1} \circ z) = d(\delta_{x}^{1}(y^{-1} \circ x)) = \frac{1}{\lambda} d(y^{-1}, x).
\]
We have thus proved that
\[
x \in y \circ \delta_{\lambda}(\alpha^{-1}) \circ \delta_{\lambda}(A_{\alpha}) \iff d(y^{-1} \circ x) < r(x) \iff x \in A_{y}.
\]
This completes the proof of (iii).

(iv) Let \( x_0 \in \mathbb{G} \setminus \Omega(\alpha) \) and consider the function
\[
v : \Omega(\alpha) \to \mathbb{R}, \quad v(x) = (d(x_0^{-1} \circ x))^{-1-Q}.
\]
Obviously the function $v$ is smooth in $\Omega(\alpha)$. Moreover, $v > 0$ and $v \in L^1(\Omega(\alpha))$. By using the left invariance of $\mathcal{L}$ on $G$ and the form of $\mathcal{L}$ for radial functions\footnote{If $w = f(d)$, then $\mathcal{L}(w) = \psi(f''(d) + \frac{Q-1}{d} f'(d))$ (see \cite{ref} Proposition 5.4.3).}, we also have

$$(\mathcal{L}v)(x) = (\mathcal{L}d^{-Q-1})(x^{-1}_0 \circ x)$$

$$= \psi(x^{-1}_0 \circ x)((Q + 1)(Q + 2) - (Q + 1)(Q - 1))d^{-Q-3}(x^{-1}_0 \circ x)$$

$$= 3(Q + 1)(\psi d^{-Q-3})(x^{-1}_0 \circ x).$$

Then $\mathcal{L}v > 0$ in a dense open set of $\Omega(\alpha)$. As a consequence, using the representation formula (2.1), we get

$$T(v)(x) - v(x) > 0 \quad \forall x \in \Omega(\alpha),$$

that is, $T(v) > v$ in $\Omega(\alpha)$. It follows that

$$\int_{\Omega(\alpha)} v \, dx < \int_{\Omega(\alpha)} T(v) \, dx = \int_{\Omega(\alpha)} \left( \int_{\Omega(\alpha)} K(x, y) v(y) \, dy \right) \, dx$$

$$= \int_{\Omega(\alpha)} v(y) \left( \int_{\Omega(\alpha)} K(x, y) \, dx \right) \, dy = c^* \int_{\Omega(\alpha)} v(y) \, dy.$$

Then

$$\int_{\Omega(\alpha)} v \, dx < c^* \int_{\Omega(\alpha)} v \, dy,$$

which implies $c^* > 1$, since $\int_{\Omega(\alpha)} v \, dx > 0$. This completes the proof of the Main Lemma. \hfill \Box

We can now conclude the proof of Theorem 1.1. Since $u$ is $\mathcal{L}$-superharmonic, we have $T(u) \leq u$ in $\Omega(\alpha)$. Therefore,

$$\int_{\Omega(\alpha)} u \, dx \geq \int_{\Omega(\alpha)} T(u) \, dx$$

(as in the proof of the Main Lemma (iv))

$$= c^* \int_{\Omega(\alpha)} u \, dx.$$

Then, since $c^* > 1$,

$$\int_{\Omega(\alpha)} u \, dx \leq 0,$$

which implies $u \equiv 0$ since $u \geq 0$ and lower semicontinuous. \hfill \Box

Proof of Proposition 1.2 Let $d$ be a gauge function for $\mathcal{L}$ and define

$$u(x) = (d(x_0^{-1} \circ x))^{-Q+2}, \quad x \in \Omega(\alpha),$$

where, as before, $x_0 \notin \overline{\Omega(\alpha)}$. The function $u$ is smooth in $\Omega(\alpha)$ and

$$\mathcal{L}(u)(x) = (\mathcal{L}d^{-Q-1})(x^{-1}_0 \circ x) = 0, \quad x \in \Omega(\alpha).$$

Moreover, $u > 0$ and $u \in L^p(\Omega(\alpha))$ since, from the assumption $\frac{Q}{2} > \frac{p}{p-1}$, it follows that

$$p(Q-2) > Q.$$
Remark 3.1. The assumption \( u \geq 0 \) in Theorem 1.1 cannot be removed. Indeed, if \( x_0 \notin \Omega(\alpha) \), the function
\[
u_k(x) := \partial_{x_N}^k (d(x_0^{-1} \circ x))^{2-Q}
\]
is \( \mathcal{L} \)-harmonic in \( \Omega(\alpha) \) for every \( k \in \mathbb{N} \), and \( \delta_\lambda \)-homogeneous of degree \( 2 - Q - k\sigma_N \).

Then, if \( k > \frac{2}{\sigma_N} \), \( u_k \in L^1(\Omega(\alpha)) \). Thus, with this choice of \( k \), \( u_k \) is a summable \( \mathcal{L} \)-harmonic function in \( \Omega(\alpha) \) and \( u_k \neq 0 \).

We would like to stress that in the previous argument we used the following properties:

(i) the differential operator \( \partial_{x_N} \) is \( \delta_\lambda \)-homogeneous of degree \( \sigma_N \) and commutes with \( \mathcal{L} \);
(ii) \( \mathcal{L} \) is left translation invariant with respect to the composition law \( \circ \);
(iii) \( d^2 - Q \) is \( \mathcal{L} \)-harmonic out of the origin.

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